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On a question of Frey and Jarden about the rank of Abelian varieties ☆

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Abstract

This paper deals with a classical question of Frey and Jarden, who asked in their 1974 paper if any non-zero Abelian variety over a number field K acquires infinite rank over the maximal Abelian extension K^{ab} of the ground field. We generalize recent results of Rosen and Wong on the subject. However, the original question in full generality remains open. Some further results on the rank in certain other infinite extensions are included.

MSC: 11G10; 14K15

Keywords: Hilbertian field; Abelian variety; Rank; Specialization

1. Introduction

Let K a field and $A|K$ an Abelian variety. By the Mordell–Weil theorem, $A(K)$ is finitely generated provided K is a finitely generated field. On the other hand it is known that $A(\overline{K})$ is of infinite rank unless K is algebraic over a finite field. (We often tacitly assume $A \neq 0$.) Interesting problems arise if one studies the rank in other infinite algebraic extensions of K. For elliptic curves $E|\mathbb{Q}$ Frey and Jarden showed that $E(\Omega)$ is of infinite rank where Ω denotes the maximal Kummer extension of $\mathbb{Q}$ of exponent 2. In the light of these facts Frey and Jarden asked in their paper [2]:

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Question. Is $\operatorname{rank}(A(K^{ab})) = \infty$ for every Abelian variety A over a number field K? Here K^{ab} denotes the maximal Abelian extension of K.

The most recent result towards this question we are aware of is due to Rosen and Wong [21]. They show (over a number field K as ground field) that $\operatorname{rank}(J_T(K^{ab})) = \infty$ for any cyclic cover $T|\mathbb{P}_1$ of positive genus. Papers [9,19,26] contain special cases of this statement. We can strengthen the above result as follows.

Theorem 1.1. *Let K a Hilbertian field and $T|K$ a smooth projective curve of positive genus. Suppose that T can be realized as a Galois cover of $\mathbb{P}_1$ with group Γ. Let B an arbitrary non-zero quotient of J_T. Then there is an infinite Galois extension $\Omega|K$ with group $\prod_{i=1}^{\infty}\Gamma$ such that* $\operatorname{rank}(B(\Omega)) = \infty$. *In particular* $\operatorname{rank}(B(K^{ab})) = \infty$ *provided Γ is Abelian.*

We can thus treat a broader class of Abelian varieties and a broader class of ground fields.

Remark 1.2. (1) Any Abelian variety A over a field K is the quotient of a Jacobian variety. See [17, 10.1] for example.

(2) Theorem 1.1 naturally leads us to the following question: Is any simple Abelian variety over a field K the quotient of the Jacobian J_T of a curve $T|K$ which can be realized as an Abelian Galois cover of $\mathbb{P}_1$? Lange pointed out that the answer to this question is not known even over the complex numbers $K = \mathbb{C}$. If the answer to this question is yes, then $A(K^{ab})$ is of infinite rank for any non-zero Abelian variety A over a Hilbertian field K.

In the proof of Theorem 1.1 we use methods totally different from the method in [21]. The key argument in our paper is a specialization theorem for Abelian varieties over Hilbertian fields (see Proposition 3.1 below). We want to mention that while reading papers [22,23] of Rubin and Silverberg on rank frequencies in families of quadratic twists of elliptic curves, it occurred to us that we might use a specialization theorem. The specialization technique also allows us to prove the following infinite rank result.

Theorem 1.3. *Let A a non-zero Abelian variety over a Hilbertian field K. Suppose that A admits a degree d projective embedding. Assume that $d \geqslant 2$. Then* $\operatorname{rank}(A(\Omega)) = \infty$ *where Ω is the compositum of all extensions of K of degree d.*

In [21] this is shown for the compositum of all extensions of K of degree $\leqslant d(4\dim(A)+2)$ instead of Ω. Finally we can slightly generalize a classical result in [2].

Theorem 1.4 (Frey–Jarden). *Let A a non-zero Abelian variety over a Hilbertian field K and $e \geqslant 1$. Write K_s for the separable closure of K. For $\sigma = (\sigma_1, \dots, \sigma_e) \in G_K^e$ denote by $K_s(\sigma)$ the fixed field in K_s of the closure of the group $\langle\sigma_1, \dots, \sigma_e\rangle \subset G_K$ generated by the components of the vector σ. Then* $\operatorname{rank}(A(K_s(\sigma))) = \infty$ *for almost all* (*in the sense of Haar measure on G_K^e*) $\sigma \in G_K^e$.

Several remarks[1] are in order.

[1] We want to thank M. Jarden for a detailed explanation of the relative roles of Theorem 1.4, [2, 9.1], [4, Theorem B] and the result in Appendix A.

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Remark 1.5. (1) In [2, 9.1] Frey and Jarden have shown Theorem 1.4 under the additional hypothesis that K is a *finitely generated* Hilbertian field.

(2) For $\sigma \in G_K^e$, denote by $K_s[\sigma]$ the maximal Galois extension of K in $K_s(\sigma)$. In Appendix A to this paper Jarden proves that the statement of the above Theorem 1.4 remains true if one replaces $K_s(\sigma)$ by its subfield $K_s[\sigma]$. This was known in case of a *finitely generated* Hilbertian field K due to work of Geyer and Jarden (see [4, Theorem B]), but the case of an arbitrary, not necessarily finitely generated Hilbertian field K as ground field is new.

(3) Let F a finitely generated field and K an infinite algebraic extension of F. Assume that K is Hilbertian. Let $A|K$ a non-zero Abelian variety. Then there is a finitely generated, Hilbertian intermediate field $F \subset K' \subset K$ with $[K' : F] < \infty$ and an Abelian variety $A'|K'$ such that $A \cong A' \otimes_{K'} K$. By the classical result [2, 9.1] mentioned above $\operatorname{rank}(A'(K'_s(\sigma))) = \infty$ for almost all $\sigma \in G_{K'}^e$. Furthermore we may identify G_K with a subgroup of $G_{K'}$. However this does *not* immediately imply $\operatorname{rank}(A(K_s(\sigma))) = \infty$ for almost all $\sigma \in G_K^e$ as G_K^e has measure zero in $G_{K'}^e$ by our hypothesis $[K : F] = \infty$. Thus our Theorem 1.4 is stronger than the classical result [2, 9.1] of Frey and Jarden. By a similar kind of reasoning Jarden's result in Appendix A is stronger than the result [4, Theorem B] of Geyer and Jarden.

(4) Im has shown in [8] that for any elliptic curve $E|\mathbb{Q}$ and any $\sigma \in G_{\mathbb{Q}}$ the Mordell–Weil group $E(\mathbb{Q}_s(\sigma))$ is of infinite rank. Furthermore, if E is an elliptic curve over a number field K and if $E(K)$ contains a point P such that $2P \neq 0$ and $3P \neq 0$, then again $\operatorname{rank}(E(K_s(\sigma))) = \infty$ for all $\sigma \in G_K$ by Im's result [7]. Larsen suspects in [13] that it might be true, that $\operatorname{rank}(A(K_s(\sigma))) = \infty$ for any non-zero Abelian variety A over an infinite, finitely generated field K and any $\sigma \in G_K^e$.

This paper is organized as follows. After summarizing some generalities on Hilbertian fields in Section 2 we prove a specialization theorem for Abelian varieties over Hilbertian fields in Section 3. In Section 4 we prove an abstract sufficient condition for infinite rank over infinite extensions. In the final section we derive the above theorems from the result in Section 4.

2. Hilbertian fields

We briefly summarize elementary but important facts about Hilbertian fields, including a notion of so-called abstract Hilbert sets. The most useful references on Hilbertian fields are [3,10].

Let K a field. Let $T = (T_1, \ldots, T_n)$ a vector of indeterminates and X a single indeterminate. For an irreducible polynomial

$$f(T, X) = \sum_{i=0}^{d} a_i(T) X^i \in K(T)[X]$$

of degree d let U_f be $\mathbb{A}_n$ with the poles of the a_i removed and let

$$H_f := \{ t \in U_f(K) \mid f(t, X) \in K[X] \text{ irreducible of degree } d \}$$

the corresponding *fundamental Hilbert set*. A *Hilbert set* is any subset of $\mathbb{A}_n(K)$ which may be written as the intersection of finitely many fundamental Hilbert sets and one non-empty open set.

K is said to be a *Hilbertian field* if (for all n) all Hilbert sets are non-empty and hence dense in $\mathbb{A}_n(K)$. Note that algebraically closed fields, local fields and finite fields are never Hilbertian.

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On the other hand number fields, fields of the form $F(u)$ where F is an arbitrary field and finite extensions of Hilbertian fields are Hilbertian.

If S is an integral scheme, then we shall denote its function field by $R(S)$ in the sequel. Let X an integral, separated, algebraic K-scheme. We shall say that $p:Y \to X$ is a *Hilbert cover* of X if Y is an integral, separated, algebraic K-scheme and p is a finite, flat, generically separable (that is, the extension of function fields $R(Y)|R(X)$ is separable) morphism. Note that we assume Y is integral but *not necessarily geometrically integral*.

If S is a K-scheme and $s \in S$ is a point, then we shall denote by $K(s)$ the residue field of s. Note that $K(s)$ is an extension field of K. Now let $p:Y \to X$ a Hilbert cover of degree d and $x \in X$ a closed point. We shall say that x is *inert for* p if $p^{-1}(x)$ is connected and geometrically reduced over $K(x)$. Thus x is inert for p iff $p^{-1}(x)$ is Spec of a finite separable extension field of $K(x)$ of degree d. If x is inert for p, then we write $s_{Y|X}(x)$ for the unique point over x. We denote the set of all closed points $x \in X$ which are inert for p by $\mathrm{Inert}(Y|X)$ and call $\mathrm{Inert}(Y|X)$ the *abstract Hilbert set* associated to p. Note that Hilbert sets consist of K-rational points in $\mathbb{A}_n(K)$ whereas an abstract Hilbert set $\mathrm{Inert}(Y|X)$ is a set of closed points of X. For example, if X is a K-variety (varieties are always meant to be geometrically integral in this paper) and $F|K$ is a finite, separable extension, then $X_F \to X$ is a Hilbert cover and $\mathrm{Inert}(X_F|X)$ consists of the closed points $x \in X$ for which $K(x)|K$ is a separable extension linearly disjoint from F.

Remark 2.1. Let X an integral, separated, algebraic K-scheme, $Y \to X$ a Hilbert cover of X and $Z \to Y$ a Hilbert cover of Y. Let $x \in X$ a closed point. Then $x \in \mathrm{Inert}(Z|X)$ if and only if $x \in \mathrm{Inert}(Y|X)$ and $s_{Y|X}(x) \in \mathrm{Inert}(Z|Y)$.

Proposition 2.2. *Let K a Hilbertian field and $n \geqslant 1$. Let $U \subset \mathbb{A}_n$ a non-empty open set and $p: X \to U$ a Hilbert cover of U. Then the set*

$$\mathrm{Inert}(X|U) \cap U(K) \subset \mathbb{A}_n(K)$$

contains a Hilbert set. In particular it is dense in $\mathbb{A}_n(K)$ and thus infinite.

Proof. We may assume that $U = \mathrm{Spec}(A)$ and hence also $X = \mathrm{Spec}(B)$ is affine. Eventually making U smaller we may even assume p étale and $B = A[b]$. Let $T = (T_1, \dots, T_n)$ the coordinates of $\mathbb{A}_n$. Then $K(T)$ is the quotient field of A. Let $f(T,Z) \in K(T)[Z]$ the minimum polynomial of b. We have $f(T,Z) \in A[Z]$, as A is normal, and $B = A[Z]/f(T,Z)A[Z]$. Now $t \in U(K)$ is inert for p iff $B \otimes_A K(t) = K[Z]/f(t,Z)$ is a field, that is iff the specialization $f(t,Z)$ is irreducible. Thus $\mathrm{Inert}(X|U) \cap U(K)$ contains the Hilbert set $H_f \cap U(K)$. $\square$

In the following we let $\mathrm{Inert}(Y_\bullet|X) = \bigcap_{i=1}^{s} \mathrm{Inert}(Y_i|X)$ if $(p_i : Y_i \to X)_{i=1,\dots,s}$ is a finite family of Hilbert covers of X. Furthermore, if $f: T \to S$ is a finite, flat morphism of schemes and $\Gamma := \mathrm{Aut}_S(T)$, then we shall say that f is a *Galois cover* if the canonical map

$$\mathrm{Mor}(S, Z) \to \mathrm{Mor}(T, Z)^{\Gamma}, \qquad h \mapsto h \circ f$$

is bijective for all schemes Z.

Remark 2.3. Let K a field and X a smooth K-variety. Then the function field $R(X)$ contains a purely transcendental subfield L over which $R(X)$ is of finite degree. Let n the transcendency degree of $L|K$. We may then construct a diagram

$$\begin{tikzcd} X' \subset X \arrow[d, "p"'] \\ U \subset \mathbb{A}_n, \end{tikzcd}$$

where the vertical map p is an étale Hilbert cover of degree $[R(X):L]$, the symbols $\subset$ stand for open immersions and p induces the inclusion $L \to R(X)$ if we identify L with $R(\mathbb{A}_n)$. (One may use [3, 6.1.5] to see this.) Once such a diagram is established, if $R(X) = R(X')$ happens to be Galois over $L = R(U)$, then p will even be an étale Galois cover with group $G(R(X')|R(U))$. See [6, Exposé V] for generalities on Galois covers of schemes.

Corollary 2.4. *Let X a smooth variety over a Hilbertian field K and $(p_i : Y_i \to X)_i$ a finite family of Hilbert covers of X. Then* $\mathrm{Inert}(Y_\bullet|X)$ *is infinite.*

Proof. Consider a diagram as in the above remark. Let $Y_i' := p_i^{-1}(X')$. Now $\mathrm{Inert}(Y_\bullet'|U)$ is infinite by Proposition 2.2 and for any $u \in \mathrm{Inert}(Y_\bullet'|U)$ there is a unique point $x \in X'$ over u which lies in $\mathrm{Inert}(Y_\bullet'|X')$ by Remark 2.1 and hence in $\mathrm{Inert}(Y_\bullet|X)$. □

The following strengthening of the corollary will be important in Section 3. Theorems similar to Proposition 2.5 below with similar proofs can be found in several places in book [3] of Fried and Jarden. See part A of the proof of [3, 18.6.1] for example.

Proposition 2.5. *Let K a Hilbertian field. Consider a diagram as in Remark* 2.3. *Let $(p_i : Y_i \to X)_i$ a finite (possibly empty) family of Hilbert covers of X. Let $F|K$ a finite, separable extension (possibly $F = K$) and fix once and for all a K-embedding $F \to K_s$. Then there is a sequence $(t_i)_{i\in\mathbb{N}}$ of geometric points in $X'(K_s)$ with the following properties*:

(1) *Each geometric point t_i is localized in a point in* $\mathrm{Inert}(Y_\bullet|X)$.
(2) *$p(t_i) \in U(K)$ is K-rational and $[K(t_i):K] = [R(X):R(U)]$ for all i.*
(3) *$(F, K(t_1), K(t_2), \ldots)$ is a linearly disjoint sequence of fields, that is $F \otimes \bigotimes_{i=1}^{\infty} K(t_i)$ is a field.*

Let Ω the composite field in K_s of F with all the $K(t_i)$. Then $X(\Omega)$ is infinite.

Proof. Suppose that geometric points $t_1, \ldots, t_m \subset X'(K_s)$ are already constructed, such that properties (1) and (2) hold for $1 \leqslant i \leqslant m$ and such that $(F, K(t_1), \ldots, K(t_m))$ is a linearly disjoint family of extensions of K. Let $E = FK(t_1)\cdots K(t_m)$ the composite field. Let $Y_i' = p_i^{-1}(X')$. Consider the composite Hilbert covers $Y_i' \to X' \to U$ and $X_E' \to X' \to U$. We may pick a K-rational point $u \in \mathrm{Inert}(Y_\bullet'|U) \cap \mathrm{Inert}(X_E'|U) \cap U(K)$. Then u is inert for p by Remark 2.1. Hence there is a unique closed point $x \in X'$ over u and, again by Remark 2.1, we have $x \in \mathrm{Inert}(Y_\bullet'|X') \cap \mathrm{Inert}(X_E'|X')$. Let $t_{m+1} \in X'(K_s)$ one of the geometric points localized at x (corresponding to a K-embedding $j : K(x) \to K_s$). Clearly (1) holds for $i = m+1$.

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Furthermore $p(t_{m+1}) = u$ is K-rational by construction. From $u \in \operatorname{Inert}(X'|U)$ it follows that $[K(t_{m+1}) : K] = \deg(p) = [R(X) : R(U)]$. Hence (2) holds for $i = m+1$. To see that also (3) holds for $i = m+1$ note that $K(x) \otimes_K E$ must be a field, because $x \in \operatorname{Inert}(X'_E|X')$. Hence $K(t_{m+1}) = j(K(x))$ is linearly disjoint from $E = FK(t_1)\cdots K(t_m)$. $\square$

Remark 2.6. (1) If $R(X)|R(U)$ and hence p happens to be Galois with group Γ, then all $K(t_i)|K$ will be Galois with group Γ. If in addition $F|K$ is Galois, then $\Omega|K$ is an infinite Galois extension with $G(\Omega|K) = G(F|K) \times \prod_{i=1}^{\infty} \Gamma$ by 2.

(2) As indicated above we may apply the theorem without the family $Y_\bullet|X$. It then states that for any diagram as in Remark 2.3, there is a sequence $(t_i)_i$ in $X(K_s)$ satisfying (2) and (3).

(3) In particular we see that for any variety X, whose function field $R(X)$ contains a purely transcendental subfield over which $R(X)$ is Galois with group Γ, there is an infinite Galois extension $\Omega|K$ with group $\prod_{i\in\mathbb{N}} \Gamma$, such that $X(\Omega)$ is infinite. Furthermore $X(K^{ab})$ is infinite provided Γ is Abelian. For example $C(K^{ab})$ is infinite for any smooth curve which can be realized as an Abelian Galois cover of $\mathbb{P}_1$.

We briefly indicate how the proof of our infinite rank results will proceed. Suppose we are given a diagram as in Remark 2.3 over a Hilbertian field K, an Abelian variety $A|K$ and a non-constant morphism $f: X \to A$. We will see by the specialization Theorem 3.1 in the next section that one can construct a finite family of Hilbert covers $Y_\bullet|X$ such that $f(X(\Omega))$ generates a subgroup of infinite rank in $A(\Omega)$, provided Ω is as in Proposition 2.5. All infinite rank results in this paper arise in that way.

3. Specialization

In this section we will discuss a specialization theorem that will play the key role in the sequel. This specialization theorem is similar to a theorem in Lang's encyclopaedia [11, I.7]. Lang does not give a proof but refers to a paper of Néron [20] containing a version weaker than [11, I.7], which is formulated in the language of Weil's foundations and therefore difficult to read for our generation. For that reason we include a proof following [24] in some places.

For the whole section let K be a Hilbertian field, $A|K$ an Abelian variety and $T|K$ a smooth, projective variety. Assume that $A(K)$ is of finite rank. For $t \in T$ there is a specialization map

$$\alpha_t : \operatorname{Mor}_K(T, A) \to A\bigl(K(t)\bigr), \qquad f \mapsto f(t).$$

In the rest of the paper we view without further mentioning $A(K)$ as the subgroup of constant morphisms in $\operatorname{Mor}_K(T, A)$ and let $M_K(T, A) := \frac{\operatorname{Mor}_K(T,A)}{A(K)}$. Then α_t induces a homomorphism

$$\bar{\alpha}_t : M_K(T, A) \to \frac{A(K(t))}{A(K)}$$

which fits into an exact diagram

$$\begin{tikzcd}
0 \arrow[r] & A(K) \arrow[r] \arrow[d, equal] & \operatorname{Mor}_K(T, A) \arrow[r] \arrow[d, "\alpha_t"'] & M_K(T, A) \arrow[r] \arrow[d, "\bar{\alpha}_t"'] & 0 \\
0 \arrow[r] & A(K) \arrow[r] & A\bigl(K(t)\bigr) \arrow[r] & \frac{A(K(t))}{A(K)} \arrow[r] & 0.
\end{tikzcd}$$

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Thus $\ker(\alpha_t) \cong \ker(\bar{\alpha}_t)$. Note that $\ker(\bar{\alpha}_t)$ is non-zero for any K-rational point $t \in T(K)$ provided $M_K(T,A) \neq 0$. Hence the specialization maps can be non-injective for infinitely many t. However we have the following specialization theorem.

Proposition 3.1. *There is a finite family of étale Hilbert covers $g_i : X_i \to T$ such that the specialization maps α_t and $\overline{\alpha_t}$ are injective for all $t \in \mathrm{Inert}(X_\bullet|T)$.*

The proof will occupy the rest of this section.

Lemma 3.2. *$M_K(T,A)$ is a free $\mathbb{Z}$-module of finite rank. In particular $\ker(\alpha_t)$ is a free $\mathbb{Z}$-module of finite rank for all $t \in T$.*

Proof. The injection $\mathrm{Mor}_K(T,A) \to \mathrm{Mor}_{\overline{K}}(T_{\overline{K}}, A_{\overline{K}})$ induces an injection $M_K(T,A) \to M_{\overline{K}}(T_{\overline{K}}, A_{\overline{K}})$. Let $J|\overline{K}$ the Albanese variety of $T_{\overline{K}}$. There is an isomorphism

$$M_{\overline{K}}(T_{\overline{K}}, A_{\overline{K}}) \cong \mathrm{Hom}_{\overline{K}}(J, A_{\overline{K}})$$

by the universal mapping property of the Albanese variety. (See [11, p. 31] and [12, Section II.3] for information on the Albanese variety.) Furthermore, by [17, 12.5] or [18, Theorem 3, p. 176], $\mathrm{Hom}_F(B_1, B_2)$ is finitely generated and $\mathbb{Z}$-free for any two Abelian varieties B_1 and B_2 over a field F. Thus $M_K(T,A)$ and $\ker(\alpha_t)$ are finitely generated and $\mathbb{Z}$-free as submodules of the finitely generated and $\mathbb{Z}$-free $\mathbb{Z}$-module $\mathrm{Hom}_{\overline{K}}(J, A_{\overline{K}})$. □

Lemma 3.3. *Let l a prime different from the characteristic. There exists a finite separable extension $F|K$ such that for all $t \in \mathrm{Inert}(T_F|T)$ the restriction*

$$\alpha_t | \mathrm{Mor}_K(T,A)_l \to A_l\big(K(t)\big)$$

of the specialization map is bijective.

Proof. Clearly $\ker(\alpha_t) \cap \mathrm{Mor}_K(T,A)_l = 0$ as $\ker(\alpha_t)$ is $\mathbb{Z}$-free. The map l_A is an étale isogeny due to our hypothesis on l. Hence the group scheme A_l is finite and étale over K. Thus there are separable extension fields E_i over K such that there is an K-isomorphism

$$A_l \cong \coprod_{i=1}^{s} \mathrm{Spec}(E_i).$$

Let $F = E_1 \cdots E_s$ the composite field and $m = |\{i \mid E_i = K\}|$. If $t \in \mathrm{Inert}(T_F|T)$, then $K(t)$ is linearly disjoint from F and hence $A_l(K(t)) = A_l(K)$. (To see this note that $|A_l(L)| = \Sigma|\mathrm{Hom}_K(E_i, L)| = m$ for all finite extension fields $L|K$ which are linearly disjoint from F, in particular for $L = K$ or $L = K(t)$.) Now $\alpha_t|\mathrm{Mor}_K(T,A)_l \to A_l(K(t)) = A_l(K)$ is clearly surjective for all $t \in \mathrm{Inert}(T_F|T)$, as $\mathrm{Mor}_K(T,A)$ contains the constant morphisms in $A_l(K)$. □

Lemma 3.4. *Let l a prime different from the characteristic. Then the group $\mathrm{Mor}_K(T,A) \otimes_{\mathbb{Z}} \mathbb{Z}/l$ is finite.*

Note that we did assume $\mathrm{rank}(A(K)) < \infty$ throughout this section, but we do not assume that $A(K)$ is finitely generated. Hence also $\mathrm{Mor}_K(T, A)$ needs not be finitely generated.

Proof[2] of Lemma 3.4. Let $M := \mathrm{Mor}_K(T, A)$. Then $M/A(K) = M_K(T, A)$ is a finitely generated, free $\mathbb{Z}$-module by Lemma 3.2. It is thus enough to show that $A(K)/l$ is finite. $F := A(K)/A(K)_{\mathrm{tor}}$ is a torsion-free Abelian group and $\mathrm{rank}(F) < \infty$ by our hypothesis $\mathrm{rank}(A(K)) < \infty$. It follows that $\dim_{\mathbb{F}_l}(F/l) \leqslant \mathrm{rank}(F) < \infty$. Hence F/l is finite. It remains to prove that $A(K)_{\mathrm{tor}}/l$ is finite. If $r = \dim(A)$, then $A(K)_{l^i}$ injects to $A(\overline{K})_{l^i} = (\mathbb{Z}/l^i)^{2r}$ and hence $\dim_{\mathbb{F}_l}(A(K)_{l^i}/l) \leqslant 2r$ for all $i \in \mathbb{N}$. Consider the l-Sylow subgroup $A(K)_{l^\infty} = \bigcup_{i=1}^{\infty} A(K)_{l^i}$. It follows that $A(K)_{l^\infty}/l$ is finite. Finally, making use of the isomorphism $A(K)_{\mathrm{tor}}/l \cong A(K)_{l^\infty}/l$, we conclude that $A(K)_{\mathrm{tor}}/l$ is finite, as desired. □

Lemma 3.5. *Let $f : X \to Y$ a finite, flat morphism of integral schemes and suppose that Y is normal. If f is of degree* 1 (*that is* $[R(X) : R(Y)] = 1$), *then f is an isomorphism.*

Proof. $f(X)$ is closed as f is a finite morphism and open as f is a flat morphism. Hence f must be surjective, as Y is connected. If $x \in X$ and U is an open, affine neighborhood of $y = f(x)$, then $V := f^{-1}(U)$ is open and affine. The homomorphism $f^\sharp : \mathcal{O}_Y(U) \to \mathcal{O}_X(V) = f_*\mathcal{O}_X(U)$ is a monomorphism of integral domains which makes $\mathcal{O}_X(V)$ a finite and hence integral algebra over $\mathcal{O}_Y(U)$. The hypothesis $[R(X) : R(Y)] = 1$ implies that both rings $\mathcal{O}_Y(U)$ and $\mathcal{O}_X(V)$ have the same quotient field. Furthermore $\mathcal{O}_Y(U)$ is normal. Hence $f^\sharp : \mathcal{O}_Y(U) \to \mathcal{O}_X(V)$ must be an isomorphism. It follows that f is an isomorphism. □

Lemma 3.6. *Let l a prime different from the characteristic. There is a finite family of étale Hilbert covers $(g_i : X_i \to T)_i$ of T such that the map*

$$\mathrm{Mor}_K(T, A) \otimes_{\mathbb{Z}} \mathbb{Z}/l \to A\big(K(t)\big) \otimes_{\mathbb{Z}} \mathbb{Z}/l$$

induced by α_t is injective for all $t \in$ $\mathrm{Inert}(X_\bullet | T)$.

Proof. Let $M := \mathrm{Mor}_K(T, A)$ and $f \in M$. We denote the multiplication by l map $A \to A$ by l_A. Note that l_A is finite and étale by [16, 8.2]. Form Cartesian squares

$$\begin{tikzcd}
F^{(f)} \arrow[r, "j"] \arrow[d, "h"] & X^{(f)} \arrow[r, "f'"] \arrow[d, "g"] & A \arrow[d, "l_A"] \\
\mathrm{Spec}\big(K(t)\big) \arrow[r, "w"] & T \arrow[r, "f"] & A,
\end{tikzcd}$$

that is $X^{(f)} = T \times_{A, l_A} A$ and $F^{(f)} = \mathrm{Spec}(K(t)) \times_T X^{(f)}$. The vertical morphisms g and h are finite and étale as l_A is.

$X^{(f)}$ must be regular by [5, IV.6.5.2] because g is a finite, étale morphism and T is regular. All local rings of $X^{(f)}$ are regular local rings and hence integral. Therefore, by [5, I.6.1.10], the connected components of $X^{(f)}$ are open and finite in number. (Note that $X^{(f)}$ is of finite type

[2] We thank C. Greither and M. Jarden for independently providing us with a proof of Lemma 3.4.

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over a field and thus Noetherian.) Hence they are also closed. It follows that $X^{(f)}$ splits up into the coproduct

$$X^{(f)} = \coprod_{i=1}^{s^{(f)}} X_i^{(f)}$$

over its connected components. Let $\nu_i : X_i^{(f)} \to X^{(f)}$ the inclusion. Then ν_i is an open immersion (and hence étale) and a closed immersion (and hence finite) at the same time. If we denote by $g_i := g \circ \nu_i : X_i^{(f)} \to T$ the restriction of the map $g : X^{(f)} \to T$ to $X_i^{(f)}$, then g_i is finite and étale as a composite of two maps which are each finite and étale. Thus the maps g_i are Hilbert covers of T.

Now let $t \in H^{(f)} := \mathrm{Inert}(X_\bullet^{(f)}|T)$. Denote by $x_i \in X_i^{(f)}$ the unique point above t. Then

$$F^{(f)} = \coprod_{i=1}^{s^{(f)}} \mathrm{Spec}\big(K(x_i)\big)$$

and the morphism $F^{(f)} \to X^{(f)}$ is the coproduct of the canonical morphisms $\mathrm{Spec}(K(x_i)) \to X_i$. Moreover we have $[K(x_i) : K(t)] = \deg(g_i)$. Assume now $\alpha_t(f) \in lA(K(t))$. This means that there is a morphism $a : \mathrm{Spec}(K(t)) \to A$ such that $l_A \circ a = f \circ w$. Using the Cartesian diagram above gives a morphism $s : \mathrm{Spec}(K(t)) \to F^{(f)}$ such that $h \circ s = \mathrm{Id}_{\mathrm{Spec}(K(t))}$ and $(f' \circ j) \circ s = a$. In other words: h has a section. Hence there is an index i where $K(x_i) = K(t)$. Then g_i must be a finite flat morphism of degree 1. By Lemma 3.5 g_i is an isomorphism.

It follows that $g : X^{(f)} \to T$ has a section $g' : T \to X^{(f)}$ and this implies $f \in l \cdot \mathrm{Mor}_K(T, A)$. Indeed, since $f \circ g = l_A \circ f'$ we have $f = f \circ g \circ g' = l_A \circ f' \circ g'$. We have shown:

$$\forall t \in H^{(f)}:\ f \in lM \quad \Longleftrightarrow \quad \alpha_t(f) \in lA\big(K(t)\big).$$

M/l is finite by Lemma 3.4. Let $R \subset M$ a system of representatives for M/l. R is finite and hence $\Sigma := \{X_i^{(f)} \mid f \in R, i \in \{1, \dots, s^{(f)}\}\}$ is a finite set. Let $H := \bigcap_{Y \in \Sigma} \mathrm{Inert}(Y|T)$. Then

$$\forall t \in H:\ \forall f \in R:\ f \in lM \quad \Longleftrightarrow \quad \alpha_t(f) \in lA\big(K(t)\big).$$

From this it is immediate that α_t induces an injection $M \otimes \mathbb{Z}/l \to A(K(t)) \otimes \mathbb{Z}/l$ for all $t \in H$. $\square$

Proof of Proposition 3.1. Let l a prime different from $\mathrm{char}(K)$ and $M := \mathrm{Mor}_K(T, A)$. Let $X_1, \dots, X_s$ as in the assertion of Lemma 3.6 and $F|K$ as in Lemma 3.3. Put $X_0 := T_F$. Let $t \in \mathrm{Inert}(X_\bullet|T)$. We show that α_t is injective. Denote by N the kernel and by I the image of α_t. There is an obvious exact sequence

$$M_l \to I_l \xrightarrow{\delta} N/l \to M/l \to I/l.$$

The map δ is explicitly given as follows: For each $i \in I_l$ choose $m \in M$ with $\alpha_t(m) = i$. Then $\alpha_t(lm) = 0$, so $lm \in N$. Map i to the residue class $lm + lN$. In the above exact sequence the left-hand map is surjective and the right-hand map is injective by Lemmas 3.3 and 3.6, respectively.

Hence $N/l = 0$. Furthermore N is finitely generated and $\mathbb{Z}$-free by Lemma 3.2. This implies $N = 0$. □

Remark 3.7. We briefly compare the specialization Theorem 3.1 with other specialization theorems in the literature.

(1) As mentioned in the introduction to this section, there is a classical specialization theorem due to Néron [20, Chapitre IV, Theorem 6, p. 133]. We restate Néron's result in our language. Let $B|R(T)$ an Abelian variety. Then B extends to an Abelian scheme $\pi : \mathcal{B} \to U$ over a non-empty open subscheme U of T and for $t \in U$ we denote by B_t the fiber $\pi^{-1}(t)$ of t. B_t is an Abelian variety over $K(t)$. Néron showed that $B_t(K(t))$ contains a copy of $B(R(T))$ for infinitely many values of t *provided B is the Jacobian variety of a curve over $R(T)$*. This is not enough for our application as we do not want to restrict our attention to Jacobians. In [11] Lang gives the statement for an arbitrary Abelian variety $B|R(T)$, but a proof is not included in [11]. Serre in [24, 11.1] proves the statement for an arbitrary Abelian variety $B|R(T)$ but under the additional hypothesis that $T = \mathbb{P}_1$ and K is a number field. This is again not enough for our application.

(2) Let $A|K$ a Jacobian variety. One may apply Néron's theorem to the case of the constant family, that is with $B = A \otimes_K R(T)$, $U = T$ and $\mathcal{B} = T \times_K A$. Then $B(R(T)) = A(R(T)) = \mathrm{Mor}_K(T, A)$ because every rational map from the smooth variety T to an Abelian variety is defined on the whole of T by [16, 3.1]. By Néron's theorem, as we assumed that A is a Jacobian, $A(K(t))$ contains a copy of $\mathrm{Mor}_K(T, A)$ for infinitely many values of t. This also follows by Proposition 3.1 together with Corollary 2.4, but in Proposition 3.1 there is no need to assume that A is a Jacobian variety.

(3) An interesting specialization theorem due to Silverman (see [1,25]) implies a statement similar to Proposition 3.1: Let $A|K$ an arbitrary Abelian variety and $\Lambda \subset \mathrm{Mor}_K(T, A)$ a subgroup for which $\Lambda \cap A(K)$ is torsion. While α_t and $\overline{\alpha_t}$ can be non-injective for infinitely many $t \in T$ as mentioned above, $\ker(\alpha_t) \cap \Lambda$ must be zero outside a set of closed points of bounded height, provided K is a global field and the Néron–Severi group of T is cyclic. Nevertheless we prefer to use Proposition 3.1 above, because we neither want to impose stricter hypothesis on K nor an additional hypothesis on T and, most importantly, because we need the injectivity of $\overline{\alpha_t}$ for sufficiently many t, which is equivalent to the injectivity of α_t on the whole of $\mathrm{Mor}_K(T, A)$. The weaker estimate of the good locus does not matter in our application.

4. Sufficient condition for infinite rank

In this section we exploit the above results to establish an infinite rank result that will imply the theorems mentioned in the introduction. For the whole section let K a Hilbertian field and $A|K$ an Abelian variety. Assume that $A(K)$ is of finite rank. Consider a diagram

$$\begin{array}{c} T' \subset T \\ \downarrow{\scriptstyle p} \\ U \subset \mathbb{A}_n, \end{array}$$

where the vertical map p is an étale Hilbert cover and the symbols $\subset$ stand for open immersions. (Recall Remark 2.3.) Furthermore let $F|K$ a finite separable extension and assume that $M_F(T_F, A_F) \neq 0$. Fix once and for all a K-embedding $F \to K_s$.

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Remark 4.1. One may assume $F = K$ for a first reading. We need the extra F when we apply the results in the important special case where T is a smooth projective curve of positive genus and $A = J_T$ is its Jacobian variety. Then $T(K)$ can be empty and $M_K(T, J_T)$ can be zero. As soon as T has a rational point over F, however, $M_F(T_F, J_{T,F})$ contains the canonical F-embedding $T_F \to J_{T,F}$ and is thus non-zero.

Theorem 4.2. *There is a sequence $(t_i)_{i\in\mathbb{N}}$ of points in $T'(K_s)$ with the following properties*:

(1) $\operatorname{rank}(A(FK(t_i))) \geqslant \operatorname{rank}(A(F)) + \operatorname{rank}(M_F(T_F, A_F))$ *for all i. Here $FK(t_i)$ is the corresponding composite field in K_s.*
(2) *$p(t_i) \in U(K)$ is K-rational and $[K(t_i) : K] = \deg(p)$ for all i.*
(3) *$(F, K(t_1), K(t_2), \ldots)$ is a linearly disjoint sequence of fields.*

Let $I \subset \mathbb{N}$ and consider the composite field $\Omega_I = F \cdot \prod_{i\in I} K(t_i)$. If I is finite, then $\operatorname{rank}(A(\Omega_I)) \geqslant \operatorname{rank}(A(F)) + |I| \operatorname{rank}(M_F(T_F, A_F))$. *Finally, if I is infinite, then* $\operatorname{rank}(A(\Omega_I)) = \infty$. (*Recall that we assumed $M_F(T_F, A_F) \neq 0$ at the beginning of this section.*)

Proof. By Proposition 3.1 there is a finite family $(p_i : Y_i \to T_F)$ of étale Hilbert covers such that

$$\overline{\alpha_x} : M_F(T_F, A_F) \to \frac{A(F(x))}{A(F)}, \qquad f \mapsto f(x)$$

is injective for every closed point $x \in \operatorname{Inert}(Y_\bullet|T_F)$. (Here $F(x)$ denotes the residue field of the closed point $x \in T_F$ which is, of course, an F-algebra.) Now consider the composite Hilbert covers $Y_i \to T_F \to T$. By Proposition 2.5 there is a sequence $(t_i)_{i\in\mathbb{N}}$ of geometric points in $T'(K_s)$, which satisfies (2) and (3) and such that each t_i is localized in a point $x_i \in \operatorname{Inert}(Y_\bullet|T)$. By Remark 2.1 there is a unique closed point $\hat{x}_i \in T'_F \subset T_F$ above x_i and $\hat{x}_i \in \operatorname{Inert}(Y_\bullet|T_F)$.

If we let f one of the non-constant morphisms $T_F \to A_F$, we obtain the following diagram:

$$\begin{tikzcd}
 & & \coprod Y_i \arrow[d] & \\
\operatorname{Spec}\big(F(\hat{x}_i)\big) \arrow[r] \arrow[d] & T'_F \arrow[r, phantom, "\subset"] \arrow[d] & T_F \arrow[r, "f"] \arrow[d] & A_F \arrow[d] \\
\operatorname{Spec}\big(K(x_i)\big) \arrow[r] & T' \arrow[r, phantom, "\subset"] \arrow[d] & T & A \\
 & U \subset \mathbb{A}_n & &
\end{tikzcd}$$

in which the two complete squares are Cartesian. Thus $F(\hat{x}_i) = F \otimes_K K(x_i) = FK(t_i)$. $\overline{\alpha_{\hat{x}_i}}$ must be injective, as $\hat{x}_i \in \operatorname{Inert}(Y_\bullet|T_F)$. Hence $\operatorname{rank}(A(FK(t_i))) \geqslant \operatorname{rank}(A(F)) + \operatorname{rank}(M_F(T_F, A_F))$. This concludes the existence proof for a sequence which satisfies (1)–(3).

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Now $(FK(t_1), FK(t_2), \ldots)$ is a linearly disjoint sequence of extensions of F for any sequence $(t_i)_{i\in\mathbb{N}}$, which satisfies (1)–(3). From this it is immediate that

$$\bigoplus_{i\in I} \frac{A(FK(t_i))}{A(F)} \to \frac{A(\Omega_I)}{A(F)}$$

is injective. This implies the statements about $\operatorname{rank}(A(\Omega_I))$. □

Remark 4.3. Let $(f_1, \ldots, f_R) \subset \operatorname{Mor}_F(T_F, A_F)$ a family of morphisms whose image in $M_F(T_F, A_F)$ is $\mathbb{Z}$-linearly independent. Then $(f_j(x_i))_{1\leqslant j\leqslant R, i\in I}$ is $\mathbb{Z}$-linearly independent in $A(\Omega_I)$ by the proof of the theorem. Note that $f_j(x_i) \in A(FK(x_i))$ but not necessarily $\in A(K(x_i))$, as f_j need not be defined over K.

Remark 4.4. If p is a Galois cover with group Γ and $F|K$ is Galois, then all $K(t_i)|K$ are Galois with group Γ and also $\Omega_I|K$ is Galois with $G(\Omega_I|K) = G(F|K) \times \prod_{i\in I}\Gamma$. Thus $\operatorname{rank}(A(K^{ab})) = \infty$ provided p is an Abelian Galois cover and $F|K$ is an Abelian extension.

5. Proof of Theorems 1.1, 1.3 and 1.4

We can now prove the theorems mentioned in the introduction.

Proof of Theorem 1.1. Let T a smooth, projective curve of positive genus over a Hilbertian field K and $p: T \to \mathbb{P}_1$ a Galois cover with group Γ. Let $a \in \mathbb{P}_1(K)$. There is a point $x \in T(K_s)$ with $p(x) = a$, because p is surjective. $F := K(x)$ is then a Galois extension of $K = K(a)$ (with group a subquotient of Γ). If we choose $a \in \operatorname{Inert}(T|\mathbb{P}_1)$ at the beginning, then $G(F|K) = \Gamma$.

Now $T(F)$ is non-empty, and thus we have a canonical F-embedding $\lambda: T_F \to J_{T,F}$, which sends $y \in T(K_s)$ to the divisor class $[y] - [x] \in J_T(K_s)$. Let B a non-zero Abelian variety and $\pi: J_T \to B$ a surjective homomorphism. Then $\pi_F \circ \lambda : T_F \to B_F$ is non-constant and thus $M_F(T_F, B_F) \neq 0$. Let $U \subset \mathbb{P}_1 \setminus \infty$ a non-empty open set such that $T' := p^{-1}U \to U$ is étale. If $B(K)$ is already of infinite rank, then there is nothing to prove. Thus we may assume $\operatorname{rank}(B(K)) < \infty$. Then $\operatorname{rank}(B(\Omega)) = \infty$ for a certain infinite Galois extension $\Omega|K$ with group $G(\Omega|K) = \prod_{i\in\mathbb{N}}\Gamma$ by Theorem 4.2 and Remark 4.4. □

Proof of Theorems 1.3 and 1.4. Let K a Hilbertian field and $A|K$ a non-zero Abelian variety. We will apply Theorem 4.2 with $T := A$ and $F := K$ in order to prove Theorems 1.3 and 1.4 simultaneously. Again we may and do assume $\operatorname{rank}(A(K)) < \infty$. Obviously $M_K(T, A) = \operatorname{End}_K(A)$ is of rank $\geqslant 1$, as it contains the identity morphism. The function field $R(A)$ contains a purely transcendental subfield L, over which it is a finite extension. If A admits a projective embedding of degree d, then we may assume $[R(A) : L] = d$ in addition. By Remark 4.1 there are non-empty open sets $T' \subset A$ and $U \subset \mathbb{A}_n$ and an étale Hilbert cover $p: T' \to U$ of degree $[R(A) : L]$. By Theorem 4.2 we can conclude that there is a linearly disjoint sequence $(K_i)_{i\in\mathbb{N}}$ of separable extensions of K, all of degree $[R(A) : L]$, such that $\operatorname{rank}(A(K_i)) \geqslant \operatorname{rank}(A(K)) + 1$. Furthermore A acquires infinite rank over the composite field of all K_i and, of course, also over the composite field of any infinite subfamily of $(K_i)_{i\in\mathbb{N}}$. Theorem 1.3 readily follows from that.

To prove Theorem 1.4 we have to show that

$$X = \left\{\sigma \in G_K^e \mid \operatorname{rank}\big(A\big(K_s(\sigma)\big)\big) = \infty\right\}$$

is of measure 1. Let $H_i = G(K_s|K_i)$. Then obviously

$$X \supset \left\{\sigma \in G_K^e \mid K_s(\sigma) \text{ contains infinitely many } K(x_i)\right\} = \bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} H_i^e.$$

It follows from the linear disjointness of $(K_i)_{i\in\mathbb{N}}$ that $(H_i^e)_{i\in\mathbb{N}}$ is an independent family of open subgroups of G_K^e. By the lemma of Borel–Cantelli [3, 18.3.5] it remains to note that the series $\sum_{i=1}^{\infty}[G_K^e : H_i^e]^{-1} = \sum_{i=1}^{\infty}[R(A) : L]^{-e}$ diverges, in order to obtain that the right-hand term has measure 1, as desired. □

Uncited references

[15]

Acknowledgments

This article is essentially a condensed version of a part of the author's PhD thesis. The author wishes to heartily thank his supervisor Prof. C. Greither for numerous very helpful discussions on the subject. Furthermore he is grateful to M. Jarden for the appendix and for several contributions to the main text. Finally he is indebted to B. Conrad, H. Lange and J. Silverman for inspiring comments on certain special questions that arose while this work was done.

Appendix A. The rank of Abelian varieties over large Galois extensions of Hilbertian fields

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We denote the absolute Galois group of a field K by $\mathrm{Gal}(K)$. For each $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ let $K_s(\boldsymbol{\sigma})$ be the fixed field of $\sigma_1, \ldots, \sigma_e$ in K_s and let $K_s[\boldsymbol{\sigma}]$ be the maximal Galois extension of K in $K_s(\boldsymbol{\sigma})$. Consider an Abelian variety A over K. Theorem B of [4] says that if K is infinite and finitely generated over its prime field (hence Hilbertian), then $\mathrm{rank}(A(K_s[\boldsymbol{\sigma}])) = \infty$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$. Theorem 1.4 of the main text asserts that if K is Hilbertian, then $\mathrm{rank}(K_s(\boldsymbol{\sigma})) = \infty$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$. The following theorem generalizes both results.

Theorem A.1. *Let K be a Hilbertian field, A an Abelian variety over K, and e a positive integer. Then* $\mathrm{rank}(A(K_s[\boldsymbol{\sigma}])) = \infty$ *for almost all* $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$.

Proof. Let $r = \dim(A)$ and let F be the function field of A over K. The stability of fields [3, Theorem 18.9.3] gives a stabilizing basis $t_1, \ldots, t_r$ for F/K. Thus, $t_1, \ldots, t_r$ are algebraically independent over K, $F/K(\boldsymbol{t})$ is a finite separable extension, and the Galois closure $\hat{F}$ of $F/K(\boldsymbol{t})$ is a regular extension of K. The latter condition implies that $\hat{F}/K$ has a projective geometrically integral model X. Choose rational maps $X \xrightarrow{\alpha} A \xrightarrow{\beta} \mathbb{A}_K^r$ corresponding to the field embeddings $K(\boldsymbol{t}) \to F \to \hat{F}$. Choose Zariski open subsets X_0 of X, A_0 of A, and U of $\mathbb{A}_K^r$ such that

[3] The author is indebted to Wulf–Dieter Geyer for help in this appendix.

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with $\alpha_0 = \alpha|_{X_0}$ and $\beta_0 = \beta|_{A_0}$, $X_0 \xrightarrow{\alpha_0} A_0 \xrightarrow{\beta_0} U$ is a sequence of surjective morphisms and $K(\boldsymbol{x})/K(\beta(\alpha(\boldsymbol{x})))$ is Galois for each closed point $\boldsymbol{x}$ of X_0. Using that K is Hilbertian, Propositions 2.5 and 3.1 of the main text give a sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots$ of closed points of X_0 such that $\boldsymbol{a}_i = \alpha(\boldsymbol{x}_i)$ and $\boldsymbol{u}_i = \beta(\boldsymbol{a}_i)$ satisfy the following conditions for each i:

(1a) $K(\boldsymbol{u}_i) = K$.
(1b) $K(\boldsymbol{x}_i)/K$ is Galois and $[K(\boldsymbol{x}_i) : K] = [\hat{F} : K(\boldsymbol{t})]$.
(1c) $K(\boldsymbol{x}_1), K(\boldsymbol{x}_2), K(\boldsymbol{x}_3), \ldots$ are linearly disjoint over K.
(1d) The map $\bar{\alpha}_{\boldsymbol{a}_i} : \mathrm{End}_K(A) \to A(K(\boldsymbol{a}_i))/A(K)$ is injective.

Here we have used the natural isomorphism $\mathrm{Mor}_K(A, A)/A(K) \cong \mathrm{End}_K(A)$. In particular, since $n \cdot \mathrm{id}_A \neq 0$, we have for each i that $n\boldsymbol{a}_i + A(K) = \alpha_{\boldsymbol{a}_i}(n \cdot \mathrm{id}_A) \neq 0$, so $\boldsymbol{a}_i + A(K)$ has infinite order, hence $\mathrm{rank}(A(K(\boldsymbol{a}_i))/A(K)) \geqslant 1$. For each finite subset I_0 of $\mathbb{N}$, induction on $|I_0|$ proves that the map $(\boldsymbol{b}_i)_{i \in I_0} \mapsto \sum_{i \in I_0} \boldsymbol{b}_i$ defines an injection

$$\bigoplus_{i \in I_0} A\bigl(K(\boldsymbol{a}_i)\bigr)/A(K) \to A\biggl(\prod_{i \in I_0} K(\boldsymbol{a}_i)\biggr)/A(K).$$

Indeed, if $\sum_{i \in I_0} \boldsymbol{b}_i + A(K) = 0$ and $I_0 \neq \emptyset$ we choose $i_0 \in I_0$ and observe that

$$\boldsymbol{b}_{i_0} \in A\bigl(K(\boldsymbol{a}_{i_0})\bigr) \cap A\biggl(\prod_{i \neq i_0} K(\boldsymbol{a}_i)\biggr) \subseteq A\bigl(K(\boldsymbol{x}_{i_0})\bigr) \cap A\biggl(\prod_{i \neq i_0} K(\boldsymbol{x}_i)\biggr) = A(K)$$

(the latter equality follows from (1c)). Hence, $\sum_{i \in I_0 \setminus \{i_0\}} \boldsymbol{b}_i + A(K) = 0$ and we may use induction to conclude that $\boldsymbol{b}_i + A(K) = 0$ for all $i \in I_0$. It follows that

$$\mathrm{rank}\left(A\biggl(\prod_{i \in I_0} K(\boldsymbol{a}_i)\biggr)\right) \geqslant |I_0| - \mathrm{rank}\bigl(A(K)\bigr).$$

Consequently, $\mathrm{rank}(A(\prod_{i \in I} K(\boldsymbol{a}_i))) = \infty$ for each infinite subset I of $\mathbb{N}$.

By Borel–Cantelli [3, Lemma 18.5.3] and by (1a) and (1c), for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ there exists an infinite subset I of $\mathbb{N}$ such that $K(\boldsymbol{x}_i) \subseteq K_s(\boldsymbol{\sigma})$ for each $i \in I$. Since each $K(\boldsymbol{x}_i)/K$ is Galois, $\prod_{i \in I} K(\boldsymbol{a}_i) \subseteq \prod_{i \in I} K(\boldsymbol{x}_i) \subseteq K_s[\boldsymbol{\sigma}]$. Consequently, $\mathrm{rank}(K_s[\boldsymbol{\sigma}]) = \infty$. □

Remark A.2 (Comparison with [4, Theorem B]). There are many Hilbertian fields which are not finitely generated over their prime fields. For example, each finite proper separable extension of a Galois extension of a Hilbertian field is Hilbertian [3, Theorem 13.9.1]. Also, each Galois extension K of a Hilbertian field K_0 such that $\mathrm{Gal}(K/K_0)$ is finitely generated is Hilbertian [3, Proposition 16.11.1]. However, if $\mathrm{rank}(A(K)) = \infty$, then $\mathrm{rank}(A(K_s[\boldsymbol{\sigma}])) = \infty$ for each $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, so Theorem A.1 is trivial in this case. Thus, Theorem A.1 gives a really new result compared to [4, Theorem B] only if the pair (K, A) consisting of a field K and an Abelian variety over K satisfies the following conditions:

(2a) K is Hilbertian but not finitely generated over its prime field.
(2b) $\mathrm{rank}(A(K)) < \infty$.

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We give three examples for pairs (K, A) satisfying condition (2).

(a) Let K be an function field of several variables over an infinite field K_0 and let A be an Abelian variety over K. Suppose $\tilde{A} = A \times_K K\tilde{K}_0$ has no non-trivial Abelian subvariety A_0 which is isomorphic to an Abelian variety defined over $\tilde{K}_0$. Then the pair (K, A) satisfies condition (2). For example, this is the case when A is an elliptic curve over K with a transcendental j-invariant.

By [3, Proposition 13.2.1], K is Hilbertian, so condition (2a) is satisfied. To settle condition (2b), we prove the stronger statement that $A(K)$ is finitely generated.

Replacing K_0 by $\tilde{K}_0$ and A by $\tilde{A}$, we may assume that K_0 is algebraically closed. By a theorem of Chow and the relative Mordell–Weil theorem [10, pp. 138–139], there exists an Abelian variety B over K_0 and a homomorphism $\tau : B \times_{K_0} K \to A$ with a finite kernel such that $A(K)/\tau(B(K_0))$ is finitely generated (see also [12, p. 213, Theorem 8]).[4] The finite kernel is necessarily defined over K_0, so we may replace B by $B/\mathrm{Ker}(\tau)$ to assume that τ is injective. If $B \neq 0$, then $\tau(B \times_{K_0} K)$ is a non-zero Abelian subvariety of A, in contradiction to our assumption on the Abelian subvarieties of A. Thus, $B = 0$ and $A(K)$ is finitely generated.

(b) Let K be a finitely generated transcendental extension of $K_0 = \tilde{\mathbb{F}}_p$ for some prime number p and let A be an Abelian variety over K. Let (B, τ) be as in (a). Then $B(K_0)$ is a torsion group. Hence, $\mathrm{rank}(A(K)) = \mathrm{rank}(A(K)/\tau(B(K_0))) < \infty$. Thus, (K, A) satisfies condition (2).

(c) In [14] Mazur gives examples of a number field K_0, a $\mathbb{Z}_p$ extension K of K_0, and an elliptic curve A over K_0 such that $A(K)$ is finitely generated. By [3, Proposition 16.11.1], K is Hilbertian. Thus, condition (2) holds for (A, K).

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