REGULAR LIFTING OF COVERS
OVER AMPLE FIELDS

by

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Introduction

Colliot-Thélène [CT] uses the technique of Kollár, Miyaoka, and Mori to prove the following result.

**Theorem A:** Let $K$ be an ample field of characteristic 0, $x$ a transcendental element over $K$, and $G$ a finite group. Then there is a Galois extension $F$ of $K(x)$ with Galois group $G$, regular over $K$. Moreover, $F$ has a $K$-rational place $\varphi$.

In fact, Colliot-Thélène proves a stronger version:

**Theorem B:** Given a Galois extension $L/K$ with Galois group $\Gamma$ which is a subgroup of $G$, one can choose $F$ and $\varphi$ so that the residue field extension of $F/K(x)$ under $\varphi$ is $L/K$.

Case $\Gamma = G$ of Theorem B means that $K$ has the arithmetic lifting property of Beckmann and Black [BB].

As the results of Kollár, Miyaoka, and Mori are valid only in characteristic 0, Colliot-Thélène’s proof works only in this case. Nonetheless, Theorem A holds in arbitrary characteristic ([Ha, Corollary 2.4] for complete fields, [Po1, Main Theorem A]; see also [Li] and [HV]). Moret-Bailly [MB], using methods of formal patching, extends Theorem B to arbitrary characteristic.

Here we use algebraic patching to prove Theorem B for arbitrary characteristic. In fact, the main ingredient of the proof is almost contained in [HJ1]. Therefore this note can be considered a sequel to [HJ1]; a large portion of it recalls the situation and facts considered there.

We also notice that if $K$ is PAC and $F$ is an arbitrary Galois extension of $K(x)$ with Galois group $G$, regular over $K$, then, for every Galois extension $L/K$ with Galois group which is a subgroup of $G$, we can choose $\varphi$ so that the residue field extension of $F/K(x)$ under $\varphi$ is $L/K$. (After the first draft of this note has been written, P. Dèbes informed us that he also made this observation in [De, Remark 3.3].) This answers a question of Harbater. Notice that this stronger property does not hold for an arbitrary ample field $K$ [CT, Appendix].
The idea (displayed in our Lemma 2.1) to use the embedding problem $G \ltimes G \to G$ in order to obtain the arithmetic lifting property has been used in [Po2]; we are grateful to F. Pop for making his notes available to us.

1. Embedding problems and decomposition groups

Let $K/K_0$ be a finite Galois extension with Galois group $\Gamma$. Let $x$ be a transcendental element over $K$. Put $E_0 = K_0(x)$. Suppose that $\Gamma$ acts (from the right) on a finite group $G$; let $\Gamma \ltimes G$ be the corresponding semidirect product and $\pi: \Gamma \ltimes G \to \Gamma$ the canonical projection. We call

$$\pi: \Gamma \ltimes G \to \Gamma = \mathcal{G}(K/K_0)$$

a finite constant split embedding problem. A solution of (1) is a Galois extension $F$ of $E_0$ such that $K \subseteq F$, $\mathcal{G}(F/E_0) = \Gamma \ltimes G$, and $\pi$ is the restriction map $\text{res}_K: \mathcal{G}(F/E_0) \to \mathcal{G}(K/K_0)$.

In [HJ1, Theorem 6.4] we reprove the following result of F. Pop [Po1]:

**Proposition 1.1:** Let $K_0$ be an ample field. Then each finite constant split embedding problem (1) has a solution $F$ such that $F$ has a $K$-rational place. (In particular, $F/K$ is regular.)

In this section we show that the proof of Proposition 1.1 in [HJ1] yields a stronger assertion.

**Lemma 1.2:** Let $F$ be a solution of (1). Put $F_0 = F_\Gamma$. Let $\varphi: F \to \overline{K}_0$ be a $K$-place extending a $K_0$-place of $E_0$. Assume that $\varphi$ is unramified in $F/E_0$ and let $D_\varphi$ be its decomposition group in $F/E_0$. Then $\varphi(F) \supseteq K$ and the following assertions are equivalent:

(a) $\varphi(F) = K$ and $\Gamma = D_\varphi$;
(b) $\Gamma \supseteq D_\varphi$;
(c) $\varphi(F_0) = K_0$;
(d) $\varphi(F) = K$ and $\varphi(f^\gamma) = \varphi(f)^\gamma$ for each $\gamma \in \Gamma$ and $f \in F$ with $\varphi(f) \neq \infty$. 

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Proof: As \( K \subseteq F \), we have \( K = \varphi(K) \subseteq \varphi(F) \). Since the inertia group of \( \varphi \) in \( F/E_0 \) is trivial, we have an isomorphism \( \theta: D_\varphi \rightarrow \mathcal{G}(\varphi(F)/K_0) \) given by

\[
(2) \quad \varphi(f^\gamma) = \varphi(f)^{\theta(\gamma)}, \quad \gamma \in D_\varphi, \; f \in F, \; \varphi(f) \neq \infty.
\]

Hence \( |D_\varphi| = [\varphi(F) : K_0] \geq [K : K_0] = |\Gamma| \). This gives (a) \( \iff \) (b).

Since \( \varphi \) is unramified over \( E_0 \), the decomposition field \( F^{D_\varphi} \) is the largest intermediate field of \( F/E_0 \) mapped by \( \varphi \) into \( K_0 \), and hence (b) \( \iff \) (c).

Clearly (d) \( \Rightarrow \) (c). If \( \varphi(F) = K \), apply (2) to \( f \in K \) to see that \( \theta(\gamma) = \gamma \) for all \( \gamma \in D_\varphi \). Hence (a) \( \Rightarrow \) (d).

Remark 1.3: Let \( K_0 \) be an ample field and let \( F \) be a solution of (1). Suppose that \( F \) has a \( K \)-rational place extending \( K_0 \)-places of \( E_0 \) and unramified over \( E_0 \) such that \( \Gamma \) is its decomposition group in \( F/E_0 \). Then \( F \) has infinitely many such places.

Indeed, put \( F_0 = F^\Gamma \). Recall that \( F_0 \) is regular over \( K_0 \). By Lemma 1.2,

(a) the assumption is that there is a \( K_0 \)-place \( \varphi: F_0 \rightarrow K_0 \) unramified over \( K_0(x) \), and

(b) we have to show that there are infinitely many such places.

But (a) \( \Rightarrow \) (b) is a property of an ample field.

**Proposition 1.4:** Let \( K_0 \) be an ample field. Then each finite constant split embedding problem (1) has a solution \( F \) with a \( K \)-rational place of \( F \) extending a \( K_0 \)-place of \( E_0 \) and unramified over \( E_0 \) such that \( \Gamma \) is its decomposition group in \( F/E_0 \).

**Proof:** Put \( E = K(x) = KK_0(x) \).

**Part A:** As in the proof of [HJ1, Theorem 6.4], we first assume that \( K_0 \) is complete with respect to a non-trivial discrete ultrametric absolute value, with infinite residue field and \( K/K_0 \) is unramified.

In this case [HJ1, Proposition 5.2] proves Proposition 1.1. Claim C of that proof shows that, for every \( b \in K_0 \) with \( |b| > 1 \), \( x \rightarrow b \) extends to a \( K \)-homomorphism \( \varphi_b: R \rightarrow K \), where \( R \) is the principal ideal ring \( K\{\frac{1}{x-c_i} \mid i \in I\} \). From there it extends to a \( K \)-place \( \varphi_b: Q \rightarrow K \cup \{\infty\} \) of the \( Q = \text{Quot}(R) \). Furthermore, [HJ1, Lemma 1.3(b)] gives an \( E \)-embedding \( \lambda: F \rightarrow Q \). The compositum \( \varphi = \varphi_b \circ \lambda \) is a \( K \)-rational place of
F. Excluding finitely many $b$’s we may assume that $\varphi$ is unramified over $E_0$. To verify that $\varphi$ satisfies condition (d) of Lemma 1.2, we first recall the relevant facts from [HJ1].

(a) [HJ1, Proposition 5.2, Construction B] The group $\Gamma = G(K/K_0)$ lifts isomorphically to $G(E/E_0)$. By the choice of the $c_i$ we have $(\frac{1}{x-c_i})^\gamma = \frac{1}{x-c_i}$, for each $\gamma \in \Gamma$. It follows that $\Gamma$ continuously acts on $R$ in the following way

$$
(a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} (\frac{1}{x-c_i})^n)^\gamma = a_0^\gamma + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in}^\gamma (\frac{1}{x-c_i})^n.
$$

This action induces an action of $\Gamma$ on $Q$.

(b) [HJ1, (7) on p. 334] The above mentioned action of $\Gamma$ on $Q$ defines an action of $\Gamma$ on the $Q$-algebra

$$
N = \text{Ind}_1^G Q = \left\{ \sum_{\theta \in G} a_{\theta} \theta \mid a_\theta \in Q \right\}
$$

in the following way:

$$
\left( \sum_{\theta \in G} a_\theta \theta \right)^\gamma = \sum_{\theta \in G} a_{\theta}^\gamma \theta^\gamma \quad a_\theta \in Q, \, \gamma \in \Gamma.
$$

Furthermore, the field $F$ is a subring of $N$ [HJ1, p. 332] and $\Gamma$ acts on it by restriction from $N$ [HJ1, Proof of Proposition 1.5, Part A].

(c) The embedding $\lambda: F \to Q$ is just the restriction to $F$ of the projection

$$
\sum_{\theta \in G} a_\theta \theta \mapsto a_1
$$

from $N = \text{Ind}_1^G Q \to Q$ [HV, Proposition 3.4].

(d) The place $\varphi_b: Q \to K \cup \{\infty\}$ is induced from the evaluation homomorphism $\varphi_b: R \to K$ given by [HJ1, Remark 3.5]

$$
\varphi_b \left( a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} (\frac{1}{x-c_i})^n \right) = a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} \left( \frac{1}{b-c_i} \right)^n.
$$

In order to prove condition (d) of Lemma 1.2 it suffices to show that both $\lambda$ and $\varphi_b$ are $\Gamma$-equivariant.
Let \( f = \sum_{\theta \in G} a_\theta \theta \in F \subseteq N \). Then, by (b) and (c),
\[
\lambda(f^\gamma) = \lambda\left( \sum_{\theta \in G} a_\theta^\gamma \theta^\gamma \right) = a_1^\gamma = \left( \lambda\left( \sum_{\theta \in G} a_\theta \theta \right) \right)^\gamma = \lambda(f)^\gamma.
\]

Furthermore, let \( r = a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} \left( \frac{1}{x - \epsilon_i^n} \right)^n \in R \). By (a) and (d),
\[
\varphi_b(r^\gamma) = \varphi_b\left( a_0^\gamma + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in}^\gamma \left( \frac{1}{x - \epsilon_i^n} \right)^n \right) = a_0^\gamma + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in}^\gamma \left( \frac{1}{b - \epsilon_i^n} \right)^n
\]
\[
= \left( a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} \left( \frac{1}{b - \epsilon_i} \right)^n \right)^\gamma = \varphi_b(r)^\gamma.
\]

Thus \( \varphi_b \) is \( \Gamma \)-equivariant.

**PART B:** \( K_0 \) is an arbitrary ample field. As in the proof of [HJ1, Theorem 6.4] let \( \hat{K}_0 \) be the field of Laurent series over \( K_0 \). Then \( \hat{K} = K\hat{K}_0 \) is an unramified extension of \( \hat{K}_0 \) with Galois group \( \Gamma \) and infinite residue field.

By Part A, \( \hat{K}_0(x) \) has a Galois extension \( \hat{F} \) which contains \( \hat{K}(x) \), such that \( \mathcal{G}(\hat{F}/\hat{K}_0(x)) = \Gamma \rtimes G \) and the restriction map \( \mathcal{G}(\hat{F}/\hat{K}_0(x)) \to \mathcal{G}(K/K_0) \) is the projection \( \pi: \Gamma \rtimes G \to \Gamma \). Furthermore, there is \( b \in \hat{K}_0 \) such that the place \( x \to b \) of \( \hat{K}_0(x) \) extends to an unramified \( \hat{K} \)-place \( \hat{\varphi}: \hat{F} \to \hat{K} \) and \( \hat{\varphi}(\hat{F}^\Gamma) = \hat{K}_0 \). Put \( m = |G| \).

Use Weak Approximation to find \( y \in \hat{F}^\Gamma \) mapped by the \( m \) distinct extensions of \( x \to b \) to \( \hat{F}^\Gamma \) into \( m \) distinct elements of the separable closure of \( \hat{K}_0 \); then \( \hat{F}^\Gamma = \hat{K}_0(x, y) \).

Thus there exist polynomials \( f \in \hat{K}_0[X, Z], g \in \hat{K}_0[X, Y], \) elements \( z \in \hat{F}, y \in \hat{F}^\Gamma, \) and elements \( b, c \in \hat{K}_0 \), such that the following conditions hold:

1. \( \hat{F} = \hat{K}_0(x, z), f(x, Z) = \text{irr}(z, \hat{K}_0(x)) \); we may therefore identify \( \mathcal{G}(f(x, Z), \hat{K}_0(x)) \) with \( \mathcal{G}(\hat{F}/\hat{K}_0(x)) \);
2. \( \hat{F}^\Gamma = \hat{K}_0(x, y), \) whence \( \hat{F} = \hat{K}(x, y), \) and \( g(x, Y) = \text{irr}(y, \hat{K}_0(x)) \); therefore \( g(X, Y) \) is absolutely irreducible;
3. \( \text{discrg}(b, Y) \neq 0 \) and \( g(b, c) = 0. \)

All of these objects depend on only finitely many parameters from \( \hat{K}_0 \). So, there are \( u_1, \ldots, u_n \in \hat{K}_0 \). So, let \( u_1, \ldots, u_n \) be elements of \( \hat{K}_0 \) such that the following conditions hold:
(4a) \( F = K_0(u, x, z) \) is a Galois extension of \( K_0(u, x) \), the coefficients of \( f(X, Z) \) lie in \( K_0[u, f(x, Z) = \text{irr}(z, K_0(u, x)), \text{and } G(f(x, Z), K_0(u, x)) = G(f(x, Z), \tilde{K}_0(x)) \);

(4b) the coefficients of \( g \) lie in \( K[u] \); hence \( g(x, Y) = \text{irr}(y, K_0(u, x)) \); furthermore, 
\[ K_0(u, x, y) = F^\Gamma; \]

(4c) \( b, c \in K_0[u] \) and \( \text{discr}(b, Y) \neq 0 \) and \( g(b, c) = 0 \).

Since \( \tilde{K}_0 \) has a \( K \)-rational place, namely, \( x \rightarrow 0 \), the field \( \tilde{K}_0 \) and therefore also \( K_0(u) \) are regular extensions of \( K_0 \). Thus, \( u \) generates an absolutely irreducible variety \( U = \text{Spec}(K_0[u]) \) over \( K_0 \). By Bertini-Noether [FJ, Proposition 8.8] the variety \( U \) has a nonempty Zariski open subset \( U' \) such that for each \( u' \in U' \) the \( K_0 \)-specialization \( u \rightarrow u' \) extends to a \( K_0 \)-homomorphism \( U' : K_0(u, x, z, y) \rightarrow K(u', x, z', y') \) such that the following conditions hold:

(5a) \( f'(x, z') = 0 \), the discriminant of \( f'(x, Z) \) is not zero, and \( F' = K_0(u', x, z') \) is the splitting field of \( f'(x, Z) \) over \( K_0(u', x) \); in particular \( F'/K_0(u', x) \) is Galois;

(5b) \( g'(X, Y) \) is absolutely irreducible and \( g'(x, y') = 0 \); so \( g'(x, Y) = \text{irr}(y', K_0(u', x)) \); furthermore, \( K_0(u', x, y') = (F')^\Gamma; \)

(5c) \( b', c' \in K_0[u'] \) and \( \text{discr}(b', Y) \neq 0 \) and \( g'(b', c') = 0 \).

As \( K_0 \) is existentially closed in \( \tilde{K}_0 \), and since \( u \in U(\tilde{K}_0) \), there is \( u' \in U(K_0) \). Now repeat the end of the proof of [HJ1, Lemma 6.2] (from “By (5a), the homomorphism. . .” to conclude that \( F' \) is a solution of (1).

Condition (5c) ensures that the place \( x \rightarrow b' \) of \( K_0(x) \) is unramified in \( (F')^\Gamma \), hence in \( F' \), and extends to a \( K_0 \)-rational place of \( (F')^\Gamma \). This ends the proof by Lemma 1.2.
2. Lifting property over ample fields

Let \( \Gamma \) be a subgroup of a finite group \( G \). Let \( \Gamma \) act on \( G \) by the conjugation in \( G \)

\[
g^\gamma = \gamma^{-1} g \gamma.
\]

and consider the semidirect product \( \Gamma \rtimes G \). To fix notation, \( \Gamma \rtimes G = \{ (\gamma, g) : \gamma \in \Gamma, g \in G \} \) and the multiplication on \( \Gamma \rtimes G \) is defined by

\[
(\gamma_1, g_1)(\gamma_2, g_2) = (\gamma_1 \gamma_2, g_1^{\gamma_2} g_2).
\]

Notice that \( \Gamma \rtimes G \cong \Gamma \times G \) by \((\gamma, g) \mapsto (\gamma, \gamma g)\). However, the above presentation gives a different splitting of the projection \( \Gamma \times G \to \Gamma \). In particular, we have an epimorphism \( \rho: \Gamma \rtimes G \to G \) given by \((\gamma, g) \mapsto \gamma g \). Let \( N \) denote its kernel.

**Lemma 2.1:** Let \( K_0 \) be a field, \( K \) a Galois extension of \( K_0 \) with Galois group \( \Gamma \), and \( x \) a transcendental element over \( K_0 \). Assume that (1) has a solution \( \hat{F} \) with a \( K \)-rational place \( \hat{\varphi} \) of \( F \) extending a \( K_0 \)-place of \( K_0(x) \) and unramified over \( K_0(x) \) such that \( \Gamma \) is its decomposition group in \( F/K_0(x) \). Let \( F = \hat{F}^N \) and let \( \varphi \) be the restriction of \( \hat{\varphi} \) to \( F \). Then

(6a) \( F \) is a Galois extension of \( K_0(x) \) and \( \mathcal{G}(F/K_0(x)) \cong G \);

(6b) \( F/K_0 \) is a regular extension;

(6c) \( \varphi \) represents a prime divisor \( p \) of \( F/K_0 \) with decomposition group \( \Gamma \) in \( F/K_0(x) \)

and residue field \( K \).

**Proof:** By assumption, \( \hat{F} \) is a Galois extension of \( K_0(x) \) containing \( K \), with Galois group \( \Gamma \rtimes G \) such that the restriction \( \mathcal{G}(\hat{F}/K_0(x)) \to \mathcal{G}(K/K_0) \) is the projection \( \Gamma \rtimes G \to \Gamma \), and \( \hat{F}/K \) is regular. Furthermore, \( \hat{\varphi}: \hat{F} \to K \) is a \( K \)-place unramified over \( K_0(x) \), with decomposition group \( \Delta = \{ (\gamma, 1) : \gamma \in \Gamma \} \cong \Gamma \) in \( \hat{F}/K_0(x) \) and residue field extension \( K/K_0 \). In particular, \( \hat{F} \) is regular over \( K \).

From the definition of \( F \) we get (6a) and \( \rho(\Delta) = \Gamma \leq G \) is the decomposition group of the restriction \( \varphi: F \to K \) of \( \hat{\varphi} \) to \( F \). As \( |\Delta| = [K : K_0] \), the residue field of \( \varphi \) is \( K \). As \( \Gamma \rtimes G = NG \), the fields \( F = \hat{F}^N \) and \( K(x) = \hat{F}^G \) are linearly disjoint over \( K_0(x) \). Therefore \( F \) is regular over \( K_0 \).
Lemma 2.1 together with Proposition 1.4 and Remark 1.3 yield the following result, originally proved by Colliot-Thélène [CT, Theorem 1] in characteristic 0:

**Theorem 2.2:** Let $K_0$ be an ample field, $G$ a finite group, $\Gamma$ a subgroup, $K$ a Galois extension of $K_0$ with Galois group $\Gamma$, and $x$ a transcendental element over $K_0$. Then there is $F$ that satisfies (6a), (6b) and

(6d) there are infinitely many prime divisors $p$ of $F/K_0$ with decomposition group $\Gamma$ in $F/K_0(x)$ and residue field $K$.

**Remark 2.3:** In case of $\Gamma = G$, Theorem 2.2 says that an ample field $K_0$ has the so-called arithmetic lifting property of Beckmann-Black [BB].

If $K_0$ is a PAC field, an even stronger property holds.

**Theorem 2.4:** Let $K_0$ be a PAC field, $G$ a finite group, $F$ a function field of one variable over $K_0$, and $E$ a subfield of $F$ such that $F/E$ is Galois with Galois group $G$. Let $\Gamma$ be a subgroup of $G$ and $K$ a Galois extension of $K_0$ with Galois group $\Gamma$. Then there are infinitely many prime divisors $p$ of $F/K_0$ with decomposition group $\Gamma$ in $F/E$ and residue field $K$.

**Proof:** By definition, $F$ is a regular extension of $K_0$. In particular, $F$ is linearly disjoint from $K$ over $K_0$. Hence,

$$\mathcal{G}(FK/E) = \mathcal{G}(FK/F) \times \mathcal{G}(FK/EK) \cong \Gamma \times G.$$ 

Consider the subgroup $\Delta = \{(\gamma, \gamma) \in \Gamma \times G \mid \gamma \in \Gamma\}$ of $\mathcal{G}(FK/E)$. It satisfies the following conditions:

(7a) $\Delta \cdot (\Gamma \times 1) = \Gamma \times \Gamma$ and $\Delta \cap (\Gamma \times 1) = 1$.

(7b) $\Delta \cdot (1 \times G) = \Gamma \times G$ and $\Delta \cap (G \times 1) = 1$.

Denote the fixed field of $\Delta$ in $FK$ by $D$ and the fixed field of the subgroup $\Gamma$ of $G = \mathcal{G}(F/E)$ by $F_0$. Condition (7) translates via Galois theory to the following one:

(8a) $D \cap F = F_0$ and $DF = FK$.

(8b) $D \cap EK = E$ and $DK = FK$.

As $F/K_0$ is regular, so is $FK/K$. Hence, by (8b), $D/K_0$ is a regular extension. Since $K_0$ is PAC, there exist infinitely many $K_0$-places $\varphi: D \to K_0$. Use (8b) to extend...
each such \( \varphi \) to a \( K \)-place \( \psi: FK \to K \). As \( [FK : D] = |\Delta| = |\Gamma| = [K : K_0] \), \( D \) is the decomposition field of \( \psi \) in \( FK/E \). By (8a), \( F_0 \) is the decomposition field of \( \psi|_F \) in \( F/E \).

**Corollary 2.5:** Let \( K_0 \) be a PAC field, \( E \) a function field of one variable over \( K_0 \), and \( G \) a finite group. For \( i = 1, \ldots, n \) let \( \Gamma_i \) be a subgroup of \( G \) and \( K_i \) a Galois extension of \( K_0 \) with Galois group \( \Gamma_i \). Then \( E \) has a Galois extension \( F \) such that

1. \( G(F/E) \cong G \).
2. \( F/K_0 \) is a regular extension.
3. For each \( i \) there exists a prime divisor \( p_i \) of \( F/K_0 \) with decomposition group over \( E \) equal to \( \Gamma_i \) and with residue field \( K_i \). Moreover, \( p_1, \ldots, p_n \) are distinct.

**Proof:** The existence of \( F \) with the properties (9a) and (9b) is well known [HJ2, Theorem 2]. Now apply Theorem 2.4 successively to \( \Gamma_i \) and \( K_i \) instead of to \( \Gamma \) and \( K \).

**References**


