A field $K$ is **ample** if it satisfies one of the following equivalent conditions [Pop2, Prop. 1.1]:

(1a) Each absolutely irreducible curve $C$ over $K$ with a simple $K$-rational point has infinitely many $K$-rational points.

(1b) Each function field of one variable $F/K$ with a prime divisor of degree 1 has infinitely many such divisors.

(1c) $K$ is existentially closed in $K((t))$.

The interest in ample fields lies in the fact that a large class of embedding problem over such fields are solvable. Thus, if $K$ is an ample field, then every finite split embedding problem over $K(t)$ (with $t$ indeterminate) is solvable (cf. [Pop1, Thm. 2.7] or [HaJ2, Thm. 4.3]). If in addition $K$ is Hilbertian, then each finite split embedding problem over $K$ is solvable (e.g. [HaJ1, Thm. 6.5]). If in addition the absolute Galois group $\text{Gal}(K)$ of $K$ is projective and $K$ is countable, then $\text{Gal}(K)$ is the free profinite group on countably many variables.

Examples of ample fields are PAC fields, Henselian fields, real closed fields, and fields with a local global principle like PRC fields, $\mathbb{P}_{\mathbb{C}}$ fields, and $\mathbb{P}_{\mathbb{S}}$ fields with $S$ being a set of primes whose completions are local fields.

The following surprising observation of Colliot-Thélène [CoT, Introduction] gives a sufficient condition for a field $K$ to be simple in terms of $\text{Gal}(K)$ alone.

**Proposition 1:** Let $K$ be a perfect field such that $\text{Gal}(K)$ is a pro-$p$ group for some prime number $p$. Then $K$ is ample.

**Proof:** Consider a function field $F$ of one variable over $K$ of genus $g$ with a prime divisor $p$ of degree 1. Let $p_1, \ldots, p_m$ be additional prime divisors of $K$. Use the weak approximation theorem to choose $f \in F$ with $v_p(f) = 1$ and $v_{p_i}(f) = 0$ for $i = 1, \ldots, m$.

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Then \( \text{div}(f) = p + \sum_{j=1}^{n} k_j q_j \), for some additional distinct prime divisors \( q_1, \ldots, q_n \). It follows that

\[
0 = \deg(\text{div}(f)) = 1 + \sum_{j=1}^{n} k_j \deg(q_j).
\]

Denote the residue field of \( F \) at \( q_j \) by \( \bar{F}_{q_j} \). As \( K \) is perfect, \( \bar{F}_{q_j}/K \) is separable. As \( \deg(q_j) = [\bar{F}_{q_j}:K] \) and \( \text{Gal}(K) \) is a pro-\( p \) group, each of the numbers \( \deg(q_j) \) is a power of \( p \). Conclude from (1) that \( \deg(q_j) = 1 \) for some \( j \) between 1 and \( n \). So, \( F/K \) has infinitely many prime divisors of degree 1. In other words, \( K \) is ample.

The goal of this note is to generalize Colliot-Thélène’s observation to fields which are not perfect.

**Lemma 2:** Let \( K \) be an infinite field, \( F \) an algebraic function field of one variable over \( K \) of genus \( g \), and \( a \) a positive divisor of \( F/K \) of degree at least \( 2g \). Then there is \( t \in F \setminus K \) with \( \text{div}_\infty(t - a) = a \) for each \( a \in K \).

**Proof:** Write \( a = \sum_{i=1}^{r} m_i p_i \) with distinct prime divisors \( p_1, \ldots, p_r \) of \( F/K \) and positive integers \( m_1, \ldots, m_r \). For each \( i \) between 1 and \( r \) let \( a_i = a - p_i \). By assumption, \( \deg(a_i) = \deg(a) - 1 \geq 2g - 1 \). Hence, by Riemann-Roch, \( \dim(\mathcal{L}(a)) = \deg(a) + 1 - g \) and \( \dim(\mathcal{L}(a_i)) = \deg(a) - g \) (We use the notation of [FrJ, §2.5].) So, \( \mathcal{L}(a_i) \) is a proper subspace of \( \mathcal{L}(a) \). As \( K \) is infinite, there is \( t \in \mathcal{L}(a) \setminus \bigcup_{i=1}^{r} \mathcal{L}(a_i) \). It satisfies \( \text{div}_\infty(t) = a \). Hence \( \text{div}_\infty(t - a) = a \) for each \( a \in K \).

Now we drop the condition “\( K \) is perfect” from Proposition 1.

**Theorem 3:** Let \( K \) be a field such that \( \text{Gal}(K) \) is a pro-\( p \) group for some prime number \( p \). Then \( K \) is ample.

**Proof:** Each finite field has finite extensions of every degree, in particular its absolute Galois group is not pro-\( p \). It follows that \( K \) is infinite.

Let \( F \) be a function field of one variable of genus \( g \) over \( K \) with a prime divisor \( p \) of degree 1. Set \( p_0 = p \) and let \( p_1, \ldots, p_m \) with \( m \geq 0 \) be additional prime divisors of \( F/K \) of degree 1. Choose a positive multiple \( k \) of \( p \) such that \( k \geq 2g \) and \( \text{char}(K)|k \) if \( \text{char}(K) \neq 0 \). Consider the divisors \( a = p + k \sum_{i=0}^{m} p_i \) and \( a_i = a - p_i, i = 0, \ldots, m, \)
of $F/K$. Then $\deg(a) > \deg(a_i) \geq k - 1 \geq 2g - 1$ for $i = 0, \ldots, m$. By Riemann-Roch, $\dim(L(a)) = \deg(a) + 1 - g$ and $\dim(L(a_i)) = \deg(a_i) + 1 - g$. Thus, $L(a_i)$ is a proper subspace of $L(a)$, $i = 0, \ldots, m$. Since $K$ is infinite, there exists $t \in L(a) \setminus \bigcup_{i=0}^{m} L(a_i)$. Hence, $\text{div}(t) + a \geq 0$ but $\text{div}(t) + a_i \not\geq 0$ for each $i$. It follows that $\text{div}_\infty(t) = a$, so $\text{div}_\infty(t - a) = a$ for each $a \in K$.

By definition

$$\deg(a) = 1 + k \sum_{i=1}^{m} \deg(p_i).$$

Hence,

$$[F : K(t - a)] = \deg(\text{div}_\infty(t - a)) = \deg(a) \equiv 1 \mod k.$$

In particular, if $\text{char}(K) \neq 0$, then $\text{char}(K) \nmid [F : K(t)]$. Thus, in each case, $F/K(t)$ is a finite separable extension.

Now choose a primitive element $x$ for $F/K(t)$, integral over $K[t]$. Let $f = \text{irr}(x, K(t))$. Then $f(T, X) \in K[T, X]$ is an absolutely irreducible polynomial separable in $X$ [FrJo08, Cor. 10.2.2(b)]. Hence, there exists $a \in K$ such that $f(a, X)$ is separable. The irreducible factors of $f(a, X)$ over $F$ correspond to zeros of $t - a$ (as an element of $F$). Therefore, $\text{div}_0(t - a) = \sum_{i=1}^{r} q_i$ and for each $i$, $q_i$ is a prime divisor of $F/K$ with residue field $\tilde{F}_{q_i}$ separable over $K$. The assumption on $K$ implies that $\deg(q_i) = [\tilde{F}_{q_i} : K]$ is a power of $p$. By (3),

$$\sum_{i=1}^{r} \deg(q_i) = \deg(\text{div}_0(t - a)) = \deg(\text{div}_\infty(t - a)) \equiv 1 \mod p.$$

Hence, there exists $i$ between 1 and $r$ with $\deg(q_i) = 1$. In addition, $q_i$ is relatively prime to $a$ (because $\text{div}_0(t - a)$ and $\text{div}_\infty(t - a)$ are relatively prime divisors), so $q_i$ differs from $p, p_1, \ldots, p_m$. Consequently, $K$ is ample.  

\[ \boxed{\phantom{0}} \]
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References


