SKOLEM DENSITY PROBLEMS OVER ALGEBRAIC P$\mathcal{S}$C FIELDS OVER RINGS*

by

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Introduction

Let $\tilde{\mathbb{Q}}$ be the field of all algebraic numbers and $\tilde{\mathbb{Z}}$ the ring of all algebraic integers. Skolem [Sko] considers a polynomial $f \in \tilde{\mathbb{Z}}[X_1, \ldots, X_n]$ and asked when does there exist $\mathbf{x} = (x_1, \ldots, x_n) \in \tilde{\mathbb{Z}}^n$ such that $f(\mathbf{x})$ is a unit of $\tilde{\mathbb{Z}}$. Skolem's answer was that this happens exactly when f is **primitive**, i.e., its coefficients have no common divisor.

Cantor and Roquette [CaR] generalize Skolem's criterion to several rational functions and strengthen it with a density statement.

Pop [Pop] considers a finite set $\mathcal{S}$ of primes and the maximal extension $N = \mathbb{Q}_{\mathrm{tot},\mathcal{S}}$ of $\mathbb{Q}$ in which each $p \in \mathcal{S}$ totally splits. He proves that N is P$\mathcal{S}$C. That is, each absolutely irreducible variety V which is defined over N and has a simple $\bar{N}$-rational point for each p-adic closure $\bar{N}$ of N, for each $p \in \mathcal{S}$, has also an N-rational point.

In [JR1] we prove that for almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^e$, the field $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ is PAC over $\mathbb{Z}$. That is, if $\varphi\colon V \to \mathbb{A}^r$ is a dominating separable rational map of an absolutely irreducible variety V of dimension r defined over $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$, then there exists $\mathbf{x} \in V(\tilde{\mathbb{Q}}(\boldsymbol{\sigma}))$ such that $\varphi(\mathbf{x}) \in \mathbb{Z}^r$.

In this note we take an axiomatic approach and consider an algebraic extension M_0 of $\mathbb{Q}$ which is PAC over $\mathbb{Z}$. Let $M = M_0 \cap N$ and denote the ring of integers of M by $\mathbb{Z}_M$. We prove that M is **weakly P$\mathcal{S}$C over** $\mathbb{Z}_M$. This means that M satisfies the following condition:

For every absolutely irreducible polynomial $h \in M[T, Y]$ which is monic in Y such that all the roots of $h(0, Y)$ are distinct and in N, and for each $g \in M[T]$ such that $g(0) \neq 0$ there exists $(a, b) \in \mathbb{Z}_M \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$.

Then we raise the level of axiomatization and consider a subfeld M of N which is weakly P$\mathcal{S}$C over $\mathbb{Z}_M$. We generalize the notion of a "Skolem problem" of Cantor and Roquette to M, and prove, as in [CaR, Thm. 5.1], that it is solvable in M if and only if it is locally everywhere solvable. In particular we prove:

THEOREM A: *Let $\mathcal{T}$ be a finite set of rational primes. Let M be a subfield of N which is weakly P$\mathcal{S}$C over $\mathbb{Z}_M$. Let $f_1, \ldots, f_m \in \tilde{\mathbb{Q}}[X_1, \ldots, X_n]$ be polynomials such that $v(f_i) = 0$ for each valuation v of $\tilde{\mathbb{Q}}$ which does not lie over $\mathcal{T}$ and each $1 \leq i \leq m$. Let $\mathbf{a} \in M^n$ and let γ be a positive integer. Then there exists $\mathbf{x} \in M^n$ such that*

(a) $v(\mathbf{x} - \mathbf{a}) > \gamma$ *for each valuation* v *of* $\tilde{\mathbb{Q}}$ *which lie over* $\mathcal{T}$*, and*

(b) $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ *are* $\mathcal{T}$*-units.*

Theorem A will be used in a subsequent paper to prove a Rumely local global principle and a strong approximation theorem for M. As a result, we prove in that paper, that M is P$\mathcal{S}$C. Indeed, it is even "P$\mathcal{S}$C over $\mathbb{Z}_M$" (which is stronger than "weakly P$\mathcal{S}$C over $\mathbb{Z}_M$"). The second author will use the strong approximation theorem (with $\mathcal{S} = \emptyset$) in his Ph.D thesis to prove that the theory of all elementary statements which are true in the ring of integers of $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$, for almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^e$, is decidable.

Since $\tilde{\mathbb{Q}}$ is PAC over $\mathbb{Z}$, Theorem A applies in particular to $M = \mathbb{Q}_{\mathrm{tot},\mathcal{S}}$.

The case where $\mathcal{S}$ is empty and $M = M_0 = \tilde{\mathbb{Q}}$ is a special case of [CaR, Thm. 5.1]. In particular, if we take in this case $m = n+1$, $f_i = 1/X_i$, $i = 1, \ldots, n$, and $f_{n+1} = f \in \tilde{\mathbb{Z}}[X_1, \ldots, X_n]$ is a primitive polynomial, we find that Skolem's original problem has a solution already over M. Thus, there exists a vector $\mathbf{x}$ of integral elements in M such that $f(\mathbf{x})$ is a unit of $\tilde{\mathbb{Z}}$.

The case where $M = \tilde{\mathbb{Q}}(\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} \in G(\mathbb{Q})^e$ chosen at random, and $\mathcal{S} = \emptyset$ appears in the unpublished manuscript [Ja1] from 1989.

It is perhaps worthwhile to make a list of the lemmas whose proofs make a direct use of the assumption "M is weakly P$\mathcal{S}$C over $\mathbb{Z}_M$". They are 1.8, 1.9, and 3.2.

As is usually the case, our results are stated and proved for an arbitrary Dedekind domain whose quotient field is global. The latter assumption is essential for the proofs because we use the strong approximation theorem and the finiteness of the class number.

1. Semi uniform density

Recall that field M is **PAC** if every nonvoid absolutely irreducible variety over M has an M-rational point. Let v be a valuation of a PAC field M. A theorem of Frey and Prestel [FJ2, Thm. 10.14] says that the Henselian closure M_v of M with respect to v coincides with its separable closure M_s. Moreover, M is w-dense in its algebraic closure for each extension w of v (Prestel [JR1, Lemma 9.1]). In this section we prove that if M is, say, PAC over $\mathbb{Z}$, then the density is even 'semi uniform'.

To fix notation consider a field K, let $\tilde{K}$ (resp., K_s) be its algebraic (resp., separable) closure, and let $G(K) = \mathcal{G}(K_s/K)$ be the absolute Galois group of K. We extend the action of $G(K)$ to $\tilde{K}$ in the unique possible way. For each valuation v of K let O_v (resp., Γ_v) be the valuation ring (resp., value group) of v. Each Henselian closure of K at v is the decomposition field K_w in K_s of some extension w of v to K_s (or to $\tilde{K}$). Choose one of these fields, denote it by K_v and let, $K_{\mathrm{t}v} = \bigcap_{\sigma\in G(K)} K_v^\sigma$. Then $K_{\mathrm{t}v}$ is a Galois extension of K. If the residue characteristic of v is 0 or K is a function field of one variable over a field K_0 and v is trivial on K_0, then $K_{\mathrm{t}v}$ is the maximal algebraic extension of K in which v totally splits. Since all fields K_w are conjugate to each other over K, the field $K_{\mathrm{t}v}$ is independent of K_v.

Consider now a set $\mathcal{S}$ of valuations of K and let $O_{\mathcal{S}} = \bigcap_{v\in\mathcal{S}} O_v$ be its holomorphy ring. If L is an algebraic extension of K, we denote the set of all valuations of L which extend those in $\mathcal{S}$ by $\mathcal{S}_L$. For $L = \tilde{K}$ we set $\tilde{\mathcal{S}} = \mathcal{S}_{\tilde{K}}$.

If L is a normal extension of K, then $\mathrm{Aut}(L/K)$ acts on $\mathcal{S}_L$ according to the following formula:

$$v^\sigma(x^\sigma) = v(x), \text{ for } v \in \mathcal{S}_L \text{ and } x \in L.$$

We may choose a subset $\mathcal{S}_0$ of $\mathcal{S}_L$ which contains exactly one extension of each valuation in $\mathcal{S}$. Then, for each $w \in \mathcal{S}_L$ there exist $v \in \mathcal{S}_0$ and $\sigma \in \mathrm{Aut}(L/K)$ such that $w = v^\sigma$. We say that $\mathcal{S}_0$ **represents** $\mathcal{S}_L$ **over** K.

We also consider the following Galois extension of K:

$$K_{\mathrm{tot},\mathcal{S}} = \bigcap_{v\in\mathcal{S}} K_{\mathrm{t}v}.$$

If v is as above for each $v \in \mathcal{S}$, then $K_{\mathrm{tot},\mathcal{S}}$ is the maximal algebraic extension of K in which each $v \in \mathcal{S}$ splits. If $\mathcal{T}$ is a set of valuations of K which contains $\mathcal{S}$, then $K_{\mathrm{tot},\mathcal{T}} \subseteq K_{\mathrm{tot},\mathcal{S}}$. If L is a separable algebraic extension of K, then $K_{\mathrm{tot},\mathcal{S}} \subseteq L_{\mathrm{tot},\mathcal{S}_L}$. In particular, if $L \subseteq K_{\mathrm{tot},\mathcal{S}}$, $w \in \mathcal{S}_L$ and $v = w|_K$, then L_w is conjugate over K to K_v. Hence, $K_{\mathrm{tot},\mathcal{S}} = L_{\mathrm{tot},\mathcal{S}_L}$. If $\mathcal{S}$ is empty, define $K_{\mathrm{tot},\mathcal{S}} = K_s$.

Data 1.1: We fix the following data for the rest of this section:

K is a field.

K' is a purely inseparable extension of K.

$N = K'_{\mathrm{tot},\mathcal{S}}$.

O is a Dedekind domain with quotient field K. For an algebraic extension M of K we denote the integral closure of O in M by O_M.

$\mathcal{S} \subseteq \mathcal{T}$ are finite sets of valuations of K which are associated with maximal ideals of O. In particular, $O \subseteq O_\mathcal{T} \subseteq O_\mathcal{S}$ and both $O_\mathcal{T}$ and $O_\mathcal{S}$ are Dedekind domains. We identify each $v \in \mathcal{T}$ with its unique extension to K'. ∎

From §2 on, we take K' to be K_{ins}, which is the maximal purely inseparable extension of K. Note that $K_{\mathrm{ins},\mathrm{t}v} = K_{\mathrm{t}v,\mathrm{ins}}$. Hence, $K_{\mathrm{ins},\mathrm{tot},\mathcal{S}} = K_{\mathrm{tot},\mathcal{S},\mathrm{ins}}$.

Lemma 1.2: *Suppose that $\mathcal{T}$ is nonempty. Let $f \in K'[X]$ be a polynomial of degree n with n distinct roots $x_1, \ldots, x_n$. Then, for each set $\{\varepsilon_v \in \Gamma_v \| v \in \mathcal{T}\}$ there exists a set $\{\delta_v \in \Gamma_v \| v \in \mathcal{T}\}$ with the following property: If*

(1a) *$g \in N[X]$ is a polynomial of degree n with $w(g - f) > \delta_w$ for each $w \in \mathcal{T}_N$, and where $\delta_w = \delta_{w|_K}$,*

then

(1b) *the roots of g are distinct, and for each $v \in \mathcal{T}$ and each $w \in \tilde{\mathcal{T}}$ which lies over v they can be enumerated as $y_1, \ldots, y_n$ such that $w(y_i - x_i) > \varepsilon_v$. Moreover, for each $v \in \mathcal{S}$ and each $w \in \tilde{\mathcal{T}}$ that lies over v we have $K'_w(x_i) = K'_w(y_i)$.*

In particular, if all roots of f belong to N, then so do the roots of g.

Proof: Choose a set $\mathcal{T}_0$ that represents $\tilde{\mathcal{T}}$ over K. By a combination of the theorem about the continuity of roots of polynomials and Krasner's lemma there exists a set $\{\delta_v \in \Gamma_v \| v \in \mathcal{T}\}$ such that if $g \in N[X]$ is a polynomial of degree n with $w(g - f) > \delta_w$

for each $w \in \mathcal{T}_0$, then (1b) holds with $\tilde{\mathcal{T}}$ replaced by $\mathcal{T}_0$ (e.g., [Ja2, Prop. 12.3]). Note that [Ja2, Prop. 12.3] is formulated for monic polynomials. However, if f and g are not monic, then we may divide them by their highest coefficients and possibly increase δ_v.

Consider now a polynomial g as in (1a). For each $w' \in \tilde{\mathcal{T}}$, there exist $w \in \mathcal{T}_0$ and $\sigma \in G(K)$ such that $w' = w^\sigma$. Since N/K' is Galois, $g^{\sigma^{-1}} \in N[X]$. Also, $w(g^{\sigma^{-1}} - f) = w'(g - f) > \delta_{w'} = \delta_w$. Moreover, σ permutes the roots of f and maps the roots of $g^{\sigma^{-1}}$ onto the roots of g. Since (1b) holds for w and $g^{\sigma^{-1}}$, it also holds for w' and for g.

If all roots of f belong to N, then they also belong to K'_w for each $v \in \mathcal{S}$ and each $w \in \tilde{\mathcal{T}}$ which lies over v. Hence, by the first part of the lemma, the same holds for g. It follows that all roots of g belong to N. ∎

Definition 1.3: (a) Let R be a subset of a field M. We say that M is **PAC over** R if it has the following property: For every absolutely irreducible variety V of dimension $r \geq 0$ and for each dominating separable rational map $\varphi\colon V \to \mathbb{A}^r$ over M there exists $\mathbf{a} \in V(M)$ such that $\varphi(\mathbf{a}) \in R^r$.

Note that if $R \subseteq R' \subseteq M$ and M is PAC over R, then it is also PAC over R'.

(b) Let M be a subextesnion of N/K'. We say that M is **weakly P$\mathcal{S}$C over** O_M if it satisfies the following condition: For each absolutely irreducible polynomial $h \in M[T,Y]$ which is monic in Y such that the roots of $h(0,Y)$ are distinct and in N, and for each $g \in M[T]$ such that $g(0) \neq 0$ there exists $(a,b) \in O_M \times M$ such that $h(a,b) = 0$ and $g(a) \neq 0$.

(c) Let M be a perfect algebraic extension of K. We say that M is **P$\mathcal{S}$C over** O_M if for every absolutely irreducible variety V of dimension r and every dominating separable rational map $\varphi\colon V \to \mathbb{A}^r$ over M there exists $\mathbf{a} \in V(M)$ such that $\varphi(\mathbf{a}) \in O_M^r$ provided that for each $v \in \mathcal{S}_M$ there exists $\mathbf{a}_v \in V_{\mathrm{simp}}(M_v)$ such that $\varphi(\mathbf{a}_v) \in O_{M,v}^r$. Here $O_{M,v}$ is the valuation ring of a Henselian closure M_v of M at v.

Note that if M is P$\mathcal{S}$C over O_M, than M is weakly P$\mathcal{S}$C over O_M. If S is empty, and M is P$\mathcal{S}$C over O_M, then M is PAC over O_M. ∎

LEMMA 1.4: *Let M_0 be an algebraic extension of K and let $M = M_0 \cap N$. Suppose that M_0 is PAC over O_M. Then M is weakly* P$\mathcal{S}$C *over O_M.*

Proof: Let $h \in M[T,Y]$ be an absolutely irreducible polynomial which is monic in Y such that all the roots of $h(0,Y)$ are distinct and in N (hence $\frac{\partial h}{\partial Y} \neq 0$), and let $g \in M[T]$ such that $g(0) \neq 0$. Let L be a finite subextension of M/K which contains the coefficients of h. Lemma 1.2, applied to $K'L$ and $\mathcal{S}_L$ instead of to K' and $\mathcal{S}$ (with $\mathcal{T} = \mathcal{S}$), gives a set $\{\delta_v \in \Gamma_v \mid v \in \mathcal{S}_L\}$ with the following property: If $k \in N[Y]$ is a polynomial of the same degree as $h(0,Y)$ such that $w(k(Y) - h(0,Y)) > \delta_{w|_L}$ for each $w \in \mathcal{S}_N$, then all the roots of k are distinct and in N. Also, there exists a set $\{\gamma_v \in \Gamma_v \mid v \in \mathcal{S}_L\}$ such that for each $w \in \mathcal{S}_N$, if $a \in N$ satisfies that $w(a) > \gamma_{w|_L}$, then $w(h(a,Y) - h(0,Y)) > \delta_{w|_L}$. Take $0 \neq m \in O_L$ such that $v(m) > \gamma_v$ for each $v \in \mathcal{S}_L$. Since M_0 is PAC over O_M, the absolutely irreducible polynomial $h(mX,Y)$ has a zero (c,b) in $O_M \times M_0$ such that $g(mc) \neq 0$ [JR1, Lemma 1.3]. Hence, $a = mc \in O_M$ satisfies $h(a,b) = 0$ and $g(a) \neq 0$. Check that $w(h(a,Y) - h(0,Y)) > \delta_{w|_L}$ for each $w \in \mathcal{S}_N$. Hence, all roots of $h(a,Y)$ belong to N. In particular $b \in M_0 \cap N = M$. Conclude that M is weakly P$\mathcal{S}$C over O_M. ∎

Remark 1.5: Our main concern in the works [JR1], the present work, and [JR2] are the concepts of PAC and P$\mathcal{S}$C over a subring. 'Weakly P$\mathcal{S}$C field over a subring' is only a technical concept. The main extra condition which goes into its definition and makes it a 'non-intrinsic' property of the field is (in the notation of Definition 1.3) that the roots of $h(0,Y)$ lie in N. However, it helps to transfer a 'PAC field over a subring' into a 'P$\mathcal{S}$C field over a subring'. Indeed, we prove in [JR1] that if O is a countable separably Hilbertian integral domain (e.g., O is the ring of integers of a global field) and e is a positive integer, then for almost all $\boldsymbol{\sigma} \in G(K)^e$, the fields $K_s(\boldsymbol{\sigma})$ and $\tilde{K}(\boldsymbol{\sigma})$ are PAC over O and therefore also over O_M. Here $K_s(\boldsymbol{\sigma})$ is the fixed field of $\boldsymbol{\sigma}$ in K_s and $\tilde{K}(\boldsymbol{\sigma})$ is the maximal purely inseparable extension of $K_s(\boldsymbol{\sigma})$ in K_s. The close 'almost all' is used in the sense of the Haar measure of the compact group $G(K)^e$.

It follows from Lemma 1.4 that for almost all $\boldsymbol{\sigma} \in G(K)^e$ the field $M = \tilde{K}(\boldsymbol{\sigma}) \cap N$ is perfect and weakly P$\mathcal{S}$C over O_M.

One of the consequences of [JR2] is that if K is a global field and M is a perfect subextension of N/K which is weakly P$\mathcal{S}$C over O_M, then M is P$\mathcal{S}$C over O_M. Thus, for almost all $\boldsymbol{\sigma} \in G(K)^e$ the field $M = \tilde{K}(\boldsymbol{\sigma}) \cap N$ is P$\mathcal{S}$C over O_M.

Of course, we will reach this conclusion only at the end of [JR2]. ∎

We explore properties of subextensions M of N/K' which are weakly P$\mathcal{S}$C over O_M. However, except in Lemma 1.7 we apply this assumption in this note only for polynomials $h(T,Y)$ which are linear in T. Lemma 1.7 is an important ingredient in the forthcoming paper [JR2].

Example 1.6: Let M be a subextension of N/K' which is weakly P$\mathcal{S}$C over O_M. Let f,r,g be nonzero polynomials in $M[X]$ such that f is monic, all its roots lie in N and they are distinct. Suppose also that $\gcd(r(X),f(X))=1$, $\deg(r)<\deg(f)$, and $g(0)\neq 0$. Then $h(T,X)=r(X)T+f(X)$ is an absolutely irreducible polynomial which is monic in X. Moreover, the roots of $h(0,X)=f(X)$ are distinct and each of them belong to N. Hence, there exists $(a,b)\in O_M\times M$ such that $r(b)a+f(b)=0$ and $g(a)\neq 0$. ∎

Our first result on weakly P$\mathcal{S}$C fields generalizes [JR1, Lemma 1.7(b)].

Lemma 1.7: *Let M be a subextension of N/K' which is weakly* P$\mathcal{S}$C *over O_M. Consider a conservative regular extension F of transcendence degree 1 of M and let Γ be its unique nonsingular smooth model. Let t be an element in $F\smallsetminus M$ whose zeros are distinct and each of them belongs to $\Gamma(N)$. Finally, let A be a finite subset of $M^\times$. Then there exists $\mathbf{p}\in\Gamma(M)$ such that $t(\mathbf{p})\in O_M\smallsetminus A$.*

Proof: We first note that F is a separable extension of $M(t)$, since otherwise t would have multiple zeros. Let $y\in F$ be a primitive element for the extension $F/M(t)$ which is integral over $M[t]$, let $h(t,Y)=\mathrm{irr}(y,M(t))$, and let $g_0(T)=\prod_{a\in A}(T-a)$. Then $h\in M[T,Y]$ is an absolutely irreducible polynomial which is monic in Y such that the roots of $h(0,Y)$ are distinct and in N, and $g_0(0)\neq 0$. Take polynomials $g_1,g_2\in M[T,Y]$ and $0\neq g_3\in M[T]$ without a common multiple in $M[T]$ such that

$$g_1(T,Y)h(T,Y)+g_2(T,Y)\frac{\partial h}{\partial Y}(T,Y)=g_3(T). \tag{2}$$

Note that $g_3(0)\neq 0$, since otherwise $h(0,Y)$ would have multiple zeros. Let $g=g_0g_3$. Since M is weakly P$\mathcal{S}$C over O_M, there exists $(a,b)\in O_M\times M$ such that $h(a,b)=0$

and $g(a) \neq 0$. In particular, $a \notin A$ and, by (2), $\frac{\partial h}{\partial Y}(a,b) \neq 0$. Hence, we can extend the specialization $(t,y) \to (a,b)$ to an M-rational place π of F. This place corresponds to a point $\mathbf{p}$ in $\Gamma(M)$ which satisfies $t(\mathbf{p}) = a \in O_M \smallsetminus A$. ∎

Lemma 1.8 (Quasi uniform approximation): *Let M be a subextension of N/K' which is weakly* P$\mathcal{S}$C *over O_M. Let $x \in N$ and for each $v \in \mathcal{T}$ let $\varepsilon_v \in \Gamma_v$. Then M has a finite subset B such that for each $v \in \mathcal{T}$ and each $w \in \tilde{\mathcal{T}}$ that lies over v there exists $b \in B$ such that $w(b-x) > \varepsilon_v$.*

Proof: Assume without loss that $x \neq 0$ and $\mathcal{T} \neq \emptyset$. We say that a monic polynomial $h \in M[X]$ is **admissible** if it has only simple roots and each of them belongs to N. Since N/K' is a Galois extension, $\mathrm{irr}(x,M)$ is an admissible polynomial which has x as a root. Hence, it suffices to prove the following statement about admissible polynomials h:

(3) There exists a finite set $B_h \subset M$ such that for each root z of h and for each $w \in \tilde{\mathcal{T}}$ there exists $b \in B_h$ such that $w(b-z) > \varepsilon_w$. Here we set $\varepsilon_w = \varepsilon_v$ if $v = w|_K$.

The case $\deg(h) = 1$ being trivial we assume that $d = \deg(h) \geq 2$ and proceed by induction on d. Let L be a finite extension of K' which contains the coefficients of h and is contained in M. For each $v_0 \in \mathcal{T}$ and each $v \in \mathcal{T}_L$ which lies over v_0 let $\varepsilon_v = \varepsilon_{v_0}$. Note that $L_{\mathrm{tot},\mathcal{S}_L} = K'_{\mathrm{tot},\mathcal{S}} = N$. Hence, by Lemma 1.2 applied to $L, \mathcal{S}_L, \mathcal{T}_L$ instead of to $K', \mathcal{S}, \mathcal{T}$, there exists a set $\{\delta_v \in \Gamma_v \| v \in \mathcal{T}_L\}$ with the following property:

(4) Every monic polynomial $h_1 \in N[X]$ of degree d which satisfies $v(h_1 - h) > \delta_v$ for each $v \in \mathcal{T}_N$ is admissible and for each $w \in \tilde{\mathcal{T}}$ and each root z of h there exists a root y of h_1 such that $w(y-z) > \varepsilon_w$.

Choose $0 \neq m \in L$ such that $v(m) > \delta_v$ for each $v \in \mathcal{T}_L$. Since M is weakly P$\mathcal{S}$C over O_M, Example 1.6 (applied to $mT + h(Y)$ instead of to $r(Y)T + f(Y)$) gives $a \in O_M$ and $c \in M$ such that $ma + h(c) = 0$. It follows that the monic polynomial $h_1(X) = ma + h(X) \in M[X]$ of degree d satisfies $h_1(c) = 0$ and $v(h_1 - h) > \delta_v$ for each $v \in \mathcal{T}_M$. Hence, h_1 satisfies the conclusion of (4).

In particular $g(X) = h_1(X)/(X-c) \in M[X]$ is an admissible polynomial of degree $d-1$. By the induction hypothesis, there exists a finite subset $B_g \subseteq M$ such that for each root y of g and for each $w \in \tilde{\mathcal{T}}$ there exists $b \in B_g$ with $w(b-y) > \varepsilon_w$.

Let $B_h = B_g \cup \{c\}$ and consider $w \in \tilde{\mathcal{T}}$. Let z be a root of h. By (4) there exists a root y of h_1 such that $w(y-z) > \varepsilon_w$. So, $y = c$ or y is a root of g. In the later case there exists $b \in B_g$ such that $w(b-y) > \varepsilon_w$ and therefore $w(b-z) > \varepsilon_w$. In both cases the induction is complete. ∎

The next result sharpens the theorem of Frey and Prestel mentioned at the beginning of this section for PAC fields over rings. Recall that a set $\mathcal{T}$ of valuations of K are said to be **independent** if for all $v, w \in \mathcal{T}$ with $v \neq w$ there is no valuation ring of K which contains both O_v and O_w. In this case K satisfies the weak approximation theorem with respect to $\mathcal{T}$ [Ja2, Prop. 17.4].

PROPOSITION 1.9: *Let M be a subextension of N/K' which is weakly* P$\mathcal{S}$C *over O_M. Suppose that $\mathcal{T} = \mathcal{S} \cup \{v\}$ is an independent set of valuations of K and $v \notin \mathcal{S}$. Let v' be an extension of v to M. Then,*

(a) *the Henselian closure of M at v' is K'_s and*

(b) *M is $\tilde{v}$-dense in $\tilde{K}$ for each extension $\tilde{v}$ of v' to $\tilde{K}$.*

Proof: Consider $0 \neq x \in K'_s$, let $\varepsilon_v \in \Gamma_v$, and let $h_v = \mathrm{irr}(x, K')$. Let $n = \deg(h_v)$, choose n distinct elements $a_1, \ldots, a_n \in K'$, and let $h_{\mathcal{S}} = \prod_{i=1}^n (X - a_i)$. Let $\tilde{v}$ be an extension of v' to $\tilde{K}$. By Lemma 1.2, there exists a set $\{\delta_w \| w \in \mathcal{T}\}$ such that

(5a) if $h_1 \in N[X]$ is a polynomial of degree n and $w(h_1 - h_{\mathcal{S}}) > \delta_w$ for each $w \in \mathcal{S}_N$, then the roots of h_1 are distinct and lie in N, and

(5b) if $h_1 \in N[X]$ is a polynomial of degree n and $w(h_1 - h_v) > \delta_v$, for each $w \in \mathcal{T}_N$ which lies over v, then we can enumerate the roots of h_v as $x_1, \ldots, x_n$ and the roots of h_1 as $x'_1, \ldots, x'_n$ such that $\tilde{v}(x'_i - x_i) > \varepsilon_v$ and $K'_{\tilde{v}}(x'_i) = K'_{\tilde{v}}(x_i)$.

Apply the weak approximation theorem to find a polynomial $h \in K'[X]$ such that $w(h - h_{\mathcal{S}}) > \delta_w$ for each $w \in \mathcal{S}$ and $v(h - h_v) > \delta_v$. Also, choose $0 \neq m \in K$ such that $w(m) > \delta_w$ for each $w \in \mathcal{T}$. Since M is weakly P$\mathcal{S}$C over O_M, Example 1.6 supplies $c \in O_M$ and $a \in M$ such that $mc + h(a) = 0$. Thus, a is a root of the polynomial $h_1(X) = mc + h(X)$. By (5), $a \in N$ and $K'_{\tilde{v}}(a) = K'_{\tilde{v}}(x')$ for some root x' of h_v. It follows that $K'_{\tilde{v}}(x') \subseteq M_{\tilde{v}}$. In particular, if $K'(x)$ is a Galois extension of K', then

$K'_v(x) = K'_v(x') \subseteq M_{\tilde{v}}$. Conclude that $M_{\tilde{v}} = K'_s$.

In the general case (5b) gives a root b of h_1 such that $\tilde{v}(b-x) > \varepsilon_v$. Since b lies in N, we conclude that N is $\tilde{v}$-dense in K'_s. As K'_s is $\tilde{v}$-dense in $\tilde{K}$ [GeJ, Lemma 1.2], N is $\tilde{v}$-dense in $\tilde{K}$. Finally, by Lemma 1.8, M is $\tilde{v}$-dense in N. So, M is $\tilde{v}$-dense in $\tilde{K}$.

∎

In the case where $\mathcal{S}$ is empty, Lemma 1.8 takes a simpler form:

Corollary 1.10: *Let M be a separable algebraic extension of K' which is PAC over O_M. Let $x \in M_s$ and for each $v \in \mathcal{T}$ let $\varepsilon_v \in \Gamma_v$. Then M has a finite subset B such that for each $v \in \mathcal{T}$ and each $w \in \tilde{\mathcal{T}}$ that lies over v there exists $b \in B$ such that $w(b-x) > \varepsilon_v$.*

2. Units

The main results of this note depend on the assumption that K is a global field. So, we amend Data 1.1 to the following one:

Data 2.1: We fix the following data and assumption for the rest of this work:

O is a Dedekind domain with a global quotient field K;

$\mathcal{V} = \mathcal{V}_K$ is the set of all valuations of K which correspond to the nonzero prime ideals of O;

$\mathcal{S} \subseteq \mathcal{T}$ are finite subsets of $\mathcal{V}$;

$N = K_{\mathrm{tot},\mathcal{S},\mathrm{ins}}$ is the maximal purely inseparable extension of $K_{\mathrm{tot},\mathcal{S}}$;

M is a perfect subextension of N/K. We assume that M is weakly P$\mathcal{S}$C over O_M.

∎

Consider $v \in \mathcal{V}_K$ and a polynomial $f \in K[\mathbf{X}]$, where $\mathbf{X} = (X_1, \ldots, X_n)$. Then $v(f)$ is defined as the minimal value of the coefficients of f. If $f \in K(\mathbf{X})$, we write $f = g/h$ with $g, h \in K[\mathbf{X}]$ and let $v(f) = v(g) - v(h)$. Gauß' Lemma implies that v is a valuation of $K(\mathbf{X})$. If $\mathbf{f} = (f_1, \ldots, f_m)$ is a vector of rational functions in $K(\mathbf{X})$, we put $v(\mathbf{f}) = \min\{v(f_1), \ldots, v(f_m)\}$. Finally, for a subset $\mathcal{S}$ of $\mathcal{V}_K$ we set $V_{\mathcal{S}}(\mathbf{f}) = \min_{v \in \mathcal{S}} v(\mathbf{f})$. If the coefficients of the f_i's belong to some algebraic extension L of K, we set $V_{\mathcal{S}}(\mathbf{f}) = V_{\mathcal{S}_L}(\mathbf{f})$.

Let $v \in \mathcal{V}_K$ and $a \in K$. We follow an old tradition in algebraic number theory (although, it is somewhat inconsistent) and say that a is v**-integral** (resp., v**-unit**) if $v(a) \geq 0$ (resp., $v(a) = 0$). On the other hand for a subset $\mathcal{S}$ of $\mathcal{V}_K$ we say that a is $\mathcal{S}$-integral (resp., $\mathcal{S}$-unit) if $v(a) \geq 0$ (resp., $v(a) = 0$) for each $v \in \mathcal{V}_K \smallsetminus \mathcal{S}$. An element a of an algebraic extension L of K is $\mathcal{S}$**-integral** (resp., $\mathcal{S}$**-unit**) if a is $\mathcal{S}_L$-integral (resp., $\mathcal{S}_L$-unit).

We have extracted Lemma 2.2 below from the proof of [CaR, Thm. 5.1]. It produces $\mathcal{T}$-units with special properties. Among others, its proof uses the finiteness of the ideal class group of the ring of integers of global fields. Here we extend each $\sigma \in G(K)$ in the unique possible way to an element of $\mathrm{Aut}(\tilde{K}/K)$.

LEMMA 2.2: *Let ν and k_0 be positive integers. Consider elements $c_1, \ldots, c_d$ of $\tilde{K}^\times$*

which are permuted under $G(K)$. Then there exists an integer $k \geq k_0$ and elements $e_1, \ldots, e_d \in K_s^\times$ such that

(a) $w(e_i - 1) \geq \nu$ *for* $w \in \tilde{\mathcal{T}}$ *and* $i = 1, \ldots, d$,

(b) $u_i = e_i c_i^k$ *is a* $\mathcal{T}$*-unit,* $i = 1, \ldots, d$, *and*

(c) $c_i^\sigma = c_j$ *implies* $e_i^\sigma = e_j$ *for* $\sigma \in G(K)$ *and* $1 \leq i, j \leq d$.

Proof: Replace $c_1, \ldots, c_d$ by $c_1^q, \ldots, c_d^q$ for some power q of $\mathrm{char}(K)$ if necessary, to assume that $c_1, \ldots, c_d \in K_s$. Let L be a finite Galois extension of K which contains $c_1, \ldots, c_d$. For each $v \in \mathcal{T}_L$ and $1 \leq i \leq d$ consider the following fractional ideal of O_L:

$$A_i = \prod_{v \in \mathcal{T}_L} P_v^{v(c_i)},$$

where P_v is the prime ideal of O_L that corresponds to v. Each $\sigma \in \mathcal{G}(L/K)$ permutes $\mathcal{T}_L$ and satisfies $P_v^\sigma = P_{v^\sigma}$ for all $v \in \mathcal{T}_L$. Suppose that $c_i^\sigma = c_j$. Then $v(c_i) = v^\sigma(c_i^\sigma) = v^\sigma(c_j)$. Hence

$$A_i^\sigma = \prod_{v \in \mathcal{T}_L} (P_v^\sigma)^{v(c_i)} = \prod_{v \in \mathcal{T}_L} (P_{v^\sigma})^{v^\sigma(c_j)} = \prod_{w \in \mathcal{T}_L} P_w^{w(c_j)} = A_j. \tag{1}$$

Since L is a global field, the ideal class group of O_L is finite. In particular there exists a positive integer h such that A_i^h is a principal ideal. By (1) we can choose $z_1, \ldots, z_d \in L$ such that $A_i^h = O_L z_i$ and $c_i^\sigma = c_j$ implies $z_i^\sigma = z_j$ for each $\sigma \in \mathcal{G}(L/K)$. In particular $v(z_i) = 0$ for each $v \in \mathcal{V}_L \smallsetminus \mathcal{T}_L$, that is, z_i is a $\mathcal{T}$-unit. For $v \in \mathcal{T}_L$ we have $v(z_i) = v(c_i^h)$. So, $b_i = c_i^{-h} z_i$ satisfies $v(b_i) = 0$, $i = 1, \ldots, d$. Thus, b_i belongs to the group

$$U_{\mathcal{T}_L} = \{x \in L \| v(x) = 0 \text{ for each } v \in \mathcal{T}_L\} = \bigcap_{v \in \mathcal{T}_L} U_v.$$

Consider also the following subgroup of $U_{\mathcal{T}_L}$:

$$U_{\mathcal{T}_L, \nu} = \{u \in L \| v(u-1) \geq \nu \text{ for each } v \in \mathcal{T}_L\} = \bigcap_{v \in \mathcal{T}_L} U_{v,\nu}.$$

For each $v \in \mathcal{T}_L$, $(O_L/P_v)^\times$ is the multiplicative group of a finite field. It follows that $U_v/U_{v,\nu}$ is a finite group [CaF, pp. 5,6]. Since $U_{\mathcal{T}_L}/U_{\mathcal{T}_L,\nu}$ naturally embeds in $\prod_{v \in \mathcal{T}_L} U_v/U_{v,\nu}$, it is also a finite group. Thus there exists a positive integer $r \geq k_0/h$

such that $b_i^r \in U_{\mathcal{T}_L,\nu}$ for $i = 1, \ldots, d$. This means that $e_i = b_i^r$ satisfies (a). Then note that $u_i = z_i^r$ is a $\mathcal{T}$-unit. Conclude that also (b) holds with $k = rh \geq k_0$. Finally, check that (c) holds. ∎

3. Main Lemma

Lemma 3.2 below is the major step in the proof of the solvability of Skolem problems over M. Its proof, except Part D, has been extracted from the proof of [CaR, Thm. 5.1]. Its main ingredients is Lemma 2.2 and the strong approximation theorem. Data 2.1 remains in force throughout this section.

LEMMA 3.1: *The following statements on monic polynomials $f, h \in \tilde{K}[X]$ with $\mathcal{T}$-integral coefficients are equivalent.*
(a) *$f(x)$ is a $\mathcal{T}$-unit for each zero x of h.*
(b) *$h(t)$ is a $\mathcal{T}$-unit for each zero t of f.*

Proof: Let $t_1, \ldots, t_d$ be the zeros of f (with multiplicities) and let $x_1, \ldots, x_n$ be the zeros of h (with multiplicities). Then $f(x_j) = \prod_{i=1}^{d}(x_j - t_i)$ and $h(t_i) = \prod_{j=1}^{n}(t_i - x_j)$. As all t_i and x_j are $\mathcal{T}$-integral, both (a) and (b) are equivalent to "$t_i - x_j$ is a $\mathcal{T}$-unit for all i and j." ∎

LEMMA 3.2: *Consider a monic polynomial $f \in K[X]$ of degree d, an element $a \in K$, and a positive integer ε. Suppose that*
(1a) *the coefficients of f are $\mathcal{T}$-integral;*
(1b) *f has d distinct roots $t_1, \ldots, t_d \in K_s$; and*
(1c) *the discriminant of f, i.e. $\Delta = \prod_{i<j}(t_i - t_j)^2$, is a $\mathcal{T}$-unit.*
Then there exist polynomials $k_0, l \in K[X]$ with $\mathcal{T}$-integral coefficients and a positive integer γ such that k_0 is monic and if a monic polynomial $k \in N[X]$ with $\mathcal{T}$-integral coefficients satisfies $\deg(k) = \deg(k_0)$ and $V_{\mathcal{T}}(k - k_0) > \gamma$, then the polynomial $h(X) = k(X)f(X) + l(X)$ has $\mathcal{T}$-integral coefficients and each root x of h satisfies:
(2a) *$f(x)$ is a $\mathcal{T}$-unit.*
(2b) *$V_{\mathcal{T}}(x - a) > \varepsilon$, and*
(2c) *x belongs to N.*
Moreover, there exists $k \in O_M[X]$ as above such that h has a root in M.

Proof: Use the strong approximation theorem to choose distinct $\mathcal{T}$-integral elements $a_1, \ldots, a_d \in K$ such that $V_{\mathcal{T}}(a_i - a) > \varepsilon$ for $i = 1, \ldots, d$, and $a_i \neq t_j$ for all $1 \leq i, j \leq d$.

PART A: *The polynomial* $g_0(X) = \prod_{i=1}^d (X - a_i)$. By the choice of the a_i's, the coefficients of g_0 are $\mathcal{T}$-integral. Since f is monic, (1a) implies that its roots t_i are also $\mathcal{T}$-integral. Hence, by Lemma 3.1, $c_i = g_0(t_i)$ is a nonzero element of K_s which is $\mathcal{T}$-integral, $i = 1, \dots, d$.

By (1c), $t_i - t_j$ is a $\mathcal{T}$-unit for $i \neq j$. Hence, the coefficients of the polynomial

$$f_i(X) = \prod_{\substack{j=1 \\ j\neq i}}^{d} \frac{X - t_j}{t_i - t_j}$$

of degree $d-1$ lie in K_s and are $\mathcal{T}$-integral. Moreover,

(3a) $f_i(t_j) = 0$ if $i \neq j$ and $f_i(t_i) = 1$, and

(3b) $t_i^\sigma = t_j$ implies $c_i^\sigma = c_i$ and $f_i^\sigma = f_j$, for all i, j and $\sigma \in G(K)$

It follows that the two sides of (4) below coincide at $t_1, \dots, t_d$:

(4) $$g_0(X) = f(X) + \sum_{i=1}^{d} c_i f_i(X).$$

Since both of them are monic polynomials of degree d, (4) is an equality of polynomials.

By Lemma 1.2, there exists $\delta > 0$ such that if $v \in \mathcal{T}$ and if $g \in K[X]$ is a monic polynomial of degree d which satisfies

(5) $$v(g - g_0) > \delta,$$

then the roots of g are distinct and belong to $K_{\mathrm{t}v}$. Also, for each root x of g and for each $w \in \tilde{\mathcal{V}}$ over v there exists i (which depends on w) such that $w(x - a_i) > \varepsilon$. It follows from the choice of the a_i's that $w(x - a) > \varepsilon$.

PART B: *Construction of* l. By Lemma 2.2, there exist $e_1, \dots, e_d \in K_s$ and a positive integer $s \geq 2$ such that

(6a) $V_{\mathcal{T}}(e_i - 1) > \delta - \min_{1\leq j\leq d}(v(c_j) + v(f_j))$, $i = 1, \dots, d$,

(6b) $u_i = e_i c_i^s$ is a $\mathcal{T}$-unit, $i = 1, \dots, d$, and

(6c) $t_i^\sigma = t_j$ implies $e_i^\sigma = e_j$, hence $u_i^\sigma = u_j$, for $\sigma \in G(K)$ and $1 \leq i, j \leq d$.

Consider the polynomial

$$g_1(X) = f(X) + \sum_{i=1}^{d} e_i c_i f_i(X) \in K_s[X].$$

By (3b) and (6c), $g_1^\sigma(X) = g_1(X)$ for each $\sigma \in G(K)$. Hence, $g_1 \in K[X]$. By (4) and (6a), g_1 is a monic polynomial of degree d that satisfies (5) (with $g = g_1$) for each $v \in \mathcal{T}$. It follows that

(7a) for each $w \in \tilde{\mathcal{T}}$, $w(x-a) > \varepsilon$ for each root x of g_1 and

(7b) the roots of g_1 are distinct and belong to $\bigcap_{v\in\mathcal{T}} K_{\mathrm{t}v} = K_{\mathrm{tot},\mathcal{T}}$.

Also, by (3a),

(7c) $g_1(t_i) = e_i c_i$, $i = 1,\ldots,d$.

Let $l(X) = \sum_{i=1}^d u_i f_i(X) \in K[X]$ (again, use (3b) and (6c)). By (6b), the coefficients of l are $\mathcal{T}$-integral. By (3a), $l(t_i) = u_i$, $i = 1,\ldots,d$.

Part C: *Construction of γ and of k_0.* By (3a), $\sum_{i=1}^d c_i f_i(t_j) = c_j$. Hence

(8) $f(X)$ and $\sum_{i=1}^d c_i f_i(X)$ have no common root.

It follows that for each $v \in \mathcal{T}$ we may choose distinct $b_{2v},\ldots,b_{sv} \in K$ such that

(9a) $v(b_{jv} - 1) > \delta - \min_{1\le i\le d}(v(c_i) + v(f_i))$, $j = 2,\ldots,s$,

(9b) $b_{2v}\cdots b_{sv} = 1$, and

(9c) $f(t) + b_{jv}\sum_{i=1}^d c_i f_i(t) \neq 0$ for each root t of $g_1(X)$, $j = 2,\ldots,s$.

For each j between 2 and s let

$$g_{jv}(X) = f(X) + b_{jv}\sum_{i=1}^d c_i f_i(X).$$

Then $g_{jv}(X)$ is a monic polynomial of degree d in $K[X]$ which satisfies (5) and therefore has d distinct roots, each of them x belongs to $K_{\mathrm{t}v}$ and satisfies $w(x-a) > \varepsilon$ for each $w \in \tilde{\mathcal{V}}$ over v. Moreover, by (9c), $g_{jv}(X)$ and $g_1(X)$ have no root in common. Since, by (8), this is also the case for $g_{jv}(X)$ and $\sum_{i=1}^d c_i f_i(X)$, and since $b_{2v},\ldots,b_{sv}$ are distinct, $g_{jv}(X)$ and $g_{j'v}(X)$ have no root in common if $j' \neq j$. Finally, by (3a), $g_{jv}(t_i) = b_{jv}c_i$, $i = 1,\ldots,d$.

It follows, by (7b), that the monic polynomial $h_{0v}(X) = g_1(X)g_{2v}(X)\cdots g_{sv}(X)$ of degree ds has ds distinct roots in $K_{\mathrm{t}v}$ and each of them x satisfies $w(x-a) > \varepsilon$ for each $w \in \tilde{\mathcal{V}}$ over v. Moreover, by (7c), (9b), and (6b), $h_{0v}(t_i) = e_i c_i \cdot b_{2v}c_i \cdots b_{sv}c_i = e_i c_i^s = u_i$, $i = 1,\ldots,d$. It follows that the following equality holds for each monic polynomial $k_{0v} \in K[X]$ of degree $ds - d$ and for $i = 1,\ldots,d$:

(10) $$h_{0v}(t_i) = k_{0v}(t_i)f(t_i) + l(t_i).$$

Choose distinct $t_{d+1}, \dots, t_{ds}$ in K which do not belong to the set $\{t_1, \dots, t_d\}$ and solve the corresponding nonsingular system of linear equations over K to find monic polynomial $k_{0v} \in K[X]$ of degree $ds - d$ such that (10) holds in addition to $i = 1, \dots, d$ also for $i = d + 1, \dots, ds$. It follows that

$$h_{0v}(X) = k_{0v}(X)f(X) + l(X).$$

By Lemma 1.2 applied to $K' = K_{\mathrm{ins}}$ and $\mathcal{S} = \mathcal{T} = \{v\}$, there exists $\gamma_v > 0$ such that if a polynomial $h \in K_{\mathrm{t}v,\mathrm{ins}}[X]$ of degree ds satisfies $w(h - h_{0v}) > \gamma_v$ for each $w \in \tilde{\mathcal{V}}$ over v, then the roots of h are distinct and belong to $K_{\mathrm{t}v,\mathrm{ins}}$. Moreover, for each root x of h and each $w \in \tilde{\mathcal{V}}$ over v there exists a root x_0 of h_{0v} such that $w(x - x_0) > \varepsilon$ and hence $w(x - a) > \varepsilon$.

Let $\gamma \geq \max_{v \in \mathcal{T}} \gamma_v - V_{\mathcal{T}}(f)$ be a positive integer and use the strong approximation theorem to find a monic polynomial $k_0 \in K[X]$ of degree $sd - d$ with $\mathcal{T}$-integral coefficients such that $v(k_0 - k_{0v}) > \gamma$ for each $v \in \mathcal{T}$.

Part D: *Conclusion of the proof.* Consider now a monic polynomial $k \in N[X]$ with $\mathcal{T}$-integral coefficients of degree $ds - d$ such that $V_{\mathcal{T}}(k - k_0) > \gamma$ and let $h(X) = k(X)f(X) + l(X)$. Then, by (6b) and Part B, $h(t_i) = l(t_i) = u_i$ is a $\mathcal{T}$-unit, $i = 1, \dots, d$. By Lemma 3.1, (2a) holds. Also, $w(h - h_{0v}) > \gamma_v$ for each $v \in \mathcal{T}$ and each $w \in \tilde{\mathcal{V}}$ over v. Hence, each root x of h satisfies $V_{\mathcal{T}}(x - a) > \varepsilon$. This proves (2b). Moreover, the roots of h are distinct and each of them bolongs to $\bigcap_{v \in \mathcal{S}} K_{\mathrm{t}v,\mathrm{ins}} = N$. This proves (2c).

The conclusion of the preceding paragraph holds in particular for $k = k_0$ and $h_0(X) = k_0(X)f(X) + l(X)$. Thus, the roots of h_0 are distinct and each of them belongs to N. Note that $\deg(l) = d - 1 < sd = \deg(k_0 f)$. Since both k_0 and f are monic, h is a monic polynomial of degree sd. Thus, $\deg(f) = d < \deg(h)$. Finally, since $l(t_i) = u_i \neq 0$, $i = 1, \dots, d$, the polynomials $f(X)$ and $h_0(X)$ have no root in common. Choose now $0 \neq m \in O$ such that $V_{\mathcal{T}}(m) > \gamma$. Apply Example 1.6 to find $(b, x_1) \in O_M \times M$ such that $mbf(x_1) + h_0(x_1) = 0$. Let $k(X) = mb + k_0(X)$ and $h(X) = k(X)f(X) + l(X)$. Then, $k \in O_M[X]$, $\deg(k) = \deg(k_0)$, and $V_{\mathcal{T}}(k - k_0) > \gamma$. Finally note that $h(x_1) = 0$, as desired. ∎

4. Skolem density problem for polynomials

We keep Data 2.1 in force. The polynomials $f_1, \ldots, f_m \in \tilde{K}[X_1, \ldots, X_n]$, the point $\mathbf{a} \in M^n$, and the positive integer γ involved in the assumptions of Theorem 4.3 can be described as a **Skolem density problem** for M/K. We then regard the point $\mathbf{x} \in M^n$ that the theorem supplies as a **solution** to this problem.

In the proof of Theorem 4.3, it becomes necessary to enlarge $\mathcal{T}$. Lemma 4.2 takes care of this enlargement.

Data and Assumption 4.1:

(a) $f_1, \ldots, f_m$ *are polynomials in* $M[\mathbf{X}] = M[X_1, \ldots, X_n]$ *such that* $v(f_i) = 0$ *for each* $v \in \tilde{\mathcal{V}} \smallsetminus \tilde{\mathcal{T}}$, $i = 1, \ldots, m$.

(b) *For each* $\mathbf{a} \in M^n$ *and every positive integer* γ *there exists* $\mathbf{x} \in M^n$ *such that* $V_{\mathcal{T}}(\mathbf{x} - \mathbf{a}) > \gamma$, $\mathbf{x}$ *is* $\mathcal{T}$*-integral, and* $f_i(\mathbf{x})$ *is a* $\mathcal{T}$*-unit,* $i = 1, \ldots, m$.

Lemma 4.2: *Let* $f_1, \ldots, f_m$ *be as in Data 4.1(a). Consider a finite subset* $\mathcal{T}'$ *of* $\mathcal{V}_K$ *which contains* $\mathcal{T}$. *Suppose that* $f_1, \ldots, f_m$ *satisfy Assumption 4.1(b) for* $\mathcal{T}'$ *instead of* $\mathcal{T}$. *Then* $f_1, \ldots, f_m$ *satisfy Assumption 4.1(b) for* $\mathcal{T}$.

Proof: Let $\mathcal{R} = \mathcal{T}' \smallsetminus \mathcal{T}$, and consider a vector $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$ and a positive integer γ. We break the rest of the proof into three parts.

Part A: *There exists a finite subextension* L *of* M/K *which contains the coefficients of* $f_1, \ldots, f_m$ *and* $a_1, \ldots, a_n$ *such that for each* $w \in \mathcal{R}_N$ *there exists* $\mathbf{b}_w \in L^n$ *such that* $w(\mathbf{b}_w) \geq 0$ *and* $w(f_i(\mathbf{b}_w)) = 0$, $i = 1, \ldots, m$.

Choose a finite set $\mathcal{R}_0$ that represents $\mathcal{R}_N$ over K. Let $v \in \mathcal{R}_0$. The Henselian closure N_v of v with respect to v is $\tilde{K}$ (e.g., Proposition 1.9(a)). Hence the corresponding residue field $\bar{N}_v$ is algebraically closed and in particular infinite. By assumption, the reduced polynomials $\bar{f}_i \in \bar{N}_v[\mathbf{X}]$ are nonzero. Hence, we can choose $\mathbf{x}_v \in N^n$ such that $v(\mathbf{x}_v) \geq 0$ and $v(f_i(\mathbf{x}_v)) = 0$, $i = 1, \ldots, m$. Since N/K is normal, the finite subset $C_v = \{\mathbf{x}_v^\sigma \| \sigma \in \mathrm{Aut}(N/K)\}$ is contained in N. By Lemma 1.8, applied to $K' = K_{\mathrm{ins}}$, M^n has a finite subset B_v such that for each $\mathbf{c} \in C_v$ and for each $w \in \mathcal{R}_N$ which lies over $v|_K$ there exists $\mathbf{b} \in B_v$ such that $w(\mathbf{b} - \mathbf{c}) > 0$.

Choose a finite extension L of K which is contained in M, contains the coefficients of $f_1, \ldots, f_m$ and the elements $a_1, \ldots, a_n$ and contains B_v for all $v \in \mathcal{R}_0$. For each $w \in \mathcal{R}_N$ there exists $v \in \mathcal{R}_0$ and $\sigma \in \mathrm{Aut}(N/K)$ such that $w = v^\sigma$. Choose $\mathbf{b}_w \in B_v$ such that $w(\mathbf{b}_w - \mathbf{x}_v^\sigma) > 0$. Then $w(\mathbf{x}_v^\sigma) \ge 0$ and $w(\mathbf{b}_w) \ge 0$. Hence, since the coefficients of f_i are w-integral, $w(f_i(\mathbf{b}_w) - f_i(\mathbf{x}_v^\sigma)) > 0$. Also, $w(f_i(\mathbf{x}_v^\sigma)) = v^\sigma(f_i^\sigma(\mathbf{x}_v^\sigma)) = v(f_i(\mathbf{x}_v)) = 0$. Hence $w(f_i(\mathbf{b}_w)) = 0$, as claimed.

Part B: *There exists $\mathbf{y} \in L^n$ such that $V_\mathcal{T}(\mathbf{y} - \mathbf{a}) > \gamma$, $V_\mathcal{R}(\mathbf{y}) \ge 0$, and $V_\mathcal{R}(f_i(\mathbf{y})) = 0$, $i = 1, \ldots, m$.* Indeed, choose a finite set $\mathcal{R}_1$ which represents $\mathcal{R}_N$ over L and choose a finite set $\mathcal{T}_1$ which represents $\mathcal{T}_N$ over L. For each $v \in \mathcal{R}_1$ Part A gives $\mathbf{b}_v \in L^n$ such that $v(\mathbf{b}_v) \ge 0$ and $v(f_i(\mathbf{b}_v)) = 0$, $i = 1, \ldots, m$. By the weak approximation theorem, there exists $\mathbf{y} \in L^n$ such that

$$v(\mathbf{y} - \mathbf{a}) > \gamma \text{ for each } v \in \mathcal{T}_1 \text{ and}$$
$$v(\mathbf{y} - \mathbf{b}_v) > 0 \text{ for each } v \in \mathcal{R}_1.$$

Now let $w \in \mathcal{T}_N$. Then there exists $\sigma \in \mathrm{Aut}(N/L)$ and $v \in \mathcal{T}_1$ such that $w = v^\sigma$. Since $\mathbf{a}, \mathbf{y} \in L^n$, we have $w(\mathbf{y} - \mathbf{a}) = v(\mathbf{y} - \mathbf{a}) > \gamma$. If $w \in \mathcal{R}_N$, then there exists $\sigma \in \mathrm{Aut}(N/L)$ and $v \in \mathcal{R}_1$ such that $w = v^\sigma$. As in Part A, $w(\mathbf{y}) \ge 0$ and $w(f_i(\mathbf{y})) = v(f_i(\mathbf{y})) = v(f_i(\mathbf{b}_v)) = 0$, $i = 1, \ldots, m$ as claimed.

Part C: *Conclusion of the proof.* By assumption, there exists $\mathbf{x} \in M^n$ such that $V_{\mathcal{T}'}(\mathbf{x} - \mathbf{y}) > \gamma$, $w(\mathbf{x}) \ge 0$, and $w(f_i(\mathbf{x})) = 0$ for each $w \in \mathcal{V}_N \smallsetminus \mathcal{T}'_N$, $i = 1, \ldots, m$. In particular, if $w \in \mathcal{T}_N$, then, by Part B, $w(\mathbf{x} - \mathbf{a}) > \gamma$. If $w \in \mathcal{R}_N$, then $w(\mathbf{x} - \mathbf{y}) > 0$ and therefore, by Part B and since the coefficients of f_i are w-integral, $w(\mathbf{x}) \ge 0$ and $w(f_i(\mathbf{x})) = w(f_i(\mathbf{y})) = 0$. Conclude that $w(\mathbf{x}) \ge 0$ and $w(f_i(\mathbf{x})) = 0$ for each $w \in \mathcal{V}_N \smallsetminus \mathcal{T}_N$, $i = 1, \ldots, m$, as desired. ∎

Theorem 4.3: *Let $f_1, \ldots, f_m \in \tilde{K}[X_1, \ldots, X_n]$ be polynomials such that $v(f_i) = 0$ for each $v \in \tilde{\mathcal{V}} \smallsetminus \tilde{\mathcal{T}}$ and each $1 \le i \le m$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$ and let γ be a positive integer. Then there exists $\mathbf{x} \in M^n$ such that*

(a) *$V_\mathcal{T}(\mathbf{x} - \mathbf{a}) > \gamma$, and*

(b) *$\mathbf{x}$ is $\mathcal{T}$-integral, and*

(c) $f_1(\mathbf{x}),\dots,f_m(\mathbf{x})$ *are* $\mathcal{T}$*-units.*

Proof: Let E be a finite subextension of M/K which contains $a_1,\dots,a_n$. Since $E_{\mathrm{tot},\mathcal{S}_E,\mathrm{ins}} = N$, we may replace E by K, if necessary, and assume that $\mathbf{a}\in K^n$. We break the rest of the proof into two parts.

Part A: We first prove the theorem by induction on n under the assumption that the coefficients of f_i belong to K. There are several cases to consider.

Case A1: *m=n=1.* Set $a=a_1$, $X=X_1$ and $f=f_1$. Assume first that f has only simple roots. Then its discriminant is nonzero. Let therefore $\mathcal{T}'$ be a finite subset of $\mathcal{V}_K$ which contains $\mathcal{T}$ such that the leading coefficient of f as well as its discriminant are $\mathcal{T}'$-units. Lemma 3.2 gives $x'\in M$ such that (a), (b), and (c) hold for $\mathcal{T}',f$. Lemma 4.2 then gives $x\in M$ such that (a), (b), and (c) hold.

In the general case, use Lemma 4.2 to enlarge $\mathcal{T}$, if necessary, such that the leading coefficient of f is a $\mathcal{T}$-unit. Then each of the roots of f is $\mathcal{T}$-integral.

Let $f(X)=g_1(X)^{\alpha_1}\cdots g_k(X)^{\alpha_k}$, where $g_1(X),\dots,g_k(X)$ are distinct irreducible polynomials over K. Each $g_j(X)$ has the form $h_j(X)^{q_j}$, where $q_j=1$ if $\mathrm{char}(K)=0$ and q_j is a power of $\mathrm{char}(K)$ if $\mathrm{char}(K)>0$, and $h_j(X)\in K_{\mathrm{ins}}[X]$ has only simple roots. Then $h(X)=h_1(X)\cdots h_k(X)$ has only simple roots. Since M is perfect, M/K has a finite subextension L which contains the coefficients of h.

By the first paragraph of Case A1, applied to L instead of to K, there exists $x\in M$ such that $V_{\mathcal{T}}(x-a)>\gamma$, x is $\mathcal{T}$-integral, and $h(x)$ is a $\mathcal{T}$-unit. By construction, $h(X)=c\prod_{i=1}^l(X-t_i)$, and $f(X)=b\prod_{i=1}^l(X-t_i)^{\beta_i}$ for some positive integers β_i and $\mathcal{T}$-units c and b. Since all t_i are $\mathcal{T}$-integral, $x-t_i$ is a $\mathcal{T}$-unit, $i=1,\dots,l$. Hence $f(x)$ is a $\mathcal{T}$-unit.

Case A2: *n=1 and m is arbitrary.* The polynomial $f=f_1\cdots f_m$ satisfies $v(f)=0$ for each $v\in\tilde{\mathcal{V}}\smallsetminus\tilde{\mathcal{T}}$. Let $\mathcal{T}'$ be a finite subset of $\mathcal{V}_K$ which contains $\mathcal{T}$ such that the leading coefficient of f is a $\mathcal{T}'$-unit. By Case A1, there exists $x'\in M$ such that $V_{\mathcal{T}'}(x'-a)>\gamma$, x' is $\mathcal{T}'$-integral, and $f(x')$ is a $\mathcal{T}'$-unit. As $f_1(x'),\dots,f_m(x')$ are $\mathcal{T}'$-integral and their product is a $\mathcal{T}'$-unit each of them is a $\mathcal{T}'$-unit. It follows from Lemma 4.2, that there exists $x\in M$ which satisfies (a), (b), and (c).

Case A3: *n is arbitrary.* Suppose now that the Proposition holds for $n-1$. Let $\mathcal{T}'$ be a finite subset of $\mathcal{V}_K$ which contains $\mathcal{T}$ such that all nonzero coefficients of f_i are $\mathcal{T}'$-units, $i = 1, \ldots, m$. Consider each f_i as a polynomial in $X_1, \ldots, X_{n-1}$ with coefficients in $K[X_n]$. Let $h_i(X_n)$ be a nonzero coefficient of f_i. Then all coefficients of h_i are $\mathcal{T}'$-units. By Case A2, there exists $x_n' \in M$ such that $V_{\mathcal{T}'}(x_n' - a_n) > \gamma$, x_n' is $\mathcal{T}'$-integral, and $h_i(x_n')$ is a $\mathcal{T}'$-unit, $i = 1, \ldots, m$. Thus $g_i(X_1, \ldots, X_{n-1}) = f_i(X_1, \ldots, X_{n-1}, x_n')$ satisfies $v(g_i) = 0$ for each $v \in \tilde{\mathcal{V}} \smallsetminus \tilde{\mathcal{T}}'$. Then $K' = K(x_n')$ is a finite subextension of M/K and $K'_{\mathrm{tot}, \mathcal{S}_{K'}, \mathrm{ins}} = N$. So, we may apply the induction hypothesis to the field K', to $\mathcal{S}_{K'}$, to $\mathcal{T}'_{K'}$, and to the polynomials g_i, $i = 1, \ldots, m$, to get $x_1', \ldots, x_{n-1}' \in M$ such that (a), (b), and (c) are satisfied for $\mathcal{T}'$ instead of $\mathcal{T}$. By Lemma 4.2, there exists $\mathbf{x} \in M^n$ which satisfies (a), (b), and (c). This competes the induction.

Part B: *The general case.* Let K'' be a finite extension of K which contains the coefficients of $f_1, \ldots, f_m$. The norm $g_i(\mathbf{X}) = N_{K''/K} f_i(\mathbf{X})$ can be expressed as a product $g_i(\mathbf{X}) = \prod f_{ij}(\mathbf{X})$ of polynomials f_{ij} which are conjugate to f_i over K. In particular $v(f_{ij}) = 0$ and therefore $v(g_i) = 0$ for each $v \in \tilde{\mathcal{V}} \smallsetminus \tilde{\mathcal{T}}$. Apply Part A to the polynomials $g_1(\mathbf{X}), \ldots, g_m(\mathbf{X})$ to get $\mathbf{x} \in M^n$ such that (a) and (b) hold, and $g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})$ are $\mathcal{T}$-units. Then for each i, the elements $f_{ij}(\mathbf{x})$ are $\mathcal{T}$-integral whose product is the $\mathcal{T}$-unit $g_i(\mathbf{x})$. It follows that each $f_{ij}(\mathbf{x})$ is a $\mathcal{T}$-unit and in particular each $f_i(\mathbf{x})$ is a $\mathcal{T}$-unit. ∎

5. Skolem density problem for rational functions

We keep using Data 2.1. If we replace the polynomials in a Skolem density problem by rational functions we must assume that their coefficients belong to M and that the problem is locally solvable.

THEOREM 5.1: *Consider rational functions $f_1, \ldots, f_m \in M(X_1, \ldots, X_n)$ and the vector $\mathbf{f} = (f_1, \ldots, f_m)$. Let $\mathbf{a} \in M^n$ and let γ be a positive integer. We assume that for each $v \in \tilde{\mathcal{V}}$ such that $v(\mathbf{f}) < 0$ there exists $\mathbf{x}_v \in \tilde{K}^n$ such that $v(\mathbf{f}(\mathbf{x}_v)) \geq 0$. Then there exists $\mathbf{x} \in M^n$ such that*

(1a) *$V_{\mathcal{T}}(\mathbf{x} - \mathbf{a}) > \gamma$, and*

(1b) *$\mathbf{f}(\mathbf{x})$ is $\mathcal{T}$-integral.*

Proof: Write each f_i as a quotient of polynomials with coefficients in M: $f_i = g_i/h_i$. Choose a finite subset $\mathcal{R}$ of $\mathcal{V}_K$ disjoint from $\mathcal{T}$, such that the coefficients of g_i and h_i are $\mathcal{T}'$-unit, where $\mathcal{T}' = \mathcal{R} \cup \mathcal{T}$.

Let L be a finite subextension of M/K which contains the coordinates of $\mathbf{a}$ as well as the coefficients of g_i and h_i, $i = 1, \ldots, m$. Choose a finite set $\mathcal{R}_0$ which represents $\tilde{\mathcal{R}}$ over L. Let $v \in \mathcal{R}_0$. If $v(\mathbf{f}) < 0$, then, by assumption, there exists $\mathbf{b}_v \in \tilde{K}^n$ such that $v(\mathbf{f}(\mathbf{b}_v)) \geq 0$. If $v(\mathbf{f}) \geq 0$, such $\mathbf{b}_v$ exists by [CaR, Thm. 2.2]. Since N is v-dense in $\tilde{K}$ (Proposition 1.9(b)) we may choose $\mathbf{b}_v$ to be in N^n.

Consider the finite subset $B = \{\mathbf{b}_v^\sigma \| v \in \mathcal{R}_0,\ \sigma \in G(L)\}$ of N. Let δ be an integer such that for each $\mathbf{x} \in \tilde{K}^n$ and for each $v \in \mathcal{R}_0$ the inequality $v(\mathbf{x} - \mathbf{b}_v) > \delta$ implies $v(\mathbf{f}(\mathbf{x})) \geq 0$. By Lemma 1.8, M^n has a finite subset B' such that for each $\mathbf{b} \in B$, for each $v \in \mathcal{R}_0$, and for each $w \in \tilde{\mathcal{R}}$ with $w|_L = v|_L$ there exists $\mathbf{b}' \in B'$ such that $w(\mathbf{b}' - \mathbf{b}) > \delta$.

Let $L_1 = L(B')$. Choose a finite set $\mathcal{R}_1$ which represents $\tilde{\mathcal{R}}$ over L_1. For each $w \in \mathcal{R}_1$ there exists $v \in \mathcal{R}_0$ and $\sigma \in G(L)$ such that $w = v^\sigma$. Choose $\mathbf{b}_w \in B'$ such that $w(\mathbf{b}_w - \mathbf{b}_v^\sigma) > \delta$. Then $v(\mathbf{b}_w^{\sigma^{-1}} - \mathbf{b}_v) > \delta$ and therefore, by the choice of δ, $v(\mathbf{f}(\mathbf{b}_w^{\sigma^{-1}})) \geq 0$. It follows that $w(\mathbf{f}(\mathbf{b}_w)) \geq 0$.

Let $\varepsilon > \gamma$ be an integer such that for each $\mathbf{x} \in \tilde{K}^n$, and for each $w \in \mathcal{R}_1$ the inequality $w(\mathbf{x} - \mathbf{b}_w) > \varepsilon$ implies $w(\mathbf{f}(\mathbf{x})) \geq 0$. Apply the weak approximation theorem to L_1 and find $\mathbf{b} \in (L_1)^n$ such that $V_{\mathcal{T}}(\mathbf{b}-\mathbf{a}) > \gamma$ and $w(\mathbf{b}-\mathbf{b}_w) > \varepsilon$ for each $w \in \mathcal{R}_{L_1}$.

By Theorem 4.3 there exists $\mathbf{x} \in M^n$ such that $V_{\mathcal{T}'}(\mathbf{x}-\mathbf{b}) > \varepsilon$ and $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ are $\mathcal{T}'$-units. Hence, $V_{\mathcal{T}}(\mathbf{x}-\mathbf{a}) > \gamma$. If $v \in \tilde{\mathcal{R}}$, then there exists $\sigma \in G(L_1)$ and $w \in \mathcal{R}_1$ such that $v = w^\sigma$. We have $v(\mathbf{x}-\mathbf{b}_w) > \varepsilon$. Hence, $w(\mathbf{x}^{\sigma^{-1}} - \mathbf{b}_w) > \varepsilon$ and therefore, by the preceding paragraph, $w(\mathbf{f}(\mathbf{x}^{\sigma^{-1}})) \geq 0$. It follows that $v(\mathbf{f}(\mathbf{x})) \geq 0$. Thus $\mathbf{f}(\mathbf{x})$ is $\mathcal{T}$-integral. ∎

Here are some interesting special cases of Theorems 5.1. If $M = \tilde{K}$, and $\mathcal{S}$ is empty, then Theorem 5.1 gives Theorem 5.1 of [CaR], except that we have not included archimedean primes in $\mathcal{T}$.

If only $\mathcal{S}$ is empty, then $N = \tilde{K}$. In this case Theorem 5.1 simplifies to the following result:

Theorem 5.2: *Let K be a global field and let $\mathcal{T}$ be a finite set of valuations of K. Let M be an algebraic extension of K which is PAC over O_M. Consider polynomials $f_1, \ldots, f_m \in \tilde{K}[X_1, \ldots, X_n]$ such that $v(f_i) = 0$ for each $v \in \tilde{\mathcal{V}} \smallsetminus \tilde{\mathcal{T}}$ and each $1 \leq i \leq m$. Let $\mathbf{a} \in M^n$ and let γ be a positive integer. Then there exists $\mathbf{x} \in M^n$ such that*

(a) *$V_{\mathcal{T}}(\mathbf{x}-\mathbf{a}) > \gamma$, and*

(b) *$x_1, \ldots, x_n$ and $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ are $\mathcal{T}$-units.*

The case $\mathcal{T} = \emptyset$ gives a solution to Skolem original problem in M. In this case the assumption of Theorem 5.2, namely that $v(f_i) = 0$ for each $v \in \tilde{\mathcal{V}}$ means that f_i is **primitive**.

Theorem 5.3: *Let M be an algebraic extension of K which is PAC over O_M. Let $f_1, \ldots, f_m \in \tilde{K}[X_1, \ldots, X_n]$ be primitive polynomials. Then there exist units $x_1, \ldots, x_n$ of O_M such that $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ are units of O_M.*

We denote the fixed field of an e-tuple $\boldsymbol{\sigma} \in G(K)^e$ in $\tilde{K}$ by $\tilde{K}(\boldsymbol{\sigma})$.

Corollary 5.4: *For each positive integer e and almost all $\boldsymbol{\sigma} \in G(K)^e$ the field $M = \tilde{K}(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S},\mathrm{ins}}$ satisfies the conclusions of all the theorems of Sections 4 and 5.*

Proof: By [JR1, Prop. 3.1], almost all fields $\tilde{K}(\boldsymbol{\sigma})$ are PAC over O. Hence, by Lemma 1.4, $M = \tilde{K}(\boldsymbol{\sigma}) \cap K_{\mathrm{tot},\mathcal{S},\mathrm{ins}}$ is weakly P$\mathcal{S}$C over O_M for almost all $\boldsymbol{\sigma} \in G(K)^e$. So, we may apply Theorems 4.3, 5.1, 5.2 and 5.3 to each of these fields. ∎

We return now to the general case where M is as in Data 2.1. Theorem 5.1 has the following consequence for the structure of O_M.

THEOREM 5.5: *Every finitely generated ideal of O_M is principal. In other words, O_M is a* **Bezout domain**.

Proof: Let $a_1, \ldots, a_n$ be elements of O_M. For each subextension L of M/K which contains $a_1, \ldots, a_n$ denote the ideal which these elements generate by $\mathfrak{a}_L$. We have to find $x \in O_M$ which generates $\mathfrak{a}_M$.

To this end let $E = K(a_1, \ldots, a_n)$. Then O_E is a Dedekind domain and hence $\mathfrak{a}_E$ decomposes into a product of powers of prime ideals: $\mathfrak{a}_E = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_m^{\alpha_m}$. Denote the valuation of E which corresponds to $\mathfrak{p}_i$ by v_i. Let e_i be the ramification index of $\mathfrak{p}_i$ over K. For each i find $a_i \in O_E$ such that $v_i(a_i) = \alpha_i/e_i$. Let $\gamma = \max\{\alpha_1/e_1, \ldots, \alpha_m/e_m\}$. Then use the Chinese remainder theorem to find $a \in O_E$ such that $v_i(a - a_i) > \gamma$ for $i = 1, \ldots, m$.

Note that a does not necessarily generate $\mathfrak{a}_L$ because a is not necessarily a $\{v_1, \ldots, v_m\}$-unit. So, denote the set of restrictions of $v_1, \ldots, v_m$ to K by $\mathcal{T}$. By Theorem 5.1, there exists $x \in M$ such that $V_{\mathcal{T}}(x - a) > \gamma$, and both x and x^{-1} are $\mathcal{T}$-integral.

Let now $L = E(x)$. If v is a valuation of L that lies over v_i, then $v(x) = v(a_i)$. In particular $v(x) \geq 0$. Also, x is a $\mathcal{T}$-unit. Hence x belongs to O_L and $xO_L = \mathfrak{a}_L$. Conclude that $xO_M = \mathfrak{a}_M$. ∎

Remark 5.6: Algebraic extensions. Theorem 5.5 partially generalizes Corollary 1.5(a), (c) of [Ja3], which states that for almost all $\boldsymbol{\sigma} \in G(K)^e$ the ring $O_s(\boldsymbol{\sigma}) = O_{K_s(\boldsymbol{\sigma})}$ is a Bezout domain.

Suppose that $S = \emptyset$. Then M is PAC over O_M. If M' is an algebraic extension of M, then M' is PAC over $O_{M'}$ [JR1, Cor. 2.5]. It follows that Theorems 5.1 and 5.5 hold for M' instead of M. In particular, for almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^e$, if L is an algebraic extension of $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$, then $\mathbb{Z}_L$ is a Bezout domain. This affirmatively solves Problem 1 of [Ja3]. ∎

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