A KARRASS-SOLITAR THEOREM FOR PROFINITE GROUPS*

by

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Introduction

Karrass and Solitar prove that if M is a finitely generated subgroup of a free abstract group F and M contains a nontrivial normal subgroup of F, then the index of M is finite [KaS] (see also [Imr, Sec. 6 for a graph theoretic approach] or [Gru]). In particular, if the rank of F is infinite, then F does not have a subgroup of that type. Our goal is to prove an analog of the Karrass-Solitar theorem for free profinite groups of rank at least 2:

Main theorem: *Let F be a free profinite group of rank ≥ 2 and let M be a finitely generated closed subgroup of infinite index. Then M contains no closed nontrivial normal subgroup of F.*

Note that the main theorem fails if $\text{rank}(F) = 1$, since in this case each closed subgroup is procyclic and normal. When $\text{rank}(F) = \infty$, our theorem is an immediate consequence of Haran's Diamond theorem: *If M_1, M_2 are closed normal subgroups and M is a closed subgroup of F which contains $M_1 \cap M_2$ but neither M_1 nor M_2, then M is profinite free of the same rank as F* [FrJ, Thm. 25.4.3]. Indeed let F and M be as in the main theorem. Assume M contains a nontrivial closed normal subgroup N of F. Then, F has an open normal subgroup F_0 which does not contain N. Set $M_0 = F_0 \cap M$. By the Diamond theorem, M_0 is profinite free of infinite rank. On the other hand, as an open subgroup of M, M_0 is finitely generated. We deduce from this contradiction that no group N as above exists.

We do not know whether the Diamond theorem holds for free profinite groups of finite rank [FrJ, Problem 25.4.9], so we can not use it to prove our main theorem. Instead we supply a direct proof to the main theorem for finite rank by using a theorem of Nielsen-Schreier combined with a trick of Lubotzky-v.d.Dries. Having done that, we reduce the case of an infinite rank to the case of a finite rank. In this way we supply a complete proof of the main theorem without using the Diamond theorem.

The proof of the main theorem improves a result of Lubotzky-Melnikov-v.d.Dries: If F is a free profinite group of a finite rank $e \geq 2$, M is a closed subgroup of F of infinite index, N is a nontrivial closed normal subgroup of F contained in M, and M_0

is an open subgroup of M which does not contains N, then M_0 is isomorphic to the free profinite group $\hat{F}_\omega$ of rank $\aleph_0$. The Lubotzky-Melnikov-v.d.Dries theorem handles only the case where $M = N$, that is, it demands that M is normal in F [FrJ, Prop. 24.10.3].

Our main theorem has a field theoretic application: Let K be an e-free PAC field, M an infinite separable algebraic extension, and M_0 a proper finite Galois extension of M not contained in the Galois closure of M/K. Then M_0 is Hilbertian. This is an analog of a theorem of Weissauer [FrJ, Thm. 13.9.1], where K is assumed to be Hilbertian rather than PAC and e-free.

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1. The Trick of Lubotzky-v.d.Dries

Let F be a free profinite group of finite rank $e \geq 2$ and N a closed normal subgroup of F of infinite index. If $\mathrm{rank}(F/N) < \mathrm{rank}(F)$, then, by Nielsen-Schreier, $\mathrm{rank}(E) - \mathrm{rank}(E/N)$ tends to infinity with $(F : E)$, where E ranges over all open subgroups of F containing N. Every finite embedding problem for N can be lifted to a finite embedding problem of an open subgroup E of F containing N and having a large index. Since E is free, one can solve the embedding problem for E in such a way that it will give a solution to the embedding problem of N. Applying a well known result of Iwasawa, this implies that N is isomorphic to the free profinite group $\hat{F}_\omega$ of countable rank [FrJ, Prop. 24.10.3].

We call this reasoning the **trick of Lubotzky-v.d.Dries**. In what follows we generalize it.

Throughout the note we freely use the following theorem of Nielsen-Schreier: Every open subgroup E of F is free and $\mathrm{rank}(E) = 1 + (F : E)(\mathrm{rank}(F) - 1)$ [FrJ, Prop. 17.6.2]. Moreover, if H is an open subgroup of a finitely generated profinite group G, then $\mathrm{rank}(H) \leq 1 + (G : H)(\mathrm{rank}(G) - 1)$ [FrJ, Cor. 17.6.3]. We denote the free profinite group of rank e by $\hat{F}_e$.

LEMMA 1.1: *Let F be a profinite group, H an open subgroup, M a closed subgroup of H, and M_0 an open normal subgroup of M. Then F has open subgroups $E_0 \triangleleft E$ such that $E_0 M = E \leq H$ and $E_0 \cap M = M_0$.*

Proof: By [FrJ, Lemma 1.2.5(a)], H has an open normal subgroup H_0 satisfying $H_0 \cap M \leq M_0$. Set $E_0 = H_0 M_0$ and $E = H_0 M$.

$$\begin{tikzcd}
H_0 \arrow[r, dash] \arrow[d, dash] & E_0 \arrow[r, dash] \arrow[d, dash] & E \arrow[r, dash] \arrow[d, dash] & H \arrow[r, dash] & F \\
H_0 \cap M \arrow[r, dash] & M_0 \arrow[r, dash] & M & &
\end{tikzcd}$$

Then $E_0 \cap M = M_0$ and $E_0 \triangleleft E$. ∎

LEMMA 1.2: *Let $e \geq 2$ be an integer, $F = \hat{F}_e$, M a closed subgroup of F of infinite index, and N a closed normal subgroup of F contained in M. Suppose*

$$\mathrm{rank}(F/N) < e. \tag{1}$$

Then $M \cong \hat{F}_\omega$.

Proof: By Iwasawa, it suffices to prove that each finite embedding problem for M is solvable [FrJ, Cor. 24.8.3]. Let therefore

$$\begin{tikzcd} & & & M \arrow[d, "\varphi"] & \\ 1 \arrow[r] & C \arrow[r] & B \arrow[r, "\alpha"] & A \arrow[r] & 1 \end{tikzcd} \tag{2}$$

be a finite embedding problem for M. Thus, φ and α are epimorphisms and B is a finite group. Moreover, $M_0 = \mathrm{Ker}(\varphi)$ is an open normal subgroup of M. Since $(F:M) = \infty$, F has an open subgroup F_1 containing M such that $(F:F_1) \ge \mathrm{rank}(C) + |A|$. By Lemma 1.1, F_1 has open subgroups $E_0 \triangleleft E$ satisfying $E_0 M = E$ and $E_0 \cap M = M_0$.

$$\begin{tikzcd}[arrows=dash] E_0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F_1 \arrow[r] & F \\ M_0 \arrow[r] \arrow[d] & M \arrow[d] & & \\ N_0 \arrow[r] & N & & \end{tikzcd}$$

By the choice of F_1,

(3) $(F:E) \ge \mathrm{rank}(C) + |A|$.

Extend φ to an epimorphism $\tilde{\varphi}\colon E \to A$ by defining $\tilde{\varphi}(e_0 y) = \varphi(y)$ for all $e_0 \in E_0$ and $y \in M$. In particular, $\mathrm{Ker}(\tilde{\varphi}) = E_0$. By (1), (3), and Nielsen-Schreier,

$$\begin{aligned} \mathrm{rank}(E/N) &\le 1 + (F:E)(\mathrm{rank}(F/N) - 1) \le 1 + (F:E)(e-1-1) \\ &= 1 + (F:E)(e-1) - (F:E) = \mathrm{rank}(E) - (F:E) \\ &\le \mathrm{rank}(E) - \mathrm{rank}(C) - |A|. \end{aligned} \tag{4}$$

Note that

$$N/N_0 \cong M_0N/M_0 \le M/M_0 \cong A,$$

so $\mathrm{rank}(N/N_0) \le |A|$. Since, $\mathrm{rank}(E/N_0) \le \mathrm{rank}(N/N_0) + \mathrm{rank}(E/N)$, (4) implies that

$$\mathrm{rank}(C) + \mathrm{rank}(E/N_0) \le \mathrm{rank}(E). \tag{5}$$

Let $k = \mathrm{rank}(C)$ and $m = \mathrm{rank}(E)$. By (5), $\mathrm{rank}(E/N_0) \le m - k$. Choose generators $\bar{x}_{k+1}, \ldots, \bar{x}_m$ for E/N_0 and set $\bar{x}_i = 1$ for $i = 1, \ldots, k$. Then $\bar{x}_1, \ldots, \bar{x}_m$ generate E/N_0. A lemma of Gaschütz gives generators $x_1, \ldots, x_m$ of E with $\bar{x}_i = x_i N_0$ for $i = 1, \ldots, m$ [FrJ, Lemma 17.7.2]. By [FrJ, Lemma 17.4.6(b)], $x_1, \ldots, x_m$ form a basis of E. Choose generators $b_1, \ldots, b_k$ of C and elements $b_{k+1}, \ldots, b_m \in B$ with $\alpha(b_i) = \tilde{\varphi}(x_i)$, $i = k+1, \ldots, m$. Then $b_1, \ldots, b_m$ generate B. If $1 \le i \le k$, then $x_i N_0 = \bar{x}_i = 1$. Hence, $x_i \in N_0 \le E_0$, so $\tilde{\varphi}(x_i) = 1 = \alpha(b_i)$. It follows that the map $x_i \mapsto b_i$, $i = 1, \ldots, m$, extends to an epimorphism $\tilde{\gamma}\colon E \to B$ with $\alpha \circ \tilde{\gamma} = \tilde{\varphi}$.

Finally, set $\gamma = \tilde{\gamma}|_M$. Then $\alpha \circ \gamma = \varphi$ and $\alpha \circ \gamma(M) = \varphi(M) = A$. Moreover, for $1 \le i \le k$ we have $x_i \in N_0 \le M$. Hence, $C = \langle b_1, \ldots, b_k \rangle = \langle \gamma(x_1), \ldots, \gamma(x_k) \rangle \le \gamma(M)$. Therefore, $\gamma(M) = B$. In other words, γ is a solution of embedding problem (2). ∎

Proposition 1.3: *Let $F = \hat{F}_e$ with $e \ge 2$ an integer, M a closed subgroup of infinite index, N a closed normal subgroup of F contained in M, and M_0 an open subgroup of M which does not contain N. Then $M_0 \cong \hat{F}_\omega$.*

Proof: First of all we replace M by $M_0 N$, if necessary, to assume $M = M_0 N$. Next we choose an open subgroup F_0 of F with $F_0 \cap M = M_0$ [FrJ, Lemma 1.2.5(a)]. By Nielsen-Schreier, $F_0 < F$ are free of degree at least 2. Replace F by $\langle F_0, M \rangle$, if necessary, to assume that $\langle F_0, M \rangle = F$ and $F_0 < F$. Then, $N_0 = M_0 \cap N$ is a closed subgroup of F, satisfying $F_0 \cap N = N_0$ and $F_0 N = F$.

$$\begin{array}{ccc} F_0 & \text{---} & F \\ | & & | \\ M_0 & \text{---} & M \\ | & & | \\ N_0 & \text{---} & N \end{array}$$

By Nielsen-Schreier, $\mathrm{rank}(F_0/N_0) = \mathrm{rank}(F/N) \le \mathrm{rank}(F) < \mathrm{rank}(F_0)$. It follows from Lemma 1.2 (with F_0 replacing F) that $M_0 \cong \hat{F}_\omega$. ∎

By Proposition 1.3, every finite group is a quotient of M_0. On the other hand, if we assume in Proposition 1.3 that $\mathrm{rank}(F) = \infty$, then by Haran's Diamond theorem for groups, $M_0 \cong F$ [FrJ, Thm. 25.4.3]. In particular, each finite group is a quotient of M_0. It turns out that the condition on the quotients of M_0 suffices to prove the analog of Karrass-Solitar theorem for profinite groups. We show here how to reduce the condition on the quotients of M_0 from infinite ranks to finite ranks, thus circumventing the Diamond theorem:

LEMMA 1.4: *Let F be a free profinite group of rank at least 2, M a closed subgroup of F of infinite index, N a closed normal subgroup of F contained in M, and M_0 an open subgroup of M which does not contain N. Then every finite group is a quotient of M_0.*

Proof: Let G be a finite group and set $m = \max(2, \mathrm{rank}(G))$. By Proposition 1.3, it suffices to prove the lemma in the case where $\mathrm{rank}(F) = \infty$. Let X be a basis of F in the sense of [FrJ, Def. 17.4.5]. For each finite subset Y of X with $|Y| \geq m$ let $\hat{F}_Y$ be the free profinite group with basis Y, $\varphi_Y\colon F \to \hat{F}_Y$ the epimorphism defined by $\varphi_Y(x) = x$ if $x \in Y$ and $\varphi_Y(x) = 1$ if $x \in X \smallsetminus Y$. Set $M_Y = \varphi_Y(M)$, $N_Y = \varphi_Y(N)$, and $M_{0,Y} = \varphi_Y(M_0)$. Then M_Y is a closed subgroup of $\hat{F}_Y$, N_Y is a closed normal subgroup of $\hat{F}_Y$, and $M_{0,Y}$ is an open subgroup of $\hat{F}_Y$. Unfortunately, we may have $N_Y \leq M_{0,Y}$. In order to overcome this difficulty, recall that X converges to 1. Thus $X \smallsetminus F_0$ is a finite set for each open subgroup F_0 of F. Therefore, the intersection of all kernels $\mathrm{Ker}(\varphi_Y)$, where Y ranges over all finite subsets of X with $|Y| \geq m$ is trivial.

By assumption, there is a $z \in N \smallsetminus M_0$. By the preceding paragraph, $M_0 = \bigcap_Y M_0\mathrm{Ker}(\varphi_Y)$ [FrJ, Lemma 1.2.2(b)], so there is a finite subset Y of X with $\varphi_Y(z) \in N_Y \smallsetminus M_{0,Y}$ and $|Y| \geq m$. If M_Y is open in $\hat{F}_Y$, then, by Nielsen-Schreier, $M_{0,Y}$ is free of rank at least $|Y|$, so G is a quotient of $M_{0,Y}$. If $(\hat{F}_Y : M_Y) = \infty$, then $M_{0,Y} \cong \hat{F}_\omega$ (Proposition 1.3), so G is a quotient of $M_{0,Y}$. In both cases, G is a quotient of M_0, as claimed. ∎

A theorem of Weissauer says that if M is a separable algebraic extension of a Hilbertian field K and M' is a finite separable extension of M not contained in the Galois closure of M/K, then M' is Hilbertian [FrJ, Thm. 13.9.1]. A theorem of Roquette allows us to transfer Proposition 1.3 to an analog of Weissauer's theorem in which K is not Hilbertian but rather PAC and e-free. Here we say that a field K is **PAC** if every absolutely irreducible variety over K has a K-rational point. The field K is e**-free**, if its absolute Galois group, $\mathrm{Gal}(K)$, is isomorphic to $\hat{F}_e$.

THEOREM 1.5: *Let $e \geq 2$ be an integer, K an e-free PAC field, M an infinite separable algebraic extension of K, and M' a finite extension of M not contained in the Galois closure of M/K. Then M' is PAC, $\mathrm{Gal}(M') \cong \hat{F}_\omega$, and M is Hilbertian.*

Proof: By Ax-Roquette, M' is PAC [FrJ, Cor. 11.2.5]. By Proposition 1.3, $\mathrm{Gal}(M') \cong \hat{F}_\omega$. Hence, by Roquette, M' is Hilbertian [FrJ, Cor. 27.3.3]. ∎

A similar reasoning yields the translation of Lemma 1.2 to field theory:

THEOREM 1.6: *Let $e \geq 2$ be an integer, K an e-free PAC field, M an infinite separable algebraic extension of K, and N a Galois extension of K containing M. Suppose $\mathrm{rank}(\mathrm{Gal}(N/K)) < \mathrm{rank}(\mathrm{Gal}(K))$. Then M is PAC, $\mathrm{Gal}(M) \cong \hat{F}_\omega$, and M is Hilbertian.*

2. Melnikov Formation

A family $\mathcal{C}$ of finite groups is called a **Melnikov formation** if it is closed under taking quotients, normal subgroups, and extensions. Each element of $\mathcal{C}$ is a **$\mathcal{C}$-group**. Projective limits of $\mathcal{C}$-groups are **pro-$\mathcal{C}$ groups**. See [FrJ, Sections 17.3 and 17.4] for a discussion of Melnikov formations and free pro-$\mathcal{C}$ groups.

When we speak about a free pro-$\mathcal{C}$ group of rank m for a Melnikov formation $\mathcal{C}$, we tacitly assume that $\mathcal{C}$ has a nontrivial group of rank at most m [FrJ, Remark 17.4.7].

PROPOSITION 2.1: *Let $\mathcal{C}$ be a Melnikov formation, F a free pro-$\mathcal{C}$ group of rank at least 2, M a closed subgroup of F of infinite index, N a closed normal subgroup of F contained in M, and M_0 an open subgroup of M which does not contain N. Then each $\mathcal{C}$-group is a quotient of M_0. If in addition M_0 is pro-$\mathcal{C}$, then M_0 is isomorphic to the free pro-$\mathcal{C}$ group $\hat{F}_\omega(\mathcal{C})$ of countable rank.*

Proof: Let $\hat{F}$ be the free profinite group with $\mathrm{rank}(\hat{F}) = \mathrm{rank}(F)$. Denote the intersection of all open normal subgroups L of $\hat{F}$ with $\hat{F}/L \in \mathcal{C}$ by $\hat{K}$. By [FrJ, Lemma 17.4.10], there is an epimorphism $\varphi\colon \hat{F} \to F$ with $\hat{K} = \mathrm{Ker}(\varphi)$. Set $\hat{M} = \varphi^{-1}(M)$, $\hat{M}_0 = \varphi^{-1}(M_0)$, and $\hat{N} = \varphi^{-1}(N)$. Then $\hat{M}_0$ is an open subgroup of $\hat{M}$ and $\hat{N}$ is a closed normal subgroup of $\hat{F}$ which is not contained in $\hat{M}_0$.

$$
\begin{tikzcd}
 & \hat{F} \arrow[rr, "\varphi"] \arrow[d, dash] & & F \arrow[d, dash] \\
 & \hat{M} \arrow[rr] \arrow[dl, dash] \arrow[dd, dash] & & M \arrow[dl, dash] \arrow[dd, dash] \\
\hat{M}_0 \arrow[rr] \arrow[dd, dash] & & M_0 \arrow[dd, dash] & \\
 & \hat{N} \arrow[rr] \arrow[dl, dash] & & N \arrow[dl, dash] \\
\hat{K} \arrow[rr] & & 1 &
\end{tikzcd}
$$

Let G be a $\mathcal{C}$-group. By Proposition 1.3, $\hat{M}_0$ has an open normal subgroup $\hat{M}_1$ with $\hat{M}_0/\hat{M}_1 \cong G$. Then $\hat{K}/\hat{M}_1 \cap \hat{K} \cong \hat{M}_1\hat{K}/\hat{M}_1 \triangleleft \hat{M}_0/\hat{M}_1$, so $\hat{K}/\hat{M}_1 \cap \hat{K} \in \mathcal{C}$. By [FrJ, Lemma 17.4.10], $\hat{M}_1 \cap \hat{K} = \hat{K}$, so $\hat{K} \le \hat{M}_1$ and $M_0/\varphi(\hat{M}_1) \cong \hat{M}_0/\hat{M}_1 \cong G$. Thus, G is a quotient of M_0, as claimed.

Now suppose M_0 is pro-$\mathcal{C}$. Then $\hat{M}_0/\hat{K} \cong M_0$ is also pro-$\mathcal{C}$. The argument of the preceding paragraph shows that if $\hat{M}_1$ is an open normal subgroup of $\hat{M}_0$ with $\hat{M}_0/\hat{M}_1 \in \mathcal{C}$, then $\hat{K} \leq \hat{M}_1$. Thus, $\hat{K}$ is the intersection of all open normal subgroups of $\hat{M}_0$ satisfying $\hat{M}_0/\hat{M}_1 \in \mathcal{C}$. It follows from [FrJ, Lemma 17.4.10] that $M_0 \cong \hat{M}_0/\hat{K} \cong \hat{F}_\omega(\mathcal{C})$. ∎

Lemma 2.2: *Let $\mathcal{C}$ be a Melnikov formation containing nontrivial groups. Then, for each positive integer m there is a $\mathcal{C}$-group G with* $\mathrm{rank}(G) \geq m$.

Proof: Since $\mathcal{C}$ is closed under taking quotients, $\mathcal{C}$ contains a simple group S, hence $S^n \in \mathcal{C}$ for each positive integer n. If $S \cong \mathbb{Z}/p\mathbb{Z}$ for some prime number p, then $\mathrm{rank}(S^n) = \dim_{\mathbb{F}_p}(\mathbb{Z}/p\mathbb{Z})^n = n$ and we are done. Assume S is nonabelian and there exists an integer e with $\mathrm{rank}(S^n) \leq e$ for all n. Then S^n is a quotient of $\hat{F}_e$ for each n. On the other hand, n is bounded by $\frac{|D_e(S)|}{|\mathrm{Aut}(S)|}$, where $D_e(S)$ is the set of all $(x_1, \ldots, x_e) \in S^e$ with $\langle x_1, \ldots, x_e \rangle = S$ [FrJ, Lemma 26.1.2]. It follows from this contradiction that $\mathrm{rank}(S^n)$ is not bounded. ∎

The results achieved so far give a pro-$\mathcal{C}$ analog of the Karrass-Solitar theorem:

Theorem 2.3: *Let $\mathcal{C}$ be a Melnikov formation of finite groups, F a free pro-$\mathcal{C}$ group of rank at least 2, M a closed subgroup of F of infinite index containing a nontrivial closed normal subgroup N of F. Then M is not finitely generated.*

Proof: The group F has an open normal subgroup F_0 which does not contain N. Thus, $M_0 = F_0 \cap M$ is an open normal subgroup of M which does not contain N. By Proposition 2.1, every $\mathcal{C}$-group is a quotient of M_0. Hence, by Lemma 2.2, M_0 is not finitely generated. Consequently, M is not finitely generated. ∎

Proposition 2.1 has many more consequences:

Remark 2.4: Let $\mathcal{C}$, F, M, M_0, and N be as in Theorem 2.3. By that theorem, every $\mathcal{C}$-group is a quotient of M_0. Therefore, not only M is finitely generated but M is not a small group [FrJ, Sec. 16.10]. Moreover, M is not a pro-$\mathcal{B}$ group for any Melnikov formation $\mathcal{B}$ properly contained in $\mathcal{C}$. In particular, if $\mathcal{C}$ contains finite non-abelian groups, then M is not pro-solvable, so M is non-abelian. If $\mathcal{C}$ contains non-p-groups, then M is not pro-p.

We could also deduce those consequences from [FrJ, Prop. 24.10.4 and Thm. 25.4.7]. The advantage of the present approach is that in the infinite case we are applying only simple means based on the Lubotzky-v.d.Dries trick and do not use the Diamond theorem. ∎

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