TEL-AVIV UNIVERSITY

RAYMOND AND BEVERLY SACKLER
FACULTY OF EXACT SCIENCES
SCHOOL OF MATHEMATICAL SCIENCES

An Algorithm for the Computation of the Metric Average of Two Simple Polygons and Extensions

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By

Shay Kels

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Abstract

In this work we present an algorithm that applies segment Voronoi diagrams and planar arrangements to the computation of the metric average of two simple polygons. The idea to apply segment Voronoi diagrams is due to E. Lipovetsky. The implementation of the algorithm is described and a collection of computational examples is presented. Based on the computational framework of the algorithm, the connectedness of the metric average of two simple polygons is studied. Furthermore an artifact produced by the metric average of two simple polygons is identified and a modified averaging operation that avoids this artifact is suggested and implemented. Finally, we extend the algorithm to compute the metric average of two sets that are each a collection of simple polygons with simple polygonal holes.

Contents

Introduction

The interest in the computation of the metric average of compact sets in R^2 is motivated by the problem of the reconstruction of a 3D object from a set of its 2D parallel cross-sections. This problem possesses a lot of potential applications. Some of the application areas are microscopy, computer vision, flight simulation, tomography and more. For a review of these topics see [16].

First introduced by Z.Arstein in [1], the metric average[1] of two compact sets in R^n is a union of weighted averages between any point from any of the two sets and the subset of all its closest points from the other set.

Extending the results in [1], Dyn et al. applied the metric average in the approximation of a set-valued function from a finite number of its samples [5, 7]. These results are relevant to the problem of the reconstruction an object from a finite set of its parallel cross-sections, since an n-dimensional body can be regarded as a univariate set-valued function with compact sets of dimension n-1 as images. The set-valued function can be approximated from the given samples (the cross sections) by the approximation methods of [5, 7], which are based on repeated computations of the metric average of the samples.

The applicability of approximation methods, based on the metric average, requires efficient and robust algorithms for the computation of the metric average. A first step in this direction is done in [6], where an algorithm for the computation of the metric average of two intersecting *convex* polygons is introduced. This algorithm has linear time complexity in the number of vertices of the input polygons.

In this work we introduce an algorithm that applies segment Voronoi diagrams and planar arrangements to the computation of the metric average of two simple polygons that are not necessarily convex. The main idea is that each polygon can be divided by the

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[1] The term "metric average" was introduced in [4].

segment Voronoi diagram induced by the boundary of the other polygon. The sites of such Voronoi diagram are the vertices and the edges of the other polygon. The partition of one polygon by the segment Voronoi diagram induced by the boundary of the other polygon is a collection of conic polygons, and the metric average is computed separately for each of these conic polygons. This way the metric average can be computed only for the boundary points of each conic polygon and only relative to the corresponding Voronoi site. The idea to apply segment Voronoi diagrams is due to E. Lipovetsky. Although the algorithm can compute the metric average of any two simple polygons, from the applications point of view only the case of intersecting polygons is of interest.

Our algorithm requires handling geometric objects such as segment Voronoi diagrams, arrangements of conic arcs and conic polygons. The required algorithms and their C++ implementations can be found in the CGAL library [18]. We implemented the algorithm as a C++ program, using version 3.3.1 of the CGAL library. The thesis presents in details the implementation of the algorithm, with references to the relevant sections of the CGAL library.

In some cases the metric average of two simple intersecting polygons is a collection of disconnected conic polygons. We give an example of such two simple intersecting polygons, and discuss conditions for connectedness of the metric average of two simple polygons. Necessary and sufficient conditions for connectedness are derived and proved. We apply results about connectedness to evaluate the combinatorial complexity of the metric average of two simple polygons and to derive complexity bounds for our algorithm.

We applied the algorithm to a range of pairs of simple polygons. Based on the graphic results, we point out a common artifact produced by the metric average of two simple polygons. Then we show how the geometric situation leading to the artifact can be identified and suggest a modification of the metric average that avoids this artifact. Results produced by our implementations of the metric average and of the modified metric average on several examples are compared.

Our algorithm has a natural extension to the computation of the metric average and of the modified metric average of two sets that are each a collection of simple polygons with simple polygonal holes. We describe the extended algorithms and provide several examples.

The outline of this thesis is as follows: in Chapter 1 we introduce definitions and notation. In particular we survey the geometric concepts: metric average, segment Voronoi diagrams, arrangements, conic polygons and regularized Boolean set operations.

In Chapter 2 we discuss in details the algorithm for the computation of the metric average of two simple polygons. Chapter 3 deals with the connectedness of the metric average of two simple polygons. In Chapter 4 we derive the run-time complexity bounds for our algorithm and bounds on the combinatorial complexity of the computed metric average. In Chapter 5 we introduce and investigate the modified metric average of two simple polygons, and in Chapter 6 we extend the two algorithms to two sets each consisting of a collection of simple polygons with simple polygonal holes.

Chapter 1: Preliminaries

In this chapter we introduce the basic definitions and notation for our work. First the metric average of two compact sets in R^n is introduced. Then we introduce the geometric structures and operations required by the algorithm.

1.1 The metric average

First we introduce some notation. The collection of all nonempty compact subsets of R^n is denoted by K_n.

The *Euclidean norm* of $x \in R^n$ is denoted by $|x|$.

A *linear segment* is denoted by

$$[p,q] = \{\, \lambda p + (1-\lambda) q \; : \; 0 \le \lambda \le 1 \,\}.$$

A *polyline* through the points $\{p_1, p_2, \dots p_n\}$ is denoted by

$$[p_1, p_2, \dots p_n] = \bigcup_{i=1}^{n-1} [p_i, p_{i+1}].$$

For a simple closed curve $\Gamma : [a,b] \to R^2$, $\overline{\Gamma}$ denotes the region of the plane bounded by the image of Γ.

The *convex hull* of a set $A \subset R^n$ is denoted by $co(A)$.

The *set difference* of $A, B \subset R^n$ is

$$A \setminus B = \{ p : p \in A, p \notin B \}.$$

The *complement* of a set $A \subset R^n$ is

$$\sim A = \{ p \in R^n : p \notin A \}.$$

The *closure* of a set $A \subset R^n$ is

$$Cl(A) = \{ p \in R^n : \forall r > 0. \exists q \in A. |p - q| < r \}.$$

The *boundary* of a set $A \subset R^n$ is

$$\partial A = Cl(A) \cap Cl(\sim A).$$

The *interior* of a set $A \subset R^n$ is

$$A^0 = Cl(A) \setminus \partial A .$$

The *Euclidean distance* from a point p to a set $A \in K_n$ is

$$\text{dist}(p, A) = \min\{ |p - q| \ : \ q \in A \}.$$

The set of all *projections* of a point p on a set $A \in K_n$ is

$$\Pi_A(p) = \{ q \in A : |p - q| = \text{dist}(p, A) \}.$$

We use the notion of a projection of a point p on a set A, also for non-compact A, meaning $\Pi_{Cl(A)}(p)$.

The *projection* of a set $B \subset R^n$ on a set $A \in K_n$ is

$$\Pi_A(B) = \bigcup_{p \in B} \Pi_A(p).$$

The *Hausdorff distance* between two sets A and B in K_n is defined by

$$\text{haus}(A, B) = \max\left\{ \sup_{p \in A} \text{dist}(p, B), \ \sup_{q \in B} \text{dist}(q, A) \right\}.$$

The *linear Minkowksi combination* of two sets $A, B \in K_n$ is

$$\lambda A + \mu B = \{\lambda p + \mu q \ : \ p \in A, q \in B\},$$

for $\lambda, \mu \in R$.

<u>Definition 1.1</u>: Let $A, B \in K_n$ and $0 \le t \le 1$. The t-*weighted metric average* of A and B is

$$A \oplus_t B = M_t(A, B) \cup M_{1-t}(B, A), \qquad (1.1)$$

With

$$M_t(A, B) = \{ t\{x\} + (1-t)\Pi_B(x) \ : \ x \in A \}. \qquad (1.2)$$

The linear combinations in the last equality are in the Minkowski sense. In the sequel we term the set $M_t(A, B)$ as *one-sided metric average*.

The most important properties of the metric average are presented below [4]:

Let $A, B \in K_n$ and $0 \le t \le 1, 0 \le s \le 1$. Then

1. $A \oplus_0 B = B, A \oplus_1 B = A, A \oplus_t B = B \oplus_{1-t} A$.
2. $A \oplus_t A = A$.
3. $A \bigcap B \subseteq A \oplus_t B \subseteq tA + (1-t)B \subseteq co(A \bigcup B)$.
4. $\text{haus}(A \oplus_t B, A \oplus_s B) = |t - s| \,\text{haus}(A, B)$. (1.3)
5. $A \oplus_t B = (A \bigcap B) \cup M_t(A \setminus B, B) \cup M_{1-t}(B \setminus A, A)$. (1.4)

It follows from properties 1 and 4 that

$$\text{haus}(A \oplus_t B, A) = (1-t)\,\text{haus}(A, B),\ \text{haus}(A \oplus_t B, B) = t\,\text{haus}(A, B). \quad (1.5)$$

Let $B \in K_n$, $p \in \sim B$, then

$$\Pi_B(p) = \Pi_{\partial B}(p).$$

So using (1.4) we can rewrite the metric average of $A, B \in K_n$ with $t \in [0,1]$ as

$$A \oplus_t B = (A \bigcap B) \bigcup M_t(A \setminus B, \partial B) \bigcup M_{1-t}(B \setminus A, \partial A). \quad (1.6)$$

Remark: In the sequel, when it is clear from the context, a point $p \in R^n$ may be treated as the singleton set $\{p\} \subset R^n$ and vice versa.

1.2 Conic segments, polygons and arrangements

<u>Definition 1.2</u>: A *planar conic curve C* is a locus of points in $p = (x, y) \in R^2$ satisfying the quadratic equation

$$ax^2 + by^2 + cxy + dx + ey + f = 0, \quad (1.7)$$

where $a, b, c, d, e, f \in R$.

There are 3 basic types of nondegenerate planar conic curves that can be identified by the sign of the expression $4ab - c^2$:

If $4ab - c^2 > 0$, then the curve C is an ellipse.

If $4ab - c^2 < 0$, then the curve C is a hyperbola.

If $4ab - c^2$=0, then the curve C is a parabola.

A straight line may be considered as a special case of a parabola with $a = b = c = 0$. A curve C is defined to be of *degree* 2, if a, b, c are not all 0, otherwise C is defined to be of *degree* 1.

The *positive domain* of a conic curve C=$\{a, b, c, d, e, f\}$ is the locus of points (x, y) satisfying $ax^2 + by^2 + cxy + dx + ey + f > 0$, and the *negative domain* of C is the locus of points satisfying $ax^2 + by^2 + cxy + dx + ey + f < 0$.

If C is of degree 2, then it divides the plane into a *convex domain* and a *non-convex domain.* The convex domain is bounded in a case of an ellipse, unbounded in a case of a parabola, and consists of two unbounded connected components in a case of a hyperbola.

The *orientation* of a conic curve of degree 2 is defined to be *positive* if and only if its positive domain coincides with its convex domain; otherwise its orientation is defined to be negative. A conic curve of degree 1 divides the plane into two convex half-planes, therefore its orientation is defined to be *zero*.

Definition 1.3: A *finite conic arc* τ may be one of the following:

- A full conic curve *C.* To be bounded *C* must be an ellipse.
- The tuple $\{C, O, p_b, p_e\}$, where $C \in R^6$ is the tuple of the coefficients, $O \in \{-1, 0, 1\}$ is the orientation of the underlying conic curve and $p_b, p_e \in R^n$ are the beginning and the end points of τ. We term the conic arcs of this type as *conic segments.*

If the orientation is positive then we traverse τ from p_b to p_e with the convex domain to our left. The opposite holds when the orientation of the underling conic is negative. In the context of the computation of the metric average we are interested only in conic segments.

For a review of conic curves see e.g. [14]. The treatment of conic arcs in CGAL is described in [17], and some of the above notations on conic arcs are borrowed from there.

<u>Definition 1.4</u>: A *Simple polygon* is a region of the plane bounded by a finite collection of linear segments, termed *edges*, satisfying the following properties:

- The edges are connected so that they form a single cyclic chain.
- The edges intersect only at their endpoints and at each point of intersection exactly two edges intersect.

<u>Definition 1.5</u>: Let P be a simple polygon enclosing other simple polygons $H_1,...H_n$. Suppose the following conditions hold:

- $\partial P \cap \partial H_i = \phi$
- $H_i \cap H_j = \phi$, $\forall i \neq j$

Then the set: $P \setminus \bigcup_{i=1}^{n} H_i^o$ is called a *simple polygon with holes.*

If in definitions 1.4, 1.5 we allow the edges to be conic segments, we get the definitions of *simple conic polygon* and *simple conic polygon with holes* respectively. Linear polygons, linear polygons with holes and conic polygons can be considered as degenerate cases of conic polygons with holes.

Polygons and polygons with holes can be treated as *planar arrangements*.

<u>Definition 1.6</u>: Given a set C of planar curves, the *arrangement* $A(C)$ is the subdivision of the plane induced by the curves in C into zero-dimensional, one-dimensional and two-dimensional cells, called *vertices*, *edges* and *faces* respectively.

<u>Definition 1.7</u>: The *overlay* of two arrangements $A(C_1), A(C_2)$ is the arrangement $A(C_1 \cup C_2)$ produced by edges from C_1 and C_2.

For a review of results on arrangements see e.g. [10]. The CGAL library supports construction and overlaying for planar arrangements of linear and conic arcs [9].

1.3 The segment Voronoi diagram

Another geometric concept that plays a major role in our algorithm is the segment Voronoi diagram. Voronoi diagrams are well studied geometric structures. For a review on Voronoi diagrams see e.g. [2, 13]. In our implementation of the metric average of two simple polygons, we used the implementation of the segment Voronoi diagram provided in the CGAL library [11, 12].

In general, Voronoi diagrams are defined for a set of objects S that lie in some space Ω and a distance function $d(\bullet)$ that measures the distance from a point in Ω to an object in S. For the computation of the metric average of 2-dimensional sets, we are interested in the Voronoi diagrams defined in R^2 with the Euclidean distance.

Definition 1.8: Let $S = \{S_1, S_2, ...S_n\}$ be a set of geometric objects in R^2; these are the *Voronoi sites*. The *Voronoi diagram* of S is a partition of the R^2 plane into *n* regions – the *Voronoi faces* $\{F(S_1), F(S_2), ...F(S_n)\}$, with the property that a point *p* belongs to $F(S_i)$ if and only if:

$$\text{dist}(p, F(S_i)) \leq \text{dist}(p, F(S_j)), \text{ for each } S_j \in S \text{ with } j \neq i.$$

A *Voronoi edge* is a connected set of points that belong to exactly two Voronoi faces and a *Voronoi vertex* is a point that belongs to three or more Voronoi faces. A *Voronoi bisector* is a set of points that are equidistant from two Voronoi sites. We denote the Voronoi diagram induced by the collection of sites *S* by VD_S.

Remark 1.9: The boundary of a polygon is a set of linear segments intersecting at their endpoints. The standard technique is to treat a segment as three disjoint geometric objects: two endpoints and an open segment [11, 13]. The bisector between an open segment (p_i, p_{i+1}) and a point p_i is defined by the line through p_i that is perpendicular to (p_i, p_{i+1}). With this technique all bisectors of the segment Voronoi diagram are curves.

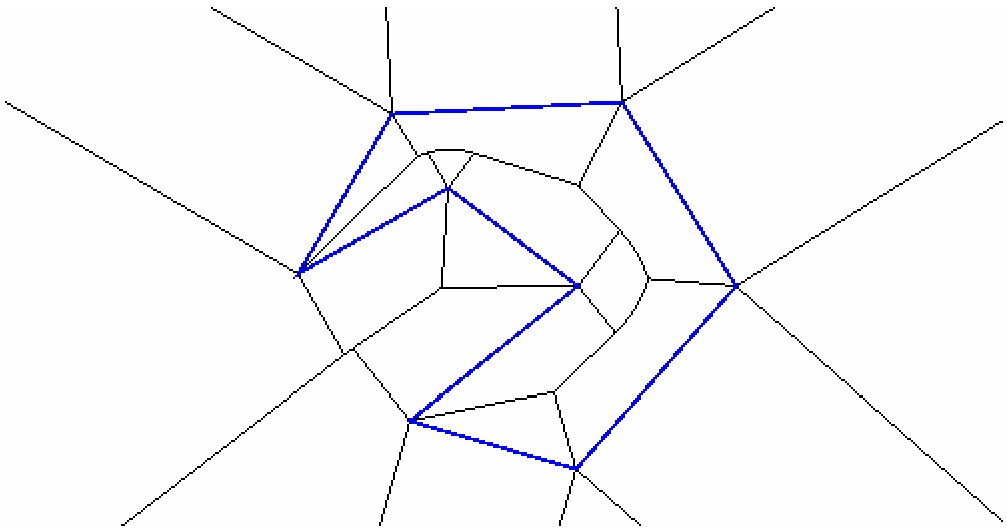


Figure 1.1 The Voronoi diagram (black) induced by the boundary of a simple polygon (bold blue)

<u>Definition 1.10</u>: Let $\partial P = \{[p_1, p_2], [p_2, p_3], \ldots [p_n, p_1]\}$ be a collection of linear segments that define a polygon $P = \overline{\partial P}$. The *segment Voronoi diagram* induced by ∂P is the Voronoi diagram defined by the collection of open segments $(p_1, p_2), (p_2, p_3), \ldots (p_n, p_1)$, termed *segment Voronoi sites,* and by the collection of points $p_1, p_2, \ldots p_n$, termed *point Voronoi sites.*

From Remark 1.9 we can conclude the following remark.

<u>Remark 1.11</u>: Let an open segment $s = (p_i, p_{i+1})$ be supported by the line l, and let a point $q \in F((p_i, p_{i+1}))$, then $\Pi_s(q) = \Pi_l(q)$.

<u>Remark 1.12</u>: The segment Voronoi diagram as defined in Definition 1.10 may have Voronoi edges of zero length or Voronoi faces of zero area. These low-dimensional features are irrelevant for the computation of the metric average and should be removed. In the sequel we consider only the Voronoi faces of the segment Voronoi diagram induced by the boundary of a simple polygon that are two-dimensional.

This degeneracy removal policy is supported by the segment Voronoi diagram implementation in CGAL [12].

From Remarks 1.9 and 1.12, we can conclude the following remark:

Remark 1.13: Let e_1, e_2 be edges of a simple polygon P that have a common endpoint, and let S_1, S_2 be the corresponding segment Voronoi sites. Then there is an edge between $F(S_1)$ and $F(S_2)$ on the segment Voronoi diagram induced by ∂P, if and only if the exterior angle between e_1 and e_2 is less than or equal to π. We term two such segment Voronoi sites S_1, S_2 as *adjacent segment Voronoi sites.*

1.4 Connectedness of sets in R^n

In this section we briefly recall the basic definitions for connectedness of sets in R^n. For details about connectedness of sets in R^n see e.g. [15].

Definition 1.14: Let $A \subseteq R^n$, A is called a *connected set* if it cannot be partitioned into two nonempty subsets such that each subset has no points in common with the closure of the other.

Remark 1.15: By Definition 1.14, if A, B are closed connected sets, then $A \bigcup B$ is a connected set if and only if $A \bigcap B \neq \phi$.

Definition 1.16: Let $B \subseteq A \subseteq R^n$, B is called a *connected component* of A, if B is a connected set and there does not exist a connected set $C \subseteq R^n$, $C \neq B$ satisfying $B \subseteq C \subseteq A$.

Definition 1.17: Let $A \subseteq R^n$, A is called *a path-connected set* if for each pair of points $p, q \in A$, there is a path in A from p to q.

Remark 1.18: Clearly simple conic polygons with holes are connected and path-connected. . It is well known that path-connected sets are connected. By Remark 1.15, if a set U is connected and can be written as a union of path-connected closed sets, then it is path-connected.

1.5 Regularized Boolean set operations

Our algorithm requires computation of *intersection, union* and *set difference* of simple conic polygons with holes. These operations are termed *Boolean set operations.* In general Boolean set operations may produce low-dimensional features, such as isolated vertices or isolated curve segments. These low-dimensional features are irregular and their presence is sensitive to infinitesimal changes in the input data. In order to keep the problem of the computation of the metric average of two simple polygons well posed, we disregard the low-dimensional features by using *regularized Boolean set operations.*

Definition 1.19: Let *op* be a Boolean set operation. The operation

$$P\ op^{*}\ Q = \text{closure (interior (}P\ op\ Q\text{))}$$

is called a *regularized Boolean set operation*.

In this way the low-dimensional features are eliminated and the resulting set is physically meaningful. The regularized Boolean set operations for conic polygons with holes are implemented in the CGAL library [8]. In the rest of this work, the traditional notation and terminology for the Boolean set operations when applied to the two-dimensional sets and unless otherwise stated, denote the regularized operations; e.g. $P \cap Q$, where P and Q are polygons, means the regularized intersection of P and Q.

Chapter 2: The algorithm

In this chapter we describe in details the algorithm for the computation of the metric average of two simple polygons based on the segment Voronoi diagrams induced by the polygons' boundaries. First we describe how a polygon can be divided by the segment Voronoi diagram induced by the boundary of the other polygon. Then we show how this partition simplifies the procedure of the computation of the metric average. Finally we describe the implementation of the algorithm and present many examples.

2.1 Partition of a polygon by the segment Voronoi diagram induced by the boundary of the other polygon

In this section we show that for two simple polygons A, B the partition of the set $A \setminus B$ by the segment Voronoi diagram induced by the boundary of *B* can be represented by a collection of conic polygons.

Suppose that we are interested in the partition of the set $A \setminus B$ by the segment Voronoi diagram induced by the boundary of the polygon *B*. There are always unbounded Voronoi faces in the segment Voronoi diagram induced by the boundary of *B*. Since we are interested only in the partition of $A \setminus B$ by $VD_{\partial B}$, we can bound the unbounded faces by adding a rectangular boundary frame. The frame should be large enough not to influence the original partition of $A \setminus B$ by $VD_{\partial B}$. The addition of the boundary frame can be embedded into the computation of $VD_{\partial B}$, if we consider the edges of the boundary frame as Voronoi sites too. Then all the unbounded Voronoi faces can be ignored. If this technique is used, the boundary frame *F* should satisfy

$$\forall p \in A \setminus B,\ \operatorname{dist}(p, \partial B) < \operatorname{dist}(p, F). \qquad (2.1)$$

In the sequel we assume that $VD_{\partial B}$ is bounded by a frame, and that the frame does not influence the partition of $A \setminus B$ by $VD_{\partial B}$.

By Definition 1.10 the Voronoi sites of the segment Voronoi diagram induced by the boundary of a polygon are points and open segments, thus the Voronoi edges are either parts of point to point bisectors, or segment to segment bisectors or point to segment bisectors.

A Voronoi edge induced by two point Voronoi sites $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ is supported by the perpendicular line to the segment $[p_1, p_2]$ at its midpoint. The line can be represented by the degenerate conic equation (with $a = b = c = 0$)

$$2(x_2 - x_1)x + 2(y_2 - y_1)y + x_1^2 + y_1^2 - (x_2^2 + y_2^2) = 0. \qquad (2.2)$$

A Voronoi edge induced by two segment Voronoi sites supported by the lines l_1, l_2 :

$$l_1 : a_1x + b_1y + c_1 = 0, \;\; l_2 : a_2x + b_2y + c_2 = 0,$$

is a part of the locus of points that are equidistant from l_1 and l_2. Thus any $p = (x, y)$ in this locus satisfies one of the two equations:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad \text{or} \quad \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}. \qquad (2.3)$$

The above two equations represent two perpendicular lines and the Voronoi edge is supported by one of these lines. Instead of (2.3) we can use the squared distance equality

$$\frac{(a_1x + b_1y + c_1)^2}{a_1^2 + b_1^2} = \frac{(a_2x + b_2y + c_2)^2}{a_2^2 + b_2^2}. \qquad (2.4)$$

This is an equation of the degenerate conic curve that comprises the two perpendicular lines. Notice that if the coefficients of l_1 and l_2 are rational numbers, then (2.4) produces a conic with rational coefficients, whereas (2.3) produces in the general case lines (degenerate conics) with irrational coefficients. This distinction is significant for computations with the CGAL library which supports conics with rational coefficients [9].

A Voronoi edge induced by a point Voronoi site $p_0 = (x_0, y_0)$ and a segment Voronoi site S supported by the line $l : ax + by + c = 0$, is supported by the locus of points equidistant from p_0 and l . A point $p = (x, y)$ on this locus satisfies the equation

$$(x - x_0)^2 + (y - y_0)^2 = \frac{(ax + by + c)^2}{a^2 + b^2}, \qquad (2.5)$$

which is an equation of a parabolic conic curve.

If $p_0 \in s$, then the corresponding bisector is a line perpendicular to S at $p_0 = (x_0, y_0)$, described by the equation:

$$bx - ay + (ay_0 - bx_0) = 0. \tag{2.6}$$

Equations (2.3) – (2.6) summarize all the cases of bisectors of the segment Voronoi diagram and they are in accordance with Definition 1.2 of a conic curve. Thus the collection of the Voronoi faces of the bounded segment Voronoi diagram is a collection of simple conic polygons.

For A, B two simple polygons, the set $A \setminus B$ is a collection of disconnected simple polygons $\{P_1, ..., P_n\}$ if $B \not\subset A$, and a simple polygon with holes (one hole) if $B \subset A$.

In case $B \not\subset A$, for a Voronoi face *F* of $VD_{\partial B}$, we get

$$F \cap (A \setminus B) = \bigcup_{i=1}^{n} F \cap P_i, \tag{2.7}$$

and therefore $F \cap (A \setminus B)$ can be represented by a collection (may be empty) of simple conic polygons.

In case $B \subset A$, for a Voronoi face *F* of $VD_{\partial B}$, the set $F \cap (A \setminus B)$ is an intersection of the polygon with holes $A \setminus B$ with F. Notice that in this case the inner boundary of A / B is the boundary of B. Since the interior of *F* does not include Voronoi sites other than $S(F)$, the polygon *B* is not included in *F*. Thus also in this case the set $F \cap (A \setminus B)$ is a collection of simple conic polygons.

Definition 2.1: Let A, B be simple polygons, $VD_{\partial B}$ be a Voronoi diagram induced by the boundary of *B*, and let F be a Voronoi face of $VD_{\partial B}$ satisfying $F \cap (A \setminus B) \neq \phi$. We call a connected component of $F \cap (A \setminus B)$ a metric face originating from F.

We denote the collection of metric faces originating from any Voronoi face of $VD_{\partial B}$ by $MF_{\partial B}(A \setminus B)$. For a metric face F in $MF_{\partial B}(A \setminus B)$, we denote by $S(F)$ the Voronoi site of $VD_{\partial B}$ corresponding to the Voronoi face that F originates from. Metric faces are the major blocks of our algorithm.

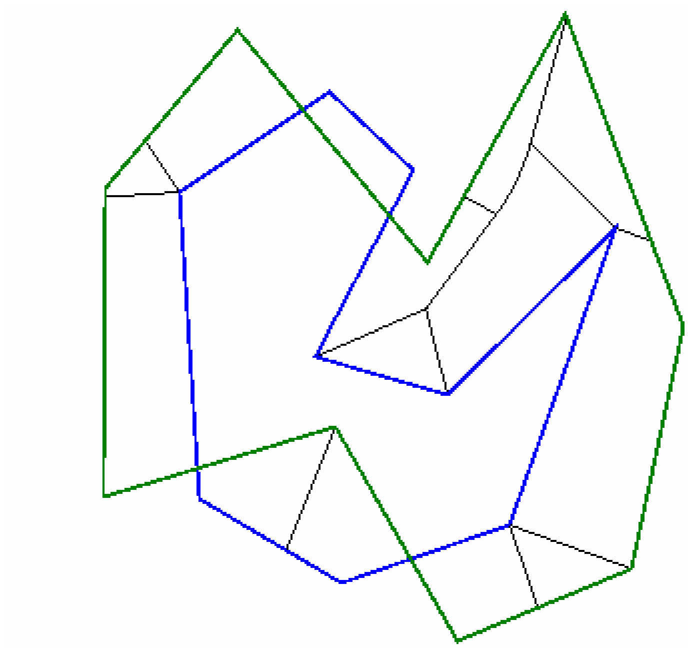


Figure 2.1: The mutual partition of two simple polygons (bold blue and bold green) to their metric faces (black): The sets A\B and B\A are partitioned by $VD_{\partial B}$ *and* $VD_{\partial A}$ *respectively.*

Notice that each of the sets A/B and $VD_{\partial B}$ constitutes of an arrangement of conic segments (see Definition 1.6), thus the metric faces can be found through the overlay of these two arrangements (see Definition 1.7). The metric faces $MF_{\partial B}(A \setminus B)$ are faces of the overlaid arrangement, which are intersection of the bounded faces of the two original arrangements. Each metric face F "inherits" the Voronoi site of the face of $VD_{\partial B}$ which contains it. The implementation of the required arrangement overlay algorithms is provided in the CGAL library [9].

2.2 Computation of the metric average with Voronoi diagrams

In this section we discuss the connection between the metric average and Voronoi diagrams that can be applied to the computation of the metric average.

Let A, B be two simple polygons. By (1.6) the set $A \oplus_t B$ with $t \in [0,1]$ can be represented as,

$$A \oplus_t B = (A \cap B) \cup M_t(A \setminus B, \partial B) \cup M_{1-t}(B \setminus A, \partial A).$$

The main part of the computation of the metric average is the computation the sets $M_t(A \setminus B, \partial B)$ and $M_{1-t}(B \setminus A, \partial A)$ with $0 < t < 1$. The set $A \setminus B$ can be written as:

$$A \setminus B = \bigcup_{F \in VD_{\partial B}} ((A \setminus B) \cap F),$$

and therefore

$$M_t(A \setminus B, \partial B) = \bigcup_{F \in VD_{\partial B}} M_t((A \setminus B) \cap F, \partial B).$$

For a point p in the interior of a Voronoi face $F \in VD_{\partial B}$ the following holds,

$$q \in \Pi_{\partial B}(p) \Rightarrow q \in S(F),$$

therefore

$$M_t(A \setminus B, \partial B) = \bigcup_{F \in VD_{\partial B}} M_t((A \setminus B) \cap F, S(F)), \qquad (2.8)$$

or in terms of metric faces

$$M_t(A \setminus B, \partial B) = \bigcup_{F \in MF_{\partial B}(A \setminus B)} M_t(F, S(F)). \qquad (2.9)$$

<u>Remark</u>: Relations (2.8) and (2.9) can be easily extended to any two compact sets A, B in R^n, for which the Voronoi diagrams of ∂A and ∂B are well defined.

If $S(F)$ is a Voronoi site of the segment Voronoi diagram $VD_{\partial B}$ i.e. a linear segment or a point, then for any $p \in R^2$ the set $\Pi_{S(F)}(p)$ is a singleton. Thus the operation $p \to tp + (1-t)\Pi_{S(F)}(p)$ can be regarded as a function G from the metric face F to R^2, which is continuous and one-to-one. Since the boundary of a metric face F is a simple closed curve, so is its mapping under G, and therefore

$$M_t(F, S(F)) = \overline{M_t(\partial F, S(F))}. \tag{2.10}$$

Consequently we can compute the metric average of two simple polygons *A, B* with a coefficient $t \in (0,1)$ as:

$$A \oplus_t B = (A \cap B) \cup M_t(A \setminus B, \partial B) \cup M_{1-t}(B \setminus A, \partial A), \tag{2.11-a}$$

where

$$M_t(A \setminus B, \partial B) = \bigcup_{F \in MF_{\partial B}(A \setminus B)} \overline{M_t(\partial F, S(F))}, \tag{2.11-b}$$

and similarly

$$M_{1-t}(B \setminus A, \partial A) = \bigcup_{F \in MF_{\partial A}(B \setminus A)} \overline{M_{1-t}(\partial F, S(F))}. \tag{2.11-c}$$

Our algorithm follows (2.11).

2.3 The algorithm

We describe the algorithmic structure of (2.11), starting with the basic computations and then developing the whole algorithm.

Computation of $\Pi_S(p)$ where p is a point in a metric face and S is its Voronoi site

If S is a point q, then $\Pi_S(p)=q$.

If S is a segment supported by the line $l: ux+vy+w=0$, then by Remark 1.11 $\Pi_S(p)=\Pi_l(p)$. Denote $\Pi_S(p)$ by (x,y) and $p=(x_0,y_0)$. Let l' be the line perpendicular to l at p, then $\Pi_S(p)$ is the intersection point of l' with l. Therefore $\Pi_S(p)$ satisfies the system of equations

$$\begin{cases} ux+vy=-w \\ -vx+uy=-vx_0+uy_0 \end{cases}$$

and it is given by

$$\Pi_S(p)=\left(\frac{-uw+v^2x_0-uvy_0}{u^2+v^2},\frac{-uvx_0+u^2y_0-vw}{u^2+v^2}\right). \qquad (2.12)$$

Computation of $M_t(p,S)$ where p is a point in a metric face and S is the corresponding Voronoi site

If S is a point q then

$$M_t(p,S)=tp+(1-t)q. \qquad (2.13)$$

If S is a segment, then we use (2.12) to find $\Pi_S(p)$ and $M_t(p,S)$ is computed by

$$M_t(p,S)=tp+(1-t)\Pi_S(p). \qquad (2.14)$$

Computation of $M_t(\tau, S)$ where τ is a conic segment on the boundary of a metric face and S is the corresponding point Voronoi site

By Definition 1.3, a conic segment τ is given by the tuple $\{\{a,b,c,d,e,f\},O,p_b,p_e\}$. The points $M_t(p_b,S)$ and $M_t(p_e,S)$ can be computed by (2.13). $M_t(\tau,S)$ is the set of points $p=(x,y)$ satisfying

$$\frac{|p_1-p|}{|p-S|}=\frac{1-t}{t}, \tag{2.15}$$

where $p_1 \in \tau$ is such that p_1, p and S are collinear (see Figure 2.2).

The point $p_1=(x_1,y_1)$ is on the conic segment τ, thus it satisfies

$$a{x_1}^2+b{y_1}^2+cxy_1+dx_1+ey_1+f=0. \tag{2.16}$$

Since the points p_1, p and $S=(x_0,y_0)$ are collinear, it follows from (2.15) that

$$\begin{aligned} x_1 &= x_0+\frac{1}{t}(x-x_0) \\ y_1 &= y_0+\frac{1}{t}(y-y_0) \end{aligned} \tag{2.17}$$

By substituting (2.17) into (2.16) and collecting terms, we observe that p lies on a conic segment.

The coefficients $\{a_1,\ldots,f_1\}$ of the equation for p can be evaluated in terms of the coefficients of the original conic τ and the coordinates of the point site S:

$$\begin{aligned} &a_1=\frac{a}{t^2},\quad b_1=\frac{b}{t^2},\quad c_1=\frac{c}{t^2}, \\ &d_1=\frac{2a(t-1)x_0}{t^2}+\frac{c(t-1)y_0}{t^2}+\frac{d}{t},\quad e_1=\frac{2b(t-1)y_0}{t^2}+\frac{c(t-1)x_0}{t^2}+\frac{e}{t}, \\ &f_1=\frac{a((t-1)x_0)^2}{t^2}+\frac{b((t-1)y_0)^2}{t^2}+\frac{c(t-1)^2x_0y_0}{t^2}+\frac{d(t-1)x_0}{t}+\frac{e(t-1)y_0}{t}+f, \end{aligned} \tag{2.18}$$

Thus $M_t(\tau,S)$ is the conic segment defined by

$$M_t(\tau,S)=\{\{a_1,b_1,c_1,d_1,e_1,f_1\},O,M_t(p_b,S),M_t(p_e,S)\}. \tag{2.19}$$

with O the direction of τ .

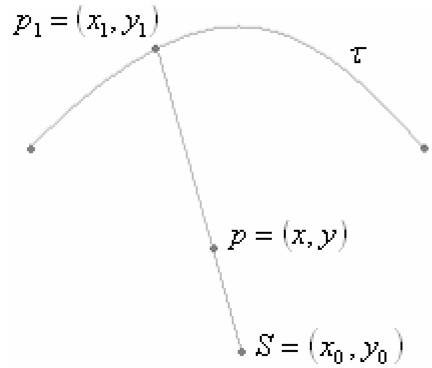


Figure 2.2: Computation of the one-sided metric average of a conic segment τ and a point Voronoi site S.

Computation of $M_t(\tau, S)$ where τ is a conic segment on the boundary of a metric face and S is the corresponding segment Voronoi site

Let $\tau = \{ \{a,b,c,d,e,f\}, O, p_b, p_e \}$ be a conic segment and let the corresponding segment Voronoi site $S = [s_1, s_2]$ be supported by the line $l : ux + vy + w = 0$. The points $M_t(p_b, S)$ and $M_t(p_e, S)$ can be computed by (2.14). $M_t(\tau, S)$ is the set of points $p = (x, y)$ satisfying

$$\frac{|p_1 - p|}{|p - p_0|} = \frac{1-t}{t}, \tag{2.20}$$

where $p_1 \in \tau$, $p_0 = \Pi_S(p)$ and the points p_1, p and p_0 are on the perpendicular line to l through p (see Figure 2.3).

The coordinates of $p_0 = (x_0, y_0)$ can be computed as in (2.12) and are given by

$$(x_0, y_0) = \left(\frac{-uw + v^2 x - uvy}{u^2 + v^2}, \frac{-uvx + u^2 y - vw}{u^2 + v^2} \right). \tag{2.21}$$

Now the computation is similar to the computation for a point Voronoi site. The coordinates of p_1 can be expressed in terms of the coordinates of p and p_0,

$$\begin{aligned} x_1 &= x_0 + \frac{1}{t}(x - x_0) \\ y_1 &= y_0 + \frac{1}{t}(y - y_0) \end{aligned} \tag{2.22}$$

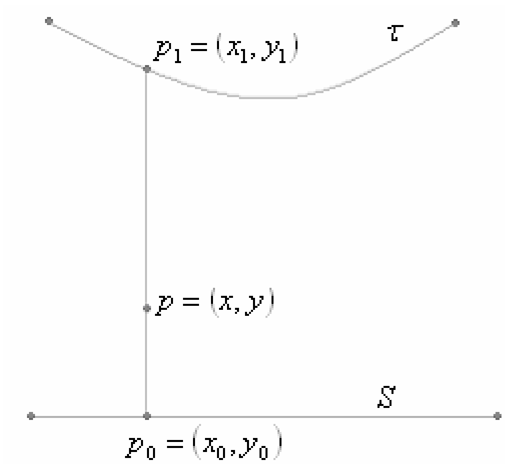


Figure 2.3: Computation of the one-sided metric average of a conic segment τ and a segment Voronoi site S.

By substituting (2.21) into (2.22) we get the expression for $p_1 = (x_1, y_1)$ in terms of *p* and l:

$$x_1 = \left(\frac{(t-1)v^2}{t(u^2+v^2)} + \frac{1}{t}\right)x + \left(\frac{(1-t)uv}{t(u^2+v^2)}\right)y + \frac{(1-t)uw}{t(u^2+v^2)}$$

$$y_1 = \left(\frac{(1-t)uv}{t(u^2+v^2)}\right)x + \left(\frac{(t-1)v^2}{t(u^2+v^2)} + \frac{1}{t}\right)y + \frac{(1-t)vw}{t(u^2+v^2)} \tag{2.23}$$

Since the point p_1 is on the conic segment τ, it satisfies (2.16). Substituting (2.23) into (2.16) and collecting terms, we get the equation (with *x, y* as variables) of the conic curve of $M_t(\tau, S)$.

First we introduce the intermediate constants:

$$\alpha = \frac{(t-1)v^2}{t(u^2+v^2)} + \frac{1}{t} \qquad \beta = \frac{(1-t)uv}{t(u^2+v^2)}$$
$$\gamma = \frac{(1-t)uw}{t(u^2+v^2)} \qquad \delta = \frac{(1-t)uv}{t(u^2+v^2)} \tag{2.24}$$
$$\eta = \frac{(t-1)u^2}{t(u^2+v^2)} + \frac{1}{t} \qquad \mu = \frac{(1-t)vw}{t(u^2+v^2)}$$

Then the coefficients $\{a_1,\dots f_1\}$ of the conic curve of $M_t(\tau,S)$ are:

$$\begin{aligned}
a_1 &= a\alpha^2 + b\delta^2 + c\alpha\delta \\
b_1 &= a\beta^2 + b\eta^2 + c\beta\eta \\
c_1 &= 2a\alpha\beta + 2b\delta\eta + c\alpha\eta + c\beta\delta \\
d_1 &= 2a\alpha\gamma + 2b\delta\mu + c\alpha\mu + c\gamma\delta + d\alpha + e\delta \\
e_1 &= 2a\beta\gamma + 2b\eta\mu + c\beta\mu + c\gamma\eta + d\beta + e\eta \\
f_1 &= a\gamma^2 + b\mu^2 + c\gamma\mu + d\gamma + e\mu + f
\end{aligned} \tag{2.25}$$

Thus

$$M_t(\tau,S) = \{\{a_1,b_1,c_1,d_1,e_1,f_1\},O,M_t(p_b,S),M_t(p_e,S)\}, \tag{2.26}$$

with O the direction of τ.

Notice that the above computations of $M_t(\tau,S)$ cover also the case when τ is a linear segment. However by (2.19), (2.25) and (2.26), if τ is a linear segment $(a=b=c=0)$ then $M_t(\tau,S)$ is also a linear segment, determined by its endpoints $M_t(p_b,S), M_t(p_e,S)$.

Remark 2.2: It follows from (2.19) and (2.26) that for a conic segment τ and the corresponding point or segment Voronoi site S the set $M_t(\tau,S)$ can be computed by a fixed number of operations, namely in $O(1)$ computational time.

By Definition 2.1 the boundary of a metric face F is a closed chain of conic segments – the edges $\{\tau_1,\dots,\tau_n\}$. Therefore $M_t(\partial F,S(F))$ is also a closed chain,

$$M_t(\partial F,S(F))=\{M_t(\tau_1,S(F)),\dots,M_t(\tau_n,S(F))\}. \tag{2.27}$$

The corresponding Voronoi site $S(F)$ may be a point or a segment, so the set $M_t(\partial F,S(F))$ can be computed as follows.

<u>Algorithm 2.1</u>: Computation of $M_t(\partial F, S(F))$ for a metric face F

1. For each conic segment τ in ∂F
   a. If $S(F)$ is a segment
      use (2.26) to compute $M(\tau, S(F))$
      Else
      use (2.19) to compute $M(\tau, S(F))$
   b. Add the result of the computation in step (a) to the collection of conic segments already computed
2. Return the resulting collection of conic segments as the boundary of a conic polygon.

<u>Algorithm 2.2</u>: Computation of the one-sided metric average $M_t(A \setminus B, \partial B)$

1. Compute the segment Voronoi diagram induced by ∂B
2. Overlay $A \setminus B$ with $VD_{\partial B}$ and obtain the collection of metric faces with their corresponding sites
3. For each metric face F in the collection found in step (2):
   a. Use Algorithm 2.1 to compute $M_t(\partial F, S(F))$
   b. Add the result of the computation in (a) to the collection of conic polygons already computed
4. Return the final collection of conic polygons

Step 1 of Algorithm 2.2 is carried out by an implementation of the segment Voronoi diagram algorithm; we use the implementation provided in [12]. Step 2 of Algorithm 2.2 can be carried out by using Boolean set operations on conic polygons or by using overlay of arrangements of conic segments. We implemented both options, using the software in [8] and [9] respectively.

<u>Algorithm 2.3</u>: Computation of the metric average $A \oplus_t B$ where A, B are simple polygons.

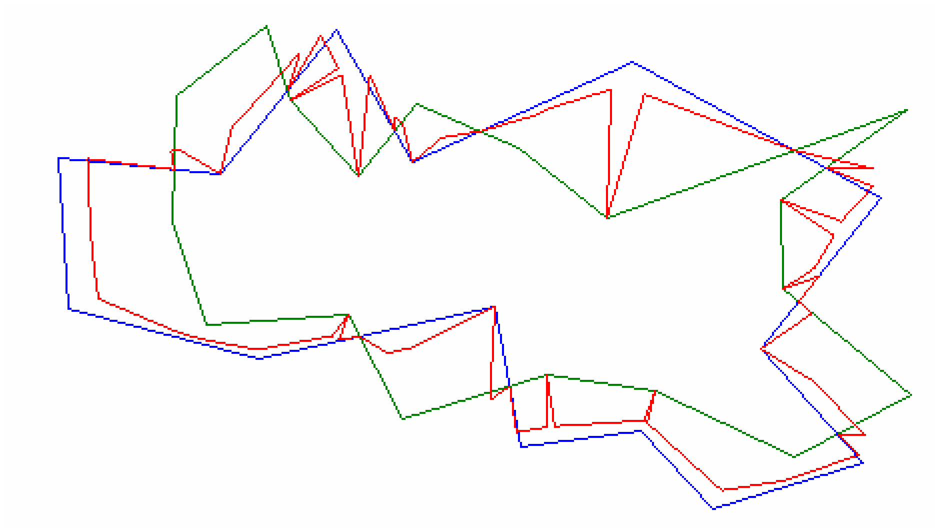
1. Compute the sets: $A \cap B$, $A \setminus B$, $B \setminus A$
2. Use Algorithm 2.2 to compute the one-sided metric average $M_t\left(A \setminus B, \partial B\right)$
3. Use Algorithm 2.2 to compute the one-sided metric average $M_{1-t}\left(B \setminus A, \partial A\right)$
4. Compute $A \cap B \cup M_t\left(A \setminus B, \partial B\right) \cup M_{1-t}\left(B \setminus A, \partial A\right)$

By (2.19) and (2.26), the one-sided metric average of a conic segment and the corresponding point or segment Voronoi site is a conic segment. Consequently the results of the computation in steps 2 and 3 of the algorithm above are collections of conic polygons. Thus steps 1 and 4 can be done by an implementation of Boolean set operations on conic polygons; we use the implementation provided in [8].

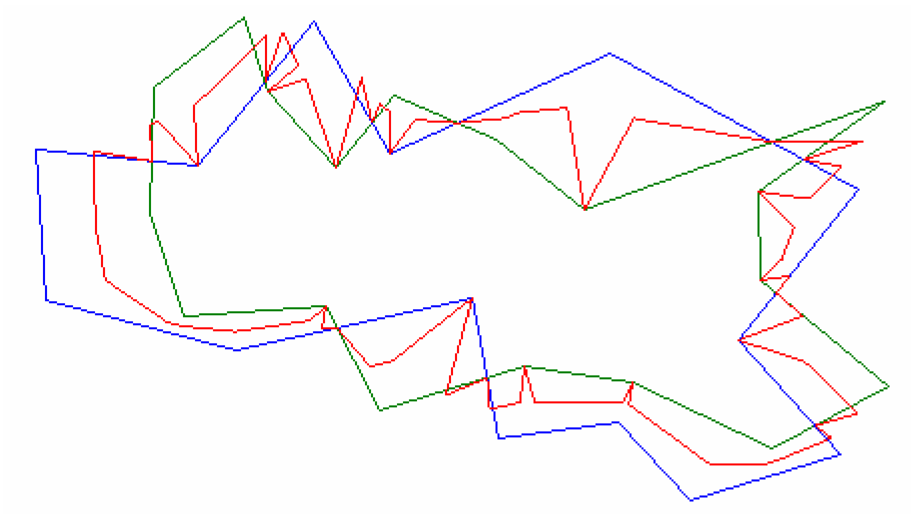
2.4 Examples of the metric average

We bring a collection of computational examples of the metric average of two simple polygons that were done by our C++ implementation of the algorithm. At the end of this section we discuss the obtained results.

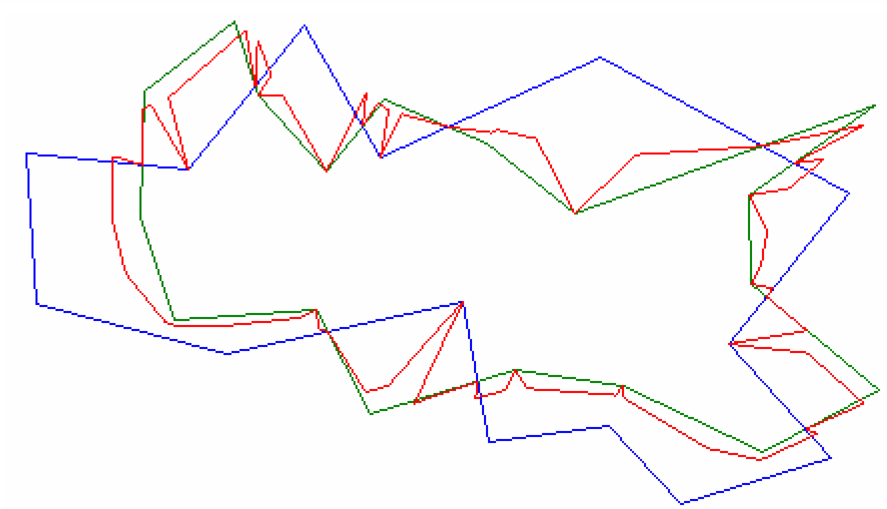
In the following examples the boundaries of the polygons A, B are colored green and blue respectively, and the boundary of the resulting metric average set $A \oplus_t B$ is colored red.



$t = 0.25$



$t = 0.5$



$t = 0.75$

Figure 2.4: The metric average of two simple polygons for different values of the averaging parameter t.

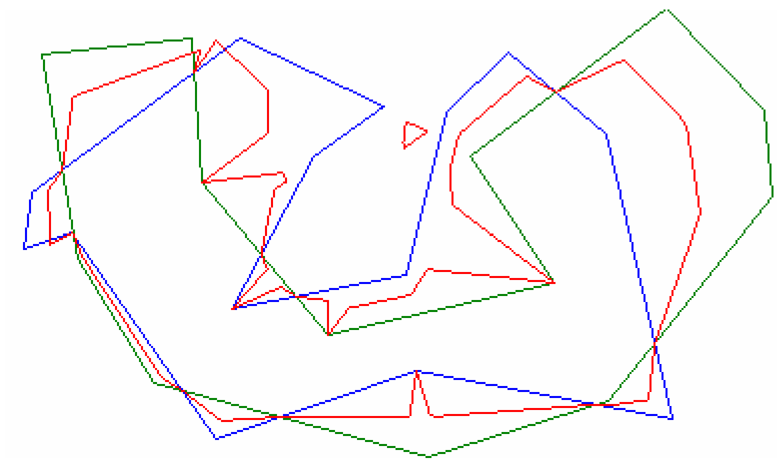


Figure 2.5: An example of two intersecting simple polygons with a metric average which is a collection of disconnected conic polygons for the value $t = 0.5$, see discussion is Chapter 3.

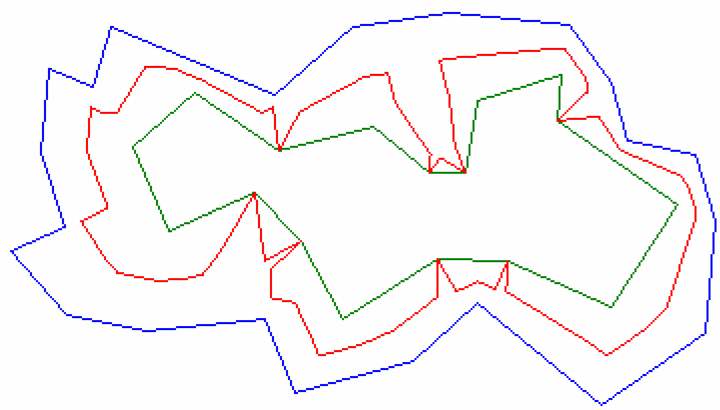


Figure 2.6: The metric average with $t = 0.5$ of two simple polygons, when the polygon A is included in the polygon B.

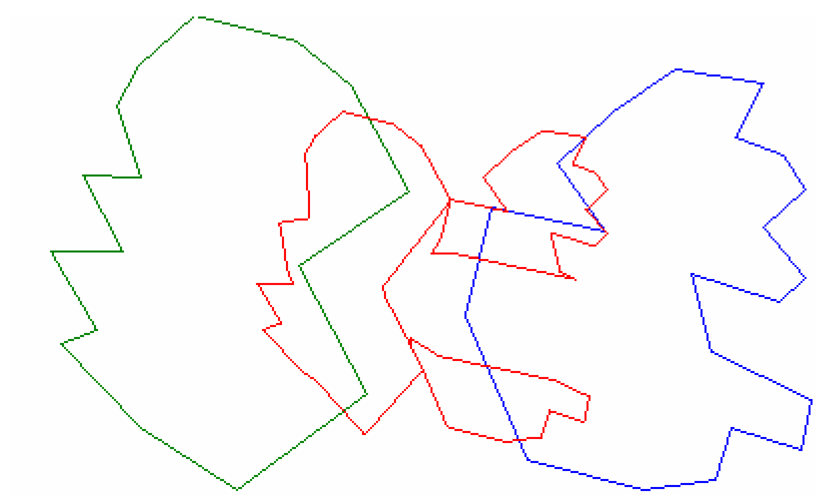


Figure 2.7: The metric average of two nonintersecting simple polygons with $t = 0.5$.

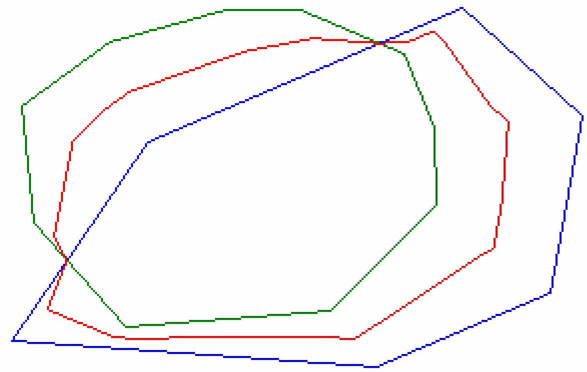
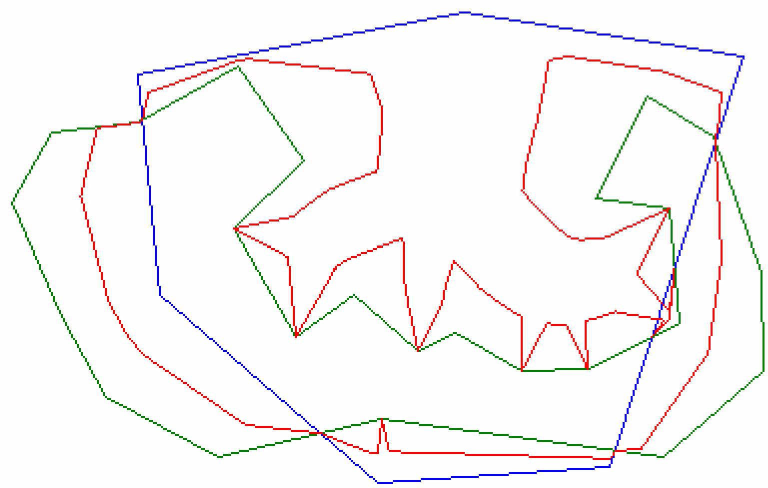


Figure 2.8: The metric average of two convex polygons with $t = 0.5$.



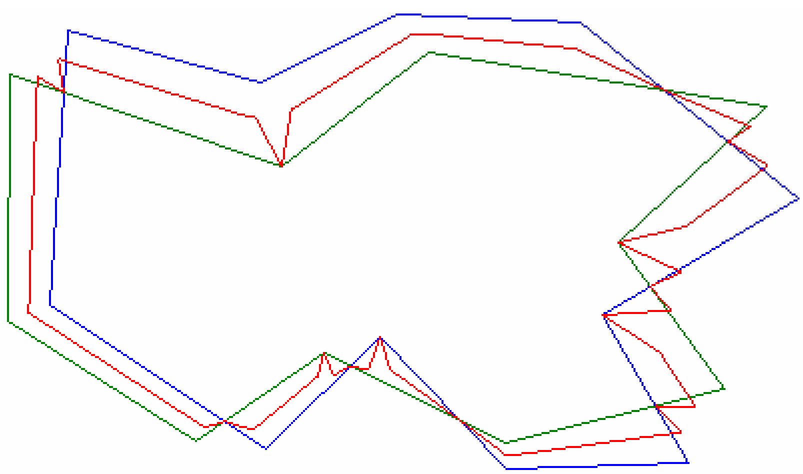


Figure 2.9: More examples of the metric average of two simple polygons with $t = 0.5$.

The obtained results demonstrate several limitations of the metric average of two simple polygons. The first limitation, which can be seen in Figure 2.4, is the splits in the obtained set near the intersection points of adjacent segments (see Remark 1.13), when the exterior angle between the segments is less than π. We address this limitation in Chapter 5, where we introduce a modification of the metric average that avoids this artifact.

Another limitation, which can be seen in Figure 2.5, is that in some cases the metric average of two intersecting polygons is a disconnected set. We discuss this issue in Chapter 3, where we give necessary and sufficient conditions for the connectedness of the metric average of two simple polygons.

Our algorithm is able to compute the metric average of two non-interesting simple polygons. An example of the metric average of two non-intersecting simple polygons can be seen in Figure 2.7.

In general, better results are obtained when the polygons are close to being convex. An example of the metric average of two simple convex polygons can be seen in Figure 2.8.

Chapter 3: Connectedness of the metric average of two simple polygons

In this chapter we derive necessary and sufficient conditions for connectedness of the metric average of two simple polygons.

By (2.11) the metric average of two simple polygons is - in the general case - a union of several conic polygons. Figure 3.1 shows an example of two simple intersecting polygons with a metric average which is a collection of disjoint conic polygons.

Our framework for examining connectedness of the metric overage of two simple polygons is based on the partition of the polygons to their intersection and their metric faces.

Remark: In this chapter the regularized Boolean set operations are denoted by op* and the traditional Boolean set operations are denoted by the usual notation (see Section 1.5).

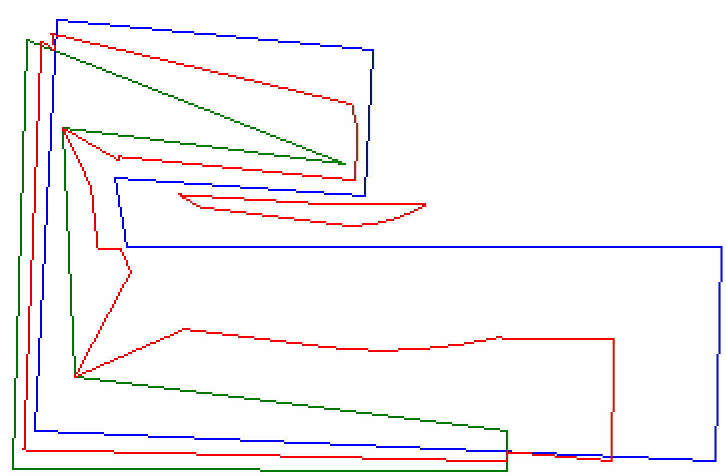


Figure 3.1: Example of two simple polygons that their metric average (red boundary) is disconnected

For A, B simple polygons, the set $A\bigcap^{*} B$ is a collection of polygons, denoted by $MF\left(A\bigcap^{*} B\right)$. We define the collection of conic polygons

$$MF(A,B)=MF\left(A\bigcap^{*} B\right)\bigcup MF_{\partial B}\left(A/^{*} B\right)\bigcup MF_{\partial A}\left(B/^{*} A\right),$$

and consider the metric average as defined on the elements of this collection.
We define the operation $\Psi_t(F)$ on the elements of the set $MF(A,B)$ by

$$\Psi_t(F)=\begin{cases} F & F\in MF\left(A\bigcap^{*} B\right) \\ M_t\left(F,S(F)\right) & F\in MF_{\partial B}\left(A\setminus^{*} B\right) \\ M_{1-t}\left(F,S(F)\right) & F\in MF_{\partial A}\left(B\setminus^{*} A\right) \end{cases} \tag{3.1}$$

We term the members of the set $MF(A,B)$ as *metric elements.*

By (1.6) and (2.9) the metric average of A and B can be computed as,

$$. \ A\oplus_t B=\bigcup_{F\in MF(A,B)}^{*}\Psi_t(F) \tag{3.2}$$

First for two metric elements F_1, F_2, we discuss the connectedness of the set $\Psi_t(F_1)\bigcup\Psi_t(F_2)$. Then we consider the metric elements as vertices of a graph with an edge between each two vertices corresponding to metric elements F_1, F_2, if the set $\Psi_t(F_1)\bigcup\Psi_t(F_2)$ is connected. In this way, the problem of the connectedness of the metric average of two simple polygons is reduced to the problem of graph connectivity.

Definition 3.1: Let A, B be simple polygons and F_1, F_2 be metric elements, $F_1, F_2 \in MF(A,B)$. F_1, F_2 are called *metric connected* if and only if the set $\Psi_t(F_1)\bigcup\Psi_t(F_2)$ is connected.

When examining the metric connectedness of two metric elements, three cases are relevant: two metric faces that are contained in the same polygon, two metric faces that belong to different polygons and a metric face with a connected component of the polygons' intersection. These cases are discussed in the following propositions.

Proposition 3.2: Let A, B be simple polygons and F_1, F_2 be metric faces contained in A. The metric faces F_1, F_2 are metric connected if and only if there is a point $p \in \partial F_1 \cap \partial F_2$, such that

$$\Pi_{S(F1)}(p) = \Pi_{S(F2)}(p) \ .$$

Proof: If there is a point $p \in \partial F_1 \cap \partial F_2$ such that $\Pi_{S(F1)}(p) = \Pi_{S(F2)}(p)$, then by the definition of the metric average

$$tp + (1-t)\Pi_{S(F1)}(p) = tp + (1-t)\Pi_{S(F2)}(p) \in M_t(F_1, S(F_1)) \cap M_t(F_2, S(F_2)),$$

and by Remark 1.15 the set $M_t(F_1, S(F_1)) \cup M_t(F_2, S(F_2))$ is connected, namely F_1, F_2 are metric connected.

To prove the other direction, assume F_1, F_2 to be metric connected. Then the set $M_t(F_1, S(F_1)) \cup M_t(F_2, S(F_2))$ is connected, and there are points $p_1 \in F_1$ and $p_2 \in F_2$, such that

$$p = tp_1 + (1-t)\Pi_{S(F1)}(p_1) = tp_2 + (1-t)\Pi_{S(F2)}(p_2). \tag{3.3}$$

Denote $s_i = \Pi_{S(Fi)}(p_i)$. Note that the point p belongs to both segments $[p_1, s_1]$ and $[p_2, s_2]$ (see Figure 3.2). Without loss of generality assume that

$$|p - s_2| \le |p - s_1|. \tag{3.4}$$

By the definition of metric faces

$$|p_1 - s_2| \ge |p_1 - s_1|. \tag{3.5}$$

By (3.4), the triangle inequality and (3.5) the following chain of inequalities holds,

$$|p_1 - p| + |p - s_1| \ge |p_1 - p| + |p - s_2| \ge |p_1 - s_2| \ge |p_1 - s_1| = |p_1 - p| + |p - s_1|, \tag{3.6}$$

where the last equality in (3.6) is due to $p \in [p_1, s_1]$. Thus there are equalities in (3.6), and therefore in (3.5) and (3.4), leading to

$$|p_1 - s_1| = |p_1 - s_2|, \ |p_1 - s_2| = |p_1 - p| + |p - s_2|,$$

from which we conclude that $p \in [p_1, s_2]$. Since $p \in [p_1, s_1]$ and $|p_1 - s_1| = |p_1 - s_2|$ it follows that $s_1 = s_2$, or equivalently

$$\Pi_{S(F1)}(p_1) = \Pi_{S(F2)}(p_2). \tag{3.7}$$

By substituting (3.7) into (3.3) we get $p_1 = p_2$. □

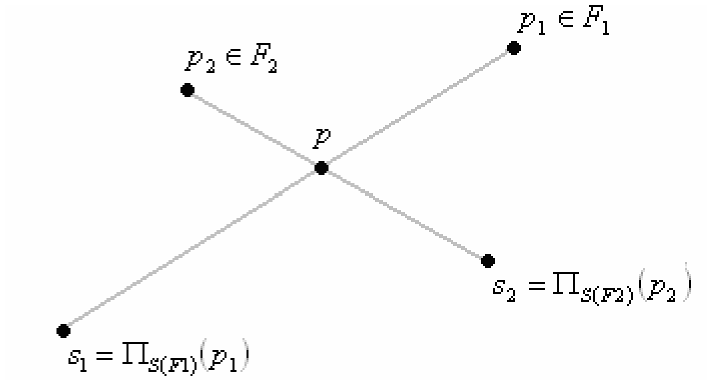


Figure 3.2: Illustration for Proposition 3.2

We now formulate the conditions of Proposition 3.2 in terms of metric faces and the corresponding Voronoi sites.

Proposition 3.2': Let A, B be simple polygons and F_1, F_2 be metric faces, $F_1, F_2 \subset A$. The metric faces F_1, F_2 are metric connected if and only if $\partial F_1 \cap \partial F_2 \neq \phi$ and the corresponding Voronoi sites satisfy one of the conditions (see Figure 3.3):

1. One of the sites is an open segment and the other is its endpoint.
2. The sites are two adjacent non-collinear segments and their common endpoint belongs to the polygon A.
3. The sites are two adjacent collinear segments.

Proof:

Assume that $\partial F_1 \cap \partial F_2 \neq \phi$ and that condition 1 holds. Let $p \in \partial F_1 \cap \partial F_2$. Without loss of generality assume that $S(F_1)$ is a segment and $S(F_2)$ is one of its endpoints. Since $p \in \partial F_1 \cap \partial F_2$,

$$\text{dist}(p, S(F_1)) = \text{dist}(p, S(F_2)),$$

Consequently,

$$\Pi_{S(F1)}(p) = \Pi_{S(F2)}(p),$$

and by Proposition 3.2 the metric faces F_1, F_2 are metric connected.

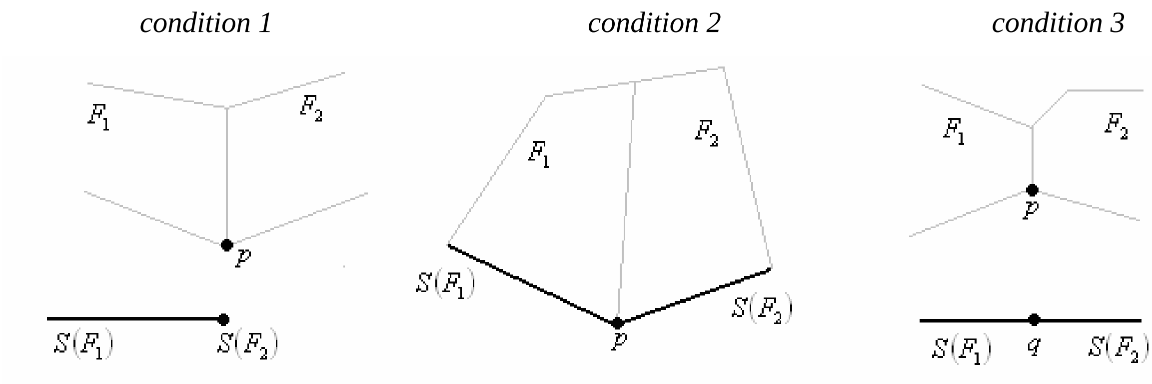


Figure 3.3: Illustration for Proposition 3.2'

Now assume that condition 2 holds. Let p be the common endpoint of $S(F_1)$ and $S(F_2)$ and $p \in A$. The point p satisfies

$$p = \Pi_{S(F1)}(p) = \Pi_{S(F2)}(p) \text{ and } p \in \partial F_1 \cap \partial F_2 .$$

Therefore it satisfies the conditions of Proposition 3.2 and the metric faces F_1, F_2 are metric connected.

Next assume that condition 3 holds, and let $p \in \partial F_1 \cap \partial F_2$. By the assumption that the sites are collinear and by Remark 1.11,

$$\Pi_{S(F1)}(p) = \Pi_{S(F2)}(p),$$

therefore by Proposition 3.2 the metric faces F_1, F_2 are metric connected.

To prove the other direction, assume F_1, F_2 to be metric connected. By Proposition 3.2 there is a point $p \in \partial F_1 \cap \partial F_2$ such that $\Pi_{S(F1)}(p) = \Pi_{S(F2)}(p)$. Thus the corresponding Voronoi sites have to satisfy:

$$Cl(S(F_1)) \cap Cl(S(F_2)) \neq \phi ,$$

so the only possible cases are a segment and its endpoint or two adjacent segments. In the first case, the proposition is proved, since condition 1 is satisfied.

In the second case, let $q = \Pi_{S(F1)}(p) = \Pi_{S(F2)}(p)$. If $q = p$, then $p \in A \cap B$ and condition 2 is necessarily true. Otherwise $p \neq q$, and the line supporting the segment $[p, q]$ is perpendicular to the line supporting $S(F_1)$ and also to the line supporting $S(F_2)$. Therefore the segment sites $S(F_1), S(F_2)$ are collinear, namely condition 3 is satisfied. □

Next we derive necessary and sufficient conditions for the metric connectedness of two metric faces that belong to different polygons.

<u>Proposition 3.3</u>: Let A, B be simple polygons and F_1, F_2 be metric faces, $F_1 \subset A, F_2 \subset B$. The metric faces F_1, F_2 are metric connected if and only if there are points $p_1 \in \partial F_1$ and $p_2 \in \partial F_2$ satisfying

$$\Pi_{S(F1)}(p_1) = p_2 \text{ and } \Pi_{S(F2)}(p_2) = p_1 .$$

Proof:

If there are points p_1, p_2 such that $\Pi_{S(F1)}(p_1) = p_2$ and $\Pi_{S(F2)}(p_2) = p_1$, then by the definition of the metric average

$$tp_1 + (1-t)\Pi_{S(F1)}(p_1) = tp_1 + (1-t)p_2 = t\Pi_{S(F2)}(p_2) + (1-t)p_2 \in M_t(F_1, S(F_1)) \cap M_{1-t}(F_2, S(F_2)),$$

and the set $M_t(F_1, S(F_1)) \cup M_{1-t}(F_2, S(F_2))$ is connected, namely F_1, F_2 are metric connected.

To prove the other direction, assume F_1, F_2 to be metric connected. Then the set $M_t(F_1, S(F_1)) \cup M_{1-t}(F_2, S(F_2))$ is connected, and there are points $p_1 \in F_1$ and $p_2 \in F_2$, such that

$$p = tp_1 + (1-t)\Pi_{S(F1)}(p_1) = (1-t)p_2 + t\Pi_{S(F2)}(p_2), \qquad (3.8)$$

Denote $s_i = \Pi_{S(Fi)}(p_i)$. Note that the point p belongs to both segments $[p_1, s_1]$ and $[p_2, s_2]$ (see Figure 3.4).

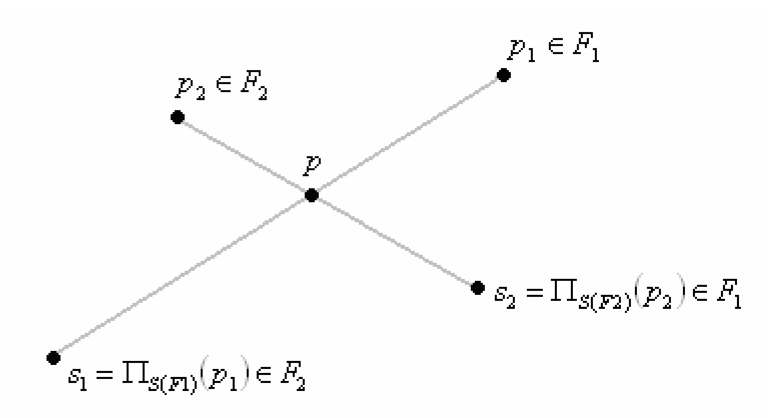


Figure 3.4: Illustration for Proposition 3.3

Without loss of generality assume that

$$|p_2 - s_2| \le |p_1 - s_1| , \tag{3.9}$$

Since $s_1 = \Pi_{S(F1)}(p_1) = \Pi_B(p_1)$ and $p_2 \in B$ it follows that

$$|p_1 - s_1| \le |p_1 - p_2| . \tag{3.10}$$

By (3.8):

$$|p - p_2| = t|p_2 - s_2| , \; |p - s_1| = t|p_1 - s_1| .$$

Thus by (3.9):

$$|p - p_2| \le |p - s_1| . \tag{3.11}$$

By (3.10), the triangle inequality and (3.11) the following chain of inequalities holds,

$$|p_1 - s_1| \le |p_1 - p_2| \le |p_1 - p| + |p - p_2| \le |p_1 - p| + |p - s_1| = |p_1 - s_1| , \tag{3.12}$$

the last equality in (3.12) is due to $p \in [p_1, s_1]$. Thus there are equalities in (3.12), leading to

$$|p_1 - s_1| = |p_1 - p_2| , \; |p_1 - p_2| = |p_1 - p| + |p - p_2| , \tag{3.13}$$

from which we conclude $p \in [p_1, p_2]$. Since $p \in [p_1, s_1]$ and $|p_1 - s_1| = |p_1 - p_2|$ it follows that

$$p_2 = s_1 ,$$

and consequently by (3.8): $p_1 = s_2$. □

We now formulate the conditions of Proposition 3.3 in terms of metric faces and the corresponding Voronoi sites.

<u>Proposition 3.3</u>':

Let A, B be simple polygons and F_1, F_2 be metric faces, $F_1 \subset A, F_2 \subset B$. The metric faces F_1, F_2 are metric connected if and only if one of the conditions holds (see Figure 3.5):

1. The Voronoi sites $S(F_1), S(F_2)$ are points and $S(F_1) \in \partial F_2, S(F_2) \in \partial F_1$.
2. The Voronoi sites $S(F_1), S(F_2)$ are a point and an open segment respectively, and $S(F_1) \in \partial F_2$, $\Pi_{S(F2)}(S(F_1)) \in \partial F_1$.
3. The Voronoi sites $S(F_1), S(F_2)$ are segments supported by parallel lines l_1, l_2 respectively, and $\Pi_{S(F2)}(S(F_1) \cap \partial F_2) \cap \partial F_1 \neq \phi$ (or the symmetric condition with the roles of 1, 2 replaced).

Proof:

Assume that condition 1 holds and denote $p_2 = S(F_1)$, $p_1 = S(F_2)$. Then by condition 1, $p_1 \in F_1$, $p_2 \in F_2$ and $\Pi_{S(F1)}(p_1) = p_2$, $\Pi_{S(F2)}(p_2) = p_1$, and by Proposition 3.3, F_1, F_2 are metric connected.

Now assume that condition 2 holds, denote $p_2 = S(F_1)$ and $p_1 = \Pi_{S(F2)}(p_2)$. We have $p_1 \in F_1$, $p_2 \in F_2$ and $\Pi_{S(F1)}(p_1) = p_2$, $\Pi_{S(F2)}(p_2) = p_1$, therefore by Proposition 3.3 F_1, F_2 are metric connected.

Next assume that condition 3 holds. There is a point $p_1 \in \Pi_{S(F2)}(S(F_1) \cap \partial F_2) \cap \partial F_1$. Let $p_2 \in S(F_1) \cap \partial F_2$ such that $p_1 = \Pi_{S(F2)}(p_2)$. Since the lines l_1, l_2 are parallel, $\Pi_{S(F2)}(p_2) = p_1$, and by Proposition 3.3 the metric faces F_1, F_2 are metric connected.

To prove the other direction, assume F_1, F_2 to be metric connected. By Proposition 3.3, there are points $p_1 \in \partial F_1$ and $p_2 \in \partial F_2$ such that $\Pi_{S(F1)}(p_1) = p_2$ and $\Pi_{S(F2)}(p_2) = p_1$.

Assume the corresponding Voronoi sites $S(F_1), S(F_2)$ to be points, then $S(F_1) = \Pi_{S(F1)}(p_1) = p_2 \in \partial F_2$, $S(F_2) = \Pi_{S(F2)}(p_2) = p_1 \in \partial F_1$, therefore condition 1 is satisfied.

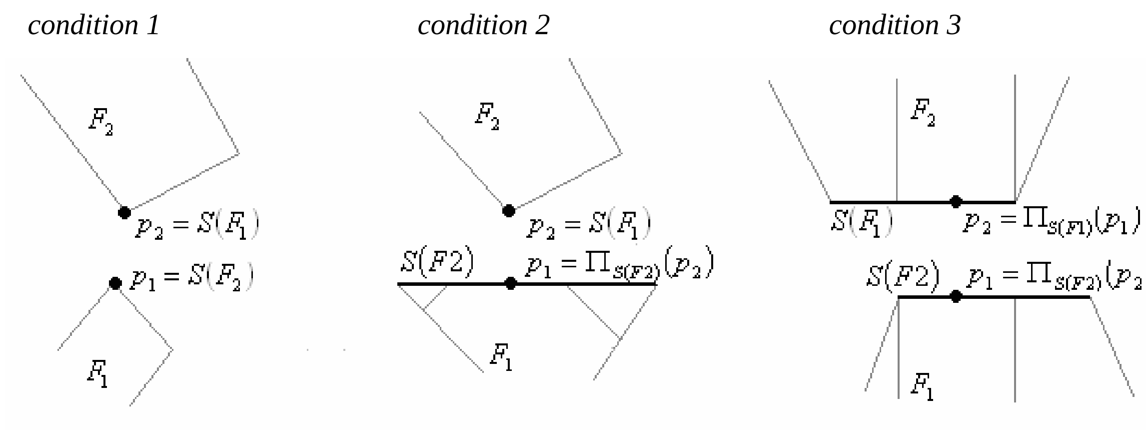


Figure 3.5: Illustration for Proposition 3.3'

Now assume $S(F_1), S(F_2)$ to be a point and a segment respectively, then $S(F_1) = \Pi_{S(F1)}(p_1) = p_2 \in \partial F_2$, and $\Pi_{S(F2)}(S(F_1)) = \Pi_{S(F2)}(p_2) = p_1 \in \partial F_1$, therefore condition 2 is satisfied.

Next assume $S(F_1), S(F_2)$ to be segments, supported by the lines l_1, l_2 respectively. Since the line supporting the segment $[p_1, p_2]$ is perpendicular to both lines l_1, l_2, the lines l_1, l_2 are parallel. The point $p_2 \in \partial F_2$ and also $p_2 = \Pi_{S(F1)}(p_1)$, namely $p_2 \in S(F_1) \cap \partial F_2$. The point $p_1 \in \partial F_1$ and also $p_1 = \Pi_{S(F2)}(p_2)$, namely $p_1 \in \Pi_{S(F2)}(S(F_1) \cap \partial F_2) \cap \partial F_1$ and condition 3 is satisfied. □

Finally we derive necessary and sufficient conditions for the metric connectedness of a metric face with a connected component of $A \cap^* B$.

Proposition 3.4: Let A, B be simple polygons and F_1, F_2 be metric elements: $F_1 \subset MF_{\partial B}(A)$, $F_2 \in MF(A \cap B)$. The metric elements F_1, F_2 are metric connected if and only if the set $F_1 \cup F_2$ is connected.

Proof:

Assume F_1, F_2 to be metric connected. Then $\Psi_t(F_1) \cap \Psi(F_2) \neq \phi$. By (3.1) $\Psi_t(F_2) = F_2$, so there is a point $p \in F_1$ such that,

$$tp + (1-t)\Pi_{S(F1)}(p) \in F_2 \subset A \cap B.$$

Then

$$p = \Pi_{S(F1)}(p),$$

from which follows that $p \in F_2$, and therefore $F_1 \cap F_2 \neq \phi$.

To prove the other direction, assume that there is a point $p \in F_1 \cap F_2$, in particular $p \in A \cap B$, so

$$tp + (1-t)\Pi_{S(F1)}(p) = p \in \Psi_t(F_1) \cap \Psi_t(F_2) \qquad \square$$

Now we can give necessary and sufficient conditions for connectedness of the metric average of two simple polygons in terms of the metric elements.

Definition 3.5: For A, B be simple polygons, the *metric connectivity graph* is a graph $G = \{V, E\}$, with the set of vertices $V = MF(A, B)$ and the set of edges E consisting of edges connecting pairs of vertices representing metric connected elements.

Proposition 3.6: Let A, B be simple polygons, the metric average $A \oplus_t B$ is connected if and only if the corresponding metric connectivity graph G is connected (there exists a path between all pairs of vertices of G).

Proof:

Let the metric connectivity graph be connected, and let $p_1, p_2 \in A \oplus_t B$, $p_1 \in \Psi_t(F')$ and $p_2 \in \Psi_t(F'')$, where F', F'' are metric elements.

If $F' = F''$, then p_1, p_2 belong to the same conic polygon, so there is a path connecting between them.

If $F' \neq F''$, then there is a path on the metric connectivity graph connecting the vertices representing F' and F'', i.e. there is a chain of metric elements $\{F_1, \ldots F_n\}$ such that $F_1 = F'$, $F_n = F''$ and $\Psi_t(F_i) \cup \Psi_t(F_{i+1})$ is a connected set for $1 \leq i \leq n-1$.

Thus the set

$$U = \bigcup_{i=1}^{n} \Psi_t\left(F_i\right),$$

is a connected set, which is a union of closed path-connected sets. Therefore by Remark 1.18 there is path connecting between p_1 and p_2, and the set $A \oplus_t B$ is connected.

To prove the other direction, assume the set $A \oplus_t B$ to be connected and let v_1, v_2 be vertices on the metric connectivity graph representing the metric elements F_1, F_2 respectively. Choose two points $p_1 \in F_1$ and $p_2 \in F_2$. By the assumption of connectedness of the set $A \oplus_t B$ and by Remark 1.18, there is a path τ connecting p_1 with p_2. By the definition of the metric connectivity graph, if τ crosses the boundary between $\Psi_t\left(F_i\right)$ and $\Psi_t\left(F_j\right)$ where F_i, F_j are metric elements, there is an edge on the metric connectivity graph between the corresponding vertices. Thus the vertices v_1, v_2 are connected by a path on the metric connectivity graph. □

Chapter 4: Complexity analysis

In this chapter we discuss the run-time complexity bounds for our algorithm and the combinatorial complexity of the obtained set. The complexity depends on the complexity of the underlying algorithms for the computation of segment Voronoi diagrams and planar arrangements. We evaluate the complexity of our algorithm based on the well-known complexity bounds for these geometric algorithms and structures. We use output sensitive bounds for the complexity of arrangement algorithms [3, 10].

The complexity bounds are based on the notion of the *combinatorial complexity of a planar arrangement*, which is the sum of the number of vertices, the number of edges, and the number of faces in the arrangement, and on the following results:

1. The segment Voronoi diagram of n linear segments can be computed in $O(n \log n)$ run-time complexity and the combinatorial complexity of the resultant diagram is $O(n)$ (see e.g. [2, 11]).
2. The maximum combinatorial complexity of an arrangement of n conic segments is $O(n^2)$ [10].
3. The arrangement of n segments (linear or conic) that intersect in k points can be computed in $O((n+k)\log n)$, where k is bounded by $O(n^2)$ [10].
4. Let $A(C_1)$, $A(C_2)$ be planar arrangements of complexity n_1 and n_2 respectively, and let $n = n_1 + n_2$. The overlay of $A(C_1)$ and $A(C_2)$ can be computed in $O((n+k)\log n)$ run-time complexity, where k - the combinatorial of the overlay, is bounded by $O(n^2)$ [3]. It is well known that the Boolean set operations on polygons can be implemented through overly of planar arrangements, so the same bounds hold for Boolean set operations on conic polygons.

We use the framework of metric elements defined in chapter 3 and apply the results about connectedness of the metric average of two simple polygons to evaluate the complexity of the obtained set.

Proposition 4.1: Let A, B be simple polygons and let n be the sum of the number of vertices in A and the number of vertices in B. In the notation of Chapter 3, consider the collections of boundary conic segments:

$$C = \bigcup_{F \in MF(A,B)} \{\tau \,:\, \tau \in \partial F\},$$

and

$$T = \bigcup_{F \in MF(A,B)} \{\tau \,:\, \tau \in \partial(\Psi_t(F))\}.$$

Let $A(C)$ and $A(T)$ be the arrangements defined by segments in C and T respectively, and let k be the combinatorial complexity of $A(C)$. Then k is $O(n^2)$, and the combinatorial complexity of $A(T)$ is $O(k)$.

Proof:

Since $A(C)$ is partial to the arrangement defined by the segments in $\partial A, \partial B, VD_{\partial A}$ and $VD_{\partial B}$, and since the number of segments within $\partial A, \partial B, VD_{\partial A}, VD_{\partial B}$ is $O(n)$, k is bounded by $O(n^2)$.

In the following we show that the combinatorial complexity of $A(T)$ is $O(k)$. Namely, we show that the number of intersection points (vertices) of $A(T)$ is $O(k)$ and also that the number of edges of $A(T)$ is $O(k)$ (and therefore the number of faces is also $O(k)$).

For each $\tau' \in T$, there is $\tau \in C$ such that τ' is the mapping of τ under the metric average. Abusing the notation of (3.1) we denote

$$\tau' = \Psi_t(\tau),\; \tau = \Psi_t^{-1}(\tau').$$

So we can rewrite T as:

$$T = \{\Psi_t(\tau) \,:\, \tau \in C\},$$

and the number of conic segments in T is $O(k)$.

For any intersection point in $A(T)$, we consider two conic segments in T which intersect at this point, and denote them by $\tau_1', \tau_2' \in T$. Let $\tau_i = \Psi_t^{-1}(\tau_i')$, $i = 1,2$ where $\tau_i \in \partial F_i$, $F_i \in MF(A,B)$. Note that any pair of metric elements F_1, F_2 satisfies one of the following conditions:

1. $F_1 = F_2$
2. $F_1 \neq F_2$, F_1, F_2 are connected components of $A \bigcap B$.
3. $F_1 \neq F_2$, F_1, F_2 are metric faces contained in the same polygon.
4. $F_1 \neq F_2$, F_1, F_2 are metric faces contained in different polygons.
5. $F_1 \neq F_2$, F_1 is a metric face and F_2 is a connected component of $A \bigcap B$.

For each of these five cases we bound the number of possible intersection points and the number of edges created by these points. Notice that an intersection point can be the intersection of more than one pair of segments, and consequently it can be attributed to more than one case.

We count first all intersection points in $A(T)$ between two segments in T originating from the same metric element (case 1). The boundary of a metric element is a simple closed chain of conic segments, so is its mapping under the metric average. So in this case, the intersection point of τ_1', τ_2' is the common endpoint. To each segment in T correspond two intersection points of this type and each such intersection point corresponds to two conic segments, then the number of intersection points of this type is equal to the number of segments in T, namely $O(k)$. Note that since each conic segment in T participates in a closed chain of conic segments which is the mapping under the metric average of the corresponding metric element, any common endpoint of two segments in T is counted in this case (see Figure 4.1). Since intersection points of this type correspond to common endpoints of conic segments that are already in T, no new edges are created.

It is clear that if F_1, F_2 are different connected components of $A \bigcap B$ (case 2), then since $\Psi_t(F_i) = F_i$, $i = 1.2$, any $\tau_1' \in \partial(\Psi_t(F_1))$ does not intersect any $\tau_2' \in \partial(\Psi_t(F_2))$.

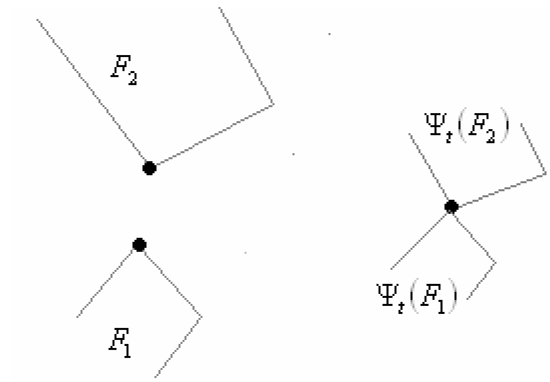


Figure 4.1: A common endpoint of more than two segments in T is already counted in its corresponding closed chains.

Now we count all intersection points in $A(T)$ between two conic segments in T originating from two metric faces contained in the same polygon (case 3). The images under the metric average of two such metric faces are either disjoint or have a common edge or vertex. In both cases no new vertices or edges are added to counting.

Next we count all intersection points in $A(T)$ between two segments in T originating from two metric faces contained in different polygons (case 4). Notice that Propositions 3.3 and 3.3' apply in this case. If τ_1', τ_2' only have a common endpoint, it is already counted in case 1. Otherwise τ_1', τ_2' intersect in the interior of at least one of them. In this case either condition 2 or condition 3 of Proposition 3.3' holds. First let us consider the case when condition 2 of Proposition 3.3' holds. Note that a vertex of one of the input polygons necessarily participates in such an intersection, and also that a vertex can participate in only one such an intersection. Consequently there are at most n intersection points of this type and at most $O(n)$ edges can be added to $A(T)$ (clearly n is $O(k)$). Next assume that condition 3 of Proposition 3.3' holds. Then the segments τ_1, τ_2 are parts of $S(F_2), S(F_1)$ respectively, which are parallel segments, and therefore τ_1', τ_2' are collinear. Since the number of vertices and edges in the arrangement of any m collinear segments is $O(m)$, the number of intersection points of this type is $O(k)$, and $O(k)$ edges can be added to $A(T)$.

Finally we count all intersection points between two segments, when one segment originates from a metric face and the other from a connected component of $A \cap B$. Notice that Proposition 3.4 applies in this case. If τ_1', τ_2' only have a common endpoint, it is already counted in condition 1. Otherwise $\tau_1' = \tau_2'$, and no intersection points are added by this case.

Summarizing the above five cases, we see that the number of intersection points (vertices) of $A(T)$ is $O(k)$, and also the number of edges of $A(T)$ is $O(k)$. Since the number of faces of a planar arrangement is bounded by the number of edges, the combinatorial complexity of $A(T)$ is $O(k)$. □

Corollary 4.3: Let A, B be simple polygons and let *n* be the sum of the number of vertices in *A* and the number of vertices in *B*. The combinatorial complexity of $A \oplus_t B$ with $t \in [0,1]$ is $O(n^2)$.

Proof:

Let

$$T = \bigcup_{F \in MF(A,B)} \{\tau \ : \ \tau \in \partial\Psi_t(F)\},$$

by Proposition 4.1, the complexity of the arrangement $A(T)$ is $O(n^2)$. Since the boundary $A \oplus_t B$ is partial to $A(T)$, its complexity is also $O(n^2)$.

Proposition 4.4: Let A, B be simple polygons and let *n* be the sum of the number of vertices in *A* and the number of vertices in *B*. Let *k* be the number of segments in the overlay of the arrangements representing the sets $\partial A, \partial B, VD_{\partial A}$ and $VD_{\partial B}$. Then the run-time complexity of the computation of the metric average $A \oplus_t B$ by Algorithm 2.3 is $O((n+k)\log n)$, where k is $O(n^2)$.

Proof:

Since the number of segments in each set within $\partial A, \partial B, VD_{\partial A}$ and $VD_{\partial B.}$ is bounded by $O(n)$, *k* is bounded by $O(n^2)$.

We start with the complexity of step 1 of Algorithm 2.3. It is the computation of Boolean set operations on the sets A, B. The complexity of the computation is bounded by $O((n+k)\log n)$ and the complexity of each set within $A \cap B$, $A \setminus B$, $B \setminus A$ is bounded by k.

Next we evaluate the complexity of Algorithm 2.2:

Step 1 of Algorithm 2.2 is the computation of the segment Voronoi diagram induced by less than n segments and can be completed in $O(n \log n)$. The complexity of the result is bounded by $O(n)$.

Step 2 of Algorithm 2.2 is an overlay of the arrangement representing the set $A \setminus B$ with the arrangement representing $VD_{\partial B}$. Notice that the complexity of the resulting arrangement is bounded by k, so step 2 can be completed in $O((n+k)\log n)$.

In Step 3 of Algorithm 2.2 we perform constant-time (by Remark 2.2) operations on edges of the resulting arrangement overlay from step 2. This can be done in $O(k)$ and the size of the result is $O(k)$.

We return now to Algorithm 2.3. Steps 2 and 3 are performed by Algorithm 2.2. In step 4 we compute the union of the resulting sets of the computations in steps 1, 2 and 3. The computation is based on the computation of the overlay of the corresponding arrangements. By Proposition 4.1, the combinatorial complexity of the resulting arrangement is $O(k)$, therefore the run-time complexity of the computations is bounded by $O((n+k)\log n)$, with $k = O(n^2)$. □

Chapter 5: The modified metric average of two simple polygons

In this chapter we point out an artifact produced by the metric average of two simple polygons and introduce a modification of the metric average that avoids this artifact.

5.1 Definition and basic properties of the modified metric average

Probably the most common type of an artifact of the metric average of two simple polygons is the one demonstrated by Figure 5.1. It occurs near the common endpoint of two adjacent segment Voronoi sites (see Remark 1.13), when the exterior angle between the sites is less than π. For two such segment Voronoi sites, the Voronoi face corresponding to the common endpoint is of zero area (see Remark 1.12). Therefore the points lying on the boundary between two corresponding metric faces are equidistant from both segment sites. Consequently they are mapped toward both sites, creating a "split" in the obtained set. Thus when computing the metric average of a metric face relative to the corresponding segment Voronoi site, we would like to treat differently an edge separating the face from a neighboring metric face corresponding to an adjacent segment Voronoi site. These observations are formalized by the following definition.

Definition 5.1: Let F be a metric face: $F \in MF_{\partial B}\left(A \setminus B\right)$ and let $\partial F = \{\tau_1,...\tau_n\}$ be a collection of conic segments. We define $\tau \in \partial F$ as a *problematic* edge if the following three conditions hold:

1. $S(F)$ is a segment.
2. τ separates F from another metric face F' such that $S(F')$ is a segment.
3. $S(F)$ and $S(F')$ are not collinear and have a common endpoint.

We denote by $J_F(\tau)$ the common endpoint of $S(F)$ and $S(F')$.

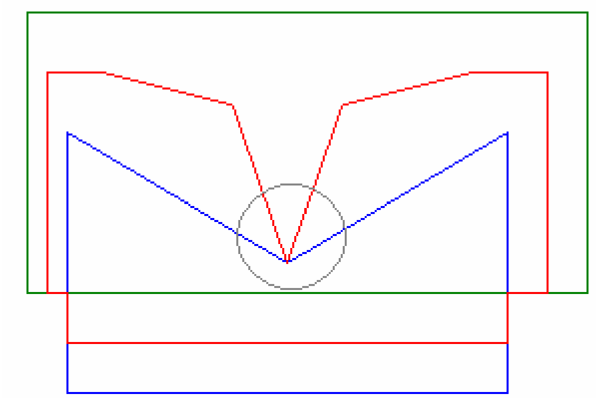


Figure 5.1: An artifact region in the metric average is marked by a circle.

Notice that if τ is a problematic segment, then τ is a part of segment to segment bisector, so it is necessarily a linear segment. Now we define an operation on a Voronoi face boundary edges, which will enable us to modify the metric average of two simple polygons.

Definition 5.2: Let F be a metric face, $F \in MF_{\partial B}(A \setminus B)$ and let $\partial F = \{\tau_1,...\tau_n\}$ be a collection of conic segments

Define the operation $ME_t(\tau, S(F))$ for any problematic edge $\tau = [p_b, p_e] \in \partial F$ by

$$ME_t(\tau, S(F)) = [M_t(p_b, S(F)), M_t(p_b, J_F(\tau)), M_t(p_e, J_F(\tau)), M_t(p_e, S(F))],$$

where $[q_{1,} q_2, q_3, q_4]$ denotes the polyline through $q_{1,} q_2, q_3, q_4$ (see Figure 5.2). For all other edges define

$$ME_t = M_t(\tau, S(F)).$$

With this modification we can define the modified metric average.

Definition 5.3: For *A, B* be simple polygons and $t \in [0,1]$, the t-weighted *modified metric average* $A \oplus_t' B$ is defined as in (2.11) with $M_t(\partial F, S(F))$ replaced by

$$M_t'(\partial F, (S(F))) = \bigcup_{\tau \in \partial F} ME_t(\tau, S(F)). \tag{5.1}$$

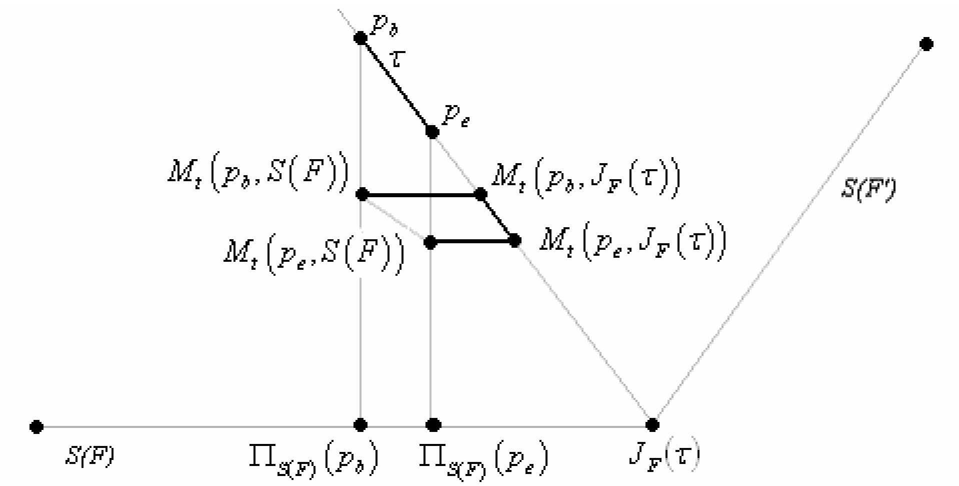


Figure 5.2: The polyline replacing the metric average of a problematic segment.

Given a problematic segment $\tau = [p_b, p_e] \in \partial F$ and a segment Voronoi site $S(F)$, the chain of linear segments $[M_t(p_b, S(F)), M_t(p_b, J_F(\tau)), M_t(p_e, J_F(\tau)), M_t(p_e, S(F))]$ can be computed using (2.13) and (2.14) . So the computation of the modified metric average is similar to the computation of the metric average. It can be done by Algorithm 2.3 with Algorithm 2.1 replaced by the following algorithm:

<u>Algorithm 5.1</u>: Computation of $M'_t(\partial F, S(F))$ for a metric face F

If $S(F)$ is a point

use Algorithm 2.1 to compute $M'_t(\partial F, S(F))$

Else for each conic segment τ in ∂F

a. If τ is problematic

use (2.13) and (2.14) to compute the polyline $[M_t(p_b, S(F)), M_t(p_b, J_F(\tau)), M_t(p_e, J_F(\tau)), M_t(p_e, S(F))]$

Else

use (2.26) to compute $ME_t(\tau, S(F))$

b. Add the result of the computation in step (a) to the collection of conic segments already computed

c. Return the resulting collection of segments as the boundary of a conic polygon

Remark: Notice that in the general case the linear segments $[M_t(p_b,S(F)), M_t(p_b,J_F(\tau))]$, $[M_t(p_e,J_F(\tau)), M_t(p_e,S(F))]$ have irrational coefficients. Since the arrangement algorithms in the CGAL library do not support segments with irrational coefficients [9], our implementation of the modified metric average approximated the segment Voronoi diagram by linear segments with rational endpoints. This type of approximate treatment is not needed in the computation of the metric average (see text below (2.4)).

Next we discuss some properties of the modified metric average.

<u>Proposition 5.4</u>: $A \oplus_t B \subseteq A \oplus'_t B$.

Proof: Let $F \in MF_{\partial B}(A \setminus B)$ and $\tau = [p_b, p_e]$ be a problematic segment of ∂F. By Remark 1.11, the points $M_t(p_s, J_F(\tau))$ and $M_t(p_t, J_F(\tau))$ satisfy:

$$M_t(p_b, J_F(\tau)) = M_t(p_b, S(F)) = p_s = J_F(\tau) \text{ or } M_t(p_b, J_F(\tau)) \in \sim M_t(F, S(F))$$

$$M_t(p_e, J_F(\tau)) = M_t(p_e, S(F)) = p_e = J_F(\tau) \text{ or } M_t(p_e, J_F(\tau)) \in \sim M_t(F, S(F)),$$

where $\sim A$ is a complement of A.

So the polyline $[M_t(p_b,S(F)), M_t(p_b,J_F(\tau)), M_t(p_e,J_F(\tau)), M_t(p_e,S(F))]$ lies outside $M_t(F,S(F))$ except for the endpoints, which coincide with the endpoints of $M_t(\tau,S(F))$. Thus

$$M_t(F,S(F)) \subseteq M'_t(F,S(F)).$$

Consequently:

$$M_t(A,B) \subseteq M'_t(A,B) \text{ and } A \oplus_t B \subseteq A \oplus'_t B \qquad \square$$

<u>Proposition 5.5</u>: Let a point $p \in M'_t(A,B)$, then:

$$\operatorname{dist}(p,B) \le t \operatorname{haus}(A,B). \qquad (5.2)$$

Proof: It is enough to show (5.2) for the boundary points of the set $M'_t(F,S(F))$ for any metric face F. Let $p \in ME_t(\tau,S(F))$, if τ is not problematic then

$$ME_t(\tau,S(F)) = M_t(\tau,S(F)).$$

Therefore by (1.5):

$$p \in M_t(A,B) \subseteq A \oplus_t B \text{ and } \operatorname{dist}(p,B) \le t \operatorname{haus}(A,B).$$

Otherwise $\tau = [p_s, p_t]$ is a problematic segment. By Thales' (intercept) theorem of Euclidean geometry, the segment Voronoi site $S(F)$ and the segments $[M_t(p_b, S(F)), M_t(p_b, J_F(\tau))]$, $[M_t(p_e, J_F(\tau)), M_t(p_e, S(F))]$ are parallel. Without loss of generality assume that:

$$\operatorname{dist}(p_b, S(F)) \le \operatorname{dist}(p_e, S(F)).$$

Then for each point $p \in ME_t(\tau, S(F))$:

$$\operatorname{dist}(p, S(F)) \le \operatorname{dist}(M_t(p_b, S(F)), S(F)) \le t \operatorname{haus}(A, B).$$

The last inequality is due to (1.5) □

In analogy with (1.5), we would like the modified metric average to satisfy:

$$\operatorname{haus}(A \oplus_t' B, B) \le t \operatorname{haus}(A, B)\ ,\ \operatorname{haus}(A \oplus_t' B, A) \le (1-t)\operatorname{haus}(A, B) \tag{5.3}$$

But in the general case, for a point $p \in M_t'(A, B)$:

$$\operatorname{dist}(p, A) \le (1-t)\operatorname{haus}(A, B)$$

is not true and (5.3) doesn't hold. However in the special case, when one of the polygons is contained in the other, the following property holds.

<u>Proposition 5.6</u>: Let *A*, *B* be simple polygons such that $B \subset A$, then the modified metric average $A \oplus_t' B$ satisfies:

$$\operatorname{haus}(A \oplus_t' B, B) \le t \operatorname{haus}(A, B) \tag{5.4}$$

$$\operatorname{haus}(A \oplus_t' B, A) \le (\max\{t, 1-t\}) \operatorname{haus}(A, B). \tag{5.5}$$

Proof:

By Definition 5.3 and since $B \subset A$,

$$A \oplus_t' B = (A \cap B) \cup M_t'(A \setminus B, B).$$

Also, for $p \in B$,

$$\operatorname{dist}(p, A \oplus_t' B) = 0.$$

Now for $p \in A \oplus_t' B$, we get by Proposition 5.5,

$$\operatorname{dist}(p, B) \le t \operatorname{haus}(A, B),$$

and (5.4) follows from the above two relations.

To prove (5.5), let $p \in A$. By Proposition 5.4 and (1.5), we get

$$\mathrm{dist}\left(p, A \oplus_t' B\right) \le \mathrm{dist}\left(p, A \oplus_t B\right) = (1-t)\mathrm{haus}(A,B). \qquad (5.6)$$

Now consider $p \in A \oplus_t' B$, by the assumption $B \subset A$ and by Proposition 5.5,

$$\mathrm{dist}(p, A) \le \mathrm{dist}(p, B) \le t\mathrm{haus}(A,B), \qquad (5.7)$$

and (5.5) follows from (5.6) and (5.7) . □

In the case $t = 1/2$ and $B \subset A$, (5.5) is equivalent to (5.3).

To obtain from the modified metric average an averaging operation which satisfies (5.3), we can intersect the modified metric average with the set of points that are close enough to the original polygons.

Definition 5.7: Let $A \in K_n$ and $r \in R, r > 0$. The union of the balls of radius r centered at points of *A*, denoted by $Bl(A,r)$ is the compact set $\left\{p \in R^n : \mathrm{dist}(p, A) \le r\right\}$.

The set $Bl(A,r)$ is the Minkowski sum of *A* with the compact ball of radius *r* centered at the origin.

Definition 5.8: Let *A, B* be simple polygons. The t-weighted *restricted modified metric average* of *A* with *B* is

$$A \oplus_t'' B = A \oplus_t' B \cap Bl\left(A, (1-t)\mathrm{haus}(A,B)\right) \cap Bl\left(B, t\,\mathrm{haus}(A,B)\right). \qquad (5.8)$$

Proposition 5.9: Let *A, B* be simple polygons, then $A \oplus_t'' B$ satisfies

$$\mathrm{haus}\left(A \oplus_t'' B, A\right) \le (1-t)\mathrm{haus}(A,B),\ \mathrm{haus}\left(A \oplus_t'' B, B\right) \le t\,\mathrm{haus}(A,B). \qquad (5.9)$$

Proof: By (1.5) and by Proposition 5.4:

$$A \oplus_t B \subseteq A \oplus_t'' B \qquad (5.10)$$

Let $p \in B$, then by (5.10) and (1.5),

$$\mathrm{dist}\left(p, A \oplus_t'' B\right) \le \mathrm{dist}\left(p, A \oplus_t B\right) \le t\,\mathrm{haus}(A,B).$$

Now for $p \in A \oplus_t'' B$, by Definitions 5.7 and 5.8,

$$\mathrm{dist}(p, B) \le t\,\mathrm{haus}(A,B).$$

Symmetric inequalities hold for A. □

5.2 Examples of the modified metric average

In this section we present examples of the modified metric average. The results are compared with the results produced by the metric average for the same input polygons.

Metric Average Modified Metric Average

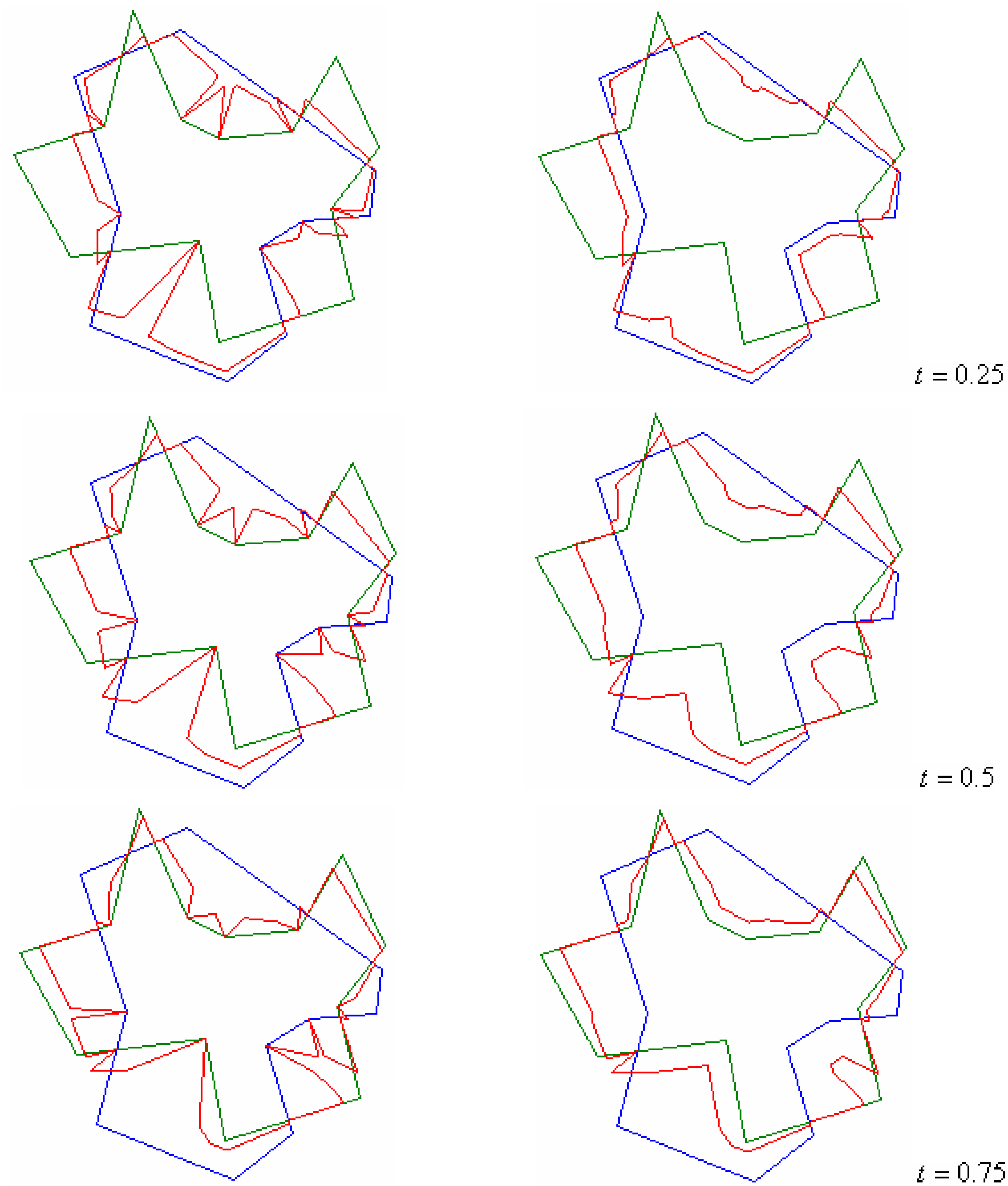


Figure 5.3: Examples of the modified metric average of two simple polygons compared to the metric average for different values of the averaging coefficient t.

Metric Average ($t = 0.5$)

Modified Metric Average ($t = 0.5$)

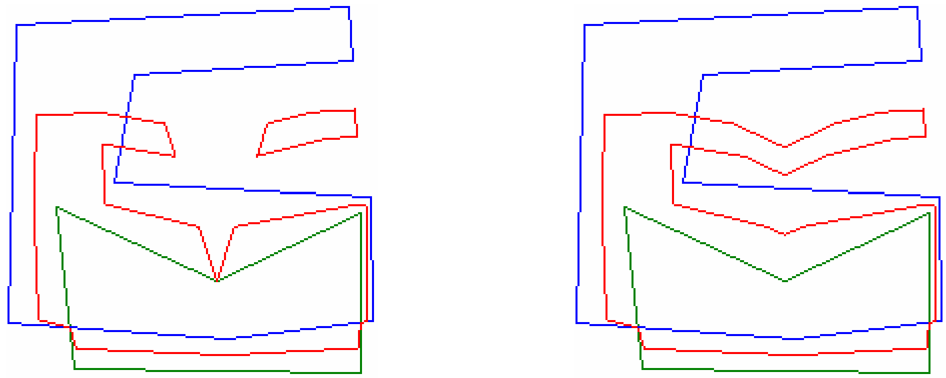


Figure 5.4: Modified metric average solves some cases of disconnectedness of the metric average

Metric Average ($t = 0.5$)

Modified Metric Average ($t = 0.5$)

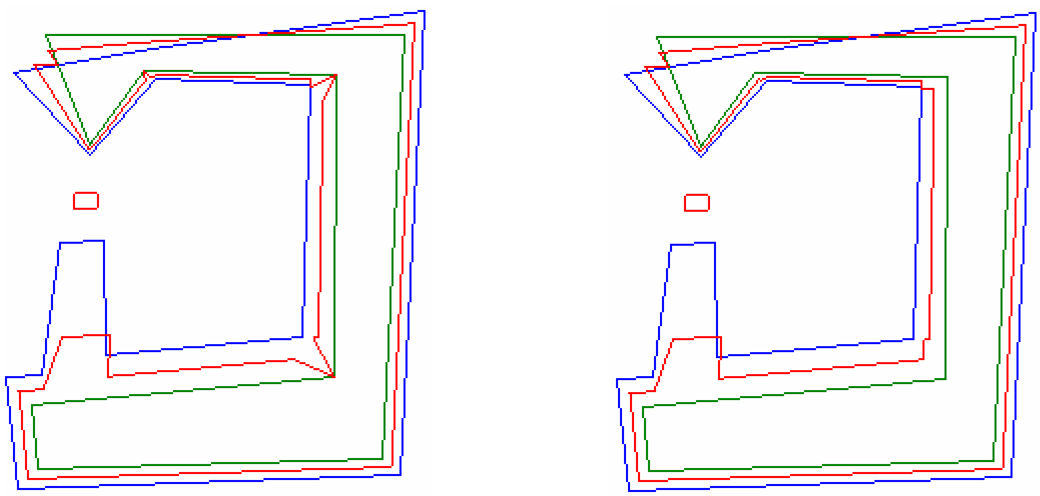


Figure 5.5: In some cases the both operations produce a disconnected result.

Metric Average ($t = 0.5$)

Modified Metric Average ($t = 0.5$)

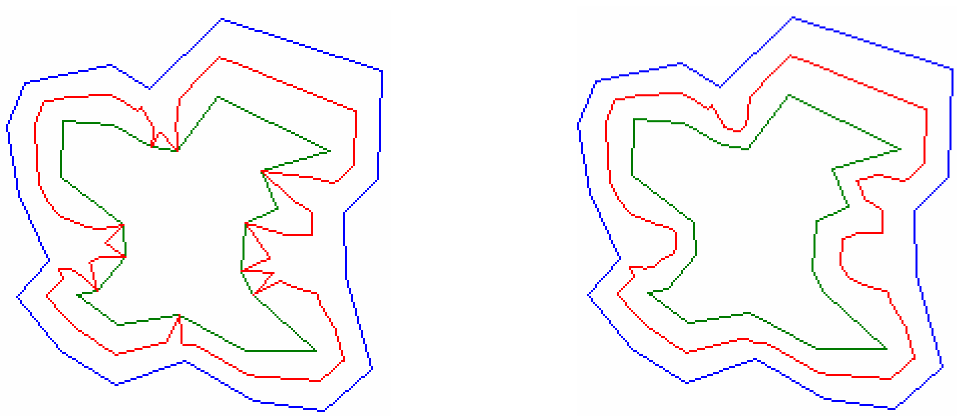


Figure 5.6: Modified metric average when one of the polygons is contained in the other.

Chapter 6: The metric average of two collections of simple polygons with holes

In this chapter we describe how the framework for the computation of the metric average of two simple polygons introduced in the previous chapters can be extended to the computation of the metric average of two sets, each consisting of a collection of simple polygons with holes.

6.1 Extension of the algorithm

First we introduce the type of sets to which the algorithm is extended. We use the definitions introduced in Chapter 1.

Definition 6.1: A set consisting of pairwise disjoint polygons with holes is termed a *simple polygonal set.*

The disjointness requirement is not a real limitation, since any collection of simple polygons with holes can be represented as a simple polygonal set.

Let the sets A, B be simple polygonal sets: $A = \{\widetilde{A}_1 \ldots \widetilde{A}_n\}$, $\{\widetilde{B}_1 \ldots \widetilde{B}_m\}$, with $\{\widetilde{A}_1 \ldots \widetilde{A}_n\}$, $\{\widetilde{B}_1 \ldots \widetilde{B}_m\}$ disjoint simple polygons with holes . A simple polygon with holes SPH is represented by a simple polygon P and a collection of simple polygons $\{H_i \mid 1 \le i \le k\}$, where P is the outer polygon and H_i are the holes. The boundary set of a simple polygon with holes is the union of the outer boundary with the boundaries of the holes. We introduce here the set of boundaries of SPH as

$$\partial(SPH) = \{\partial P, \partial H_1, \ldots, \partial H_k\}. \quad (6.1)$$

The boundary of a simple polygonal set is a union of the boundaries of its member polygons with holes i.e. a collection of linear segments. Thus the segment Voronoi diagram of the boundary of a simple polygonal set is well defined and we can apply (2.8) for the computation of metric average of two simple polygonal sets.

<u>Definition 6.2</u>: Let A, B be simple polygonal sets, $VD_{\partial B}$ be a Voronoi diagram induced by the boundary of B and F be a Voronoi face of $VD_{\partial B}$ such that $F \cap (A \setminus B) \neq \phi$. A connected component of $F \cap (A \setminus B)$ is called a *metric face* originating from F.

Definition 6.2 is consistent with Definition 2.1, but now in the general case a metric face is a simple conic polygon with holes (see Definition 1.5).

The metric average of two simple polygonal sets can be computed similarly to the computation of the metric average of two simple polygons. For Algorithms 2.2 and 2.3, the allowed input is now two simple polygonal sets, but the computations are essentially similar. We only have to change the computation of $M_t(\partial F, S(F))$ (Algorithm 2.1), now taking care of the possible holes inside a metric face. We present the extension of Algorithm 2.1 in Algorithm 6.1.

Let the metric face F be a polygon P with holes $\{H_1,...H_n\}$, the operation $p \to tp + (1-t)\Pi_{S(F)}(p)$ for $p \in F$, considered as function from F to R^2 is continuous and one-to-one. Therefore $M_t(F, S(F))$ is a conic polygon $M_t(P, S(F))$ with holes $\{M_t(H_1, S(F)),..., M_t(H_n, S(F))\}$, which can be computed by the following algorithm.

<u>Algorithm 6.1</u>: Computation of $M_t(\partial F, S(F))$ for a metric face F which is a simple conic polygon P with holes $\{H_1,...H_n\}$.

1. Use Algorithm 2.1 to compute $M_t(\partial P, S(F))$.
2. For each H in $\{H_1,...H_n\}$
   a. Use Algorithm 2.1 to compute $M_t(\partial H, S(F))$
   b. Add the result of step 2a to the collection of holes of $M_t(\partial P, S(F))$
3. Return the resulting polygon with holes.

Algorithm 6.1 is consistent with Algorithm 2.1 in the sense that if F is a simple conic polygon, then the computation is identical, thus the above algorithm can handle both simple conic polygons and simple conic polygons with holes.

If we replace the calls to Algorithm 2.1 in Algorithm 6.1 by calls to Algorithm 5.1, we get an algorithm for the computation of the modified metric average of two simple polygonal sets.

The extension of the algorithm from simple polygons to simple polygonal sets is supported by the CGAL library, as it contains algorithms for the regularized Boolean set operations of simple polygonal sets [8].

6.2 Examples of the metric average and the modified metric averages of two simple polygonal sets

As in the previous sections the boundaries of the input polygonal sets A, B are colored green and blue respectivlly, and the boundary of their metric average is colored red.

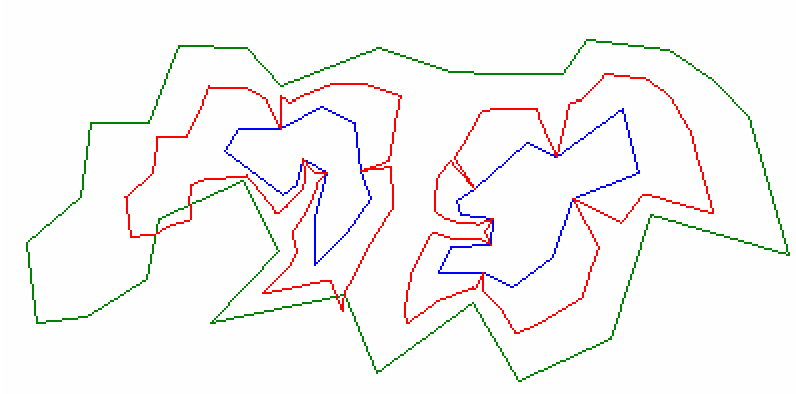


Figure 6.1: $A \oplus_{1/2} B$*, where* A *is a polygon and* B *consists of two polygons contained in* A*.*

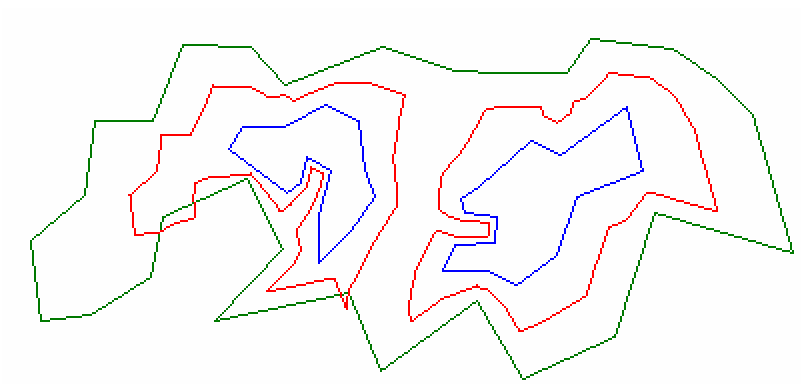


Figure 6.2: Modified metric average of the above example.

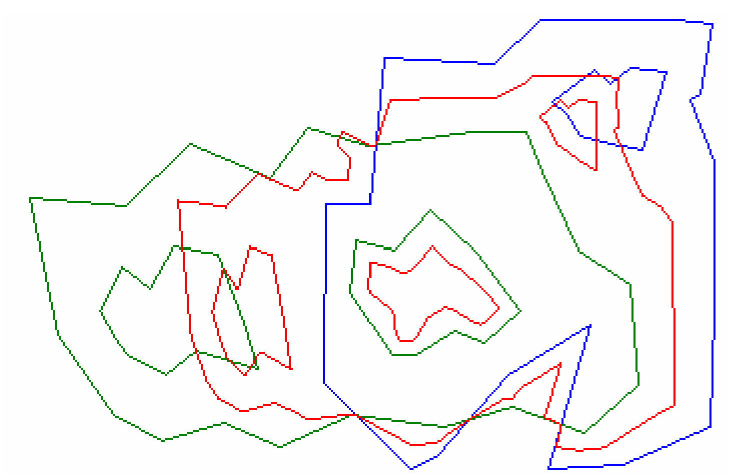


Figure 6.3: $A \oplus'_{1/2} B$*, where* A, B *are simple polygons with holes.*

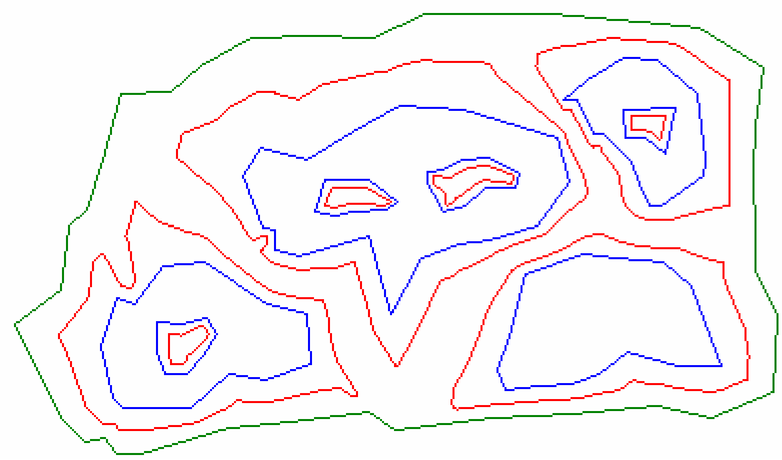


Figure 6.4: $A \oplus'_{1/2} B$, *where* A *is a polygon and* B *is a simple polygonal set contained in* A.

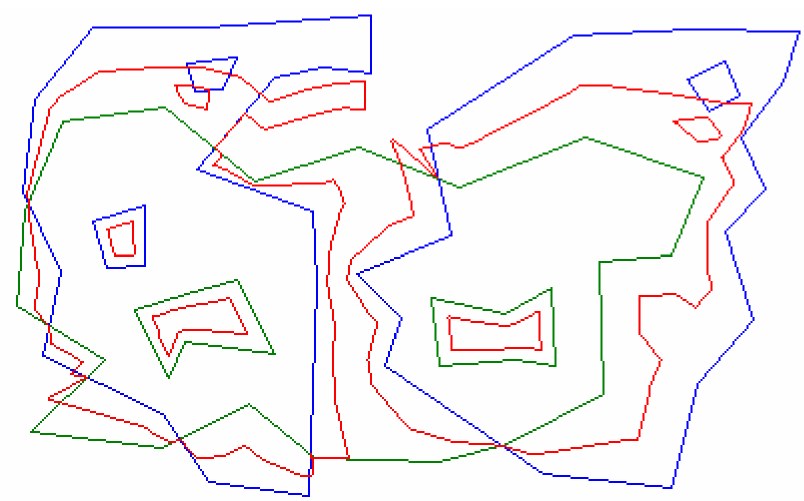


Figure 6.5: $A \oplus'_{1/2} B$, *where* A *is a polygon with holes and* B *consists of two polygons with holes.*

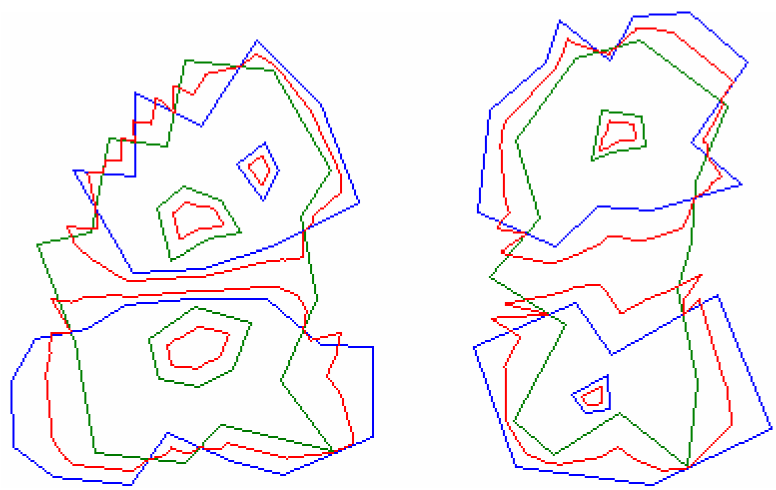


Figure 6.6: $A \oplus'_{1/2} B$, *where* A, B *are simple polygonal sets.*

Conclusions and future work

The algorithm for the computation of the metric average of two simple polygons and its extension to the computation of the metric average of two simple polygonal sets are an additional step toward the development and implementation of algorithms for the computation of the metric average of compact sets in R^2. Development and implementation of such algorithms will enable practical application of the metric average to approximation of set-valued functions [5, 7] in the 3D reconstruction problem.

The computational experiments suggest that in many cases the metric average is a reasonable averaging operation for simple polygonal sets, except for the artifacts which appear near adjacent segment Voronoi sites. When these artifacts are eliminated by using the modified metric average for simple polygonal sets, a good average of the original sets is obtained in many cases. The best geometric results are achieved when one of the sets is contained in the other. Even though the geometry of the metric average is in some cases complicated, it is of theoretical interest due to its properties under the Hausdorff distance [4, 5, 7].

The practical application of the metric average for the approximation of set-valued functions with images in R^n requires repeated computations of the metric averages. We have seen that in the general case the metric average of two simple polygons has conic edges, thus only one computation can be done by the algorithms introduced in this work. This problem can be solved by approximating the conic segments by chains of linear segments or by using the straight skeleton instead of the Voronoi diagram [2].

An issue that should be addressed is that in some cases the metric average and the modified metric average of two connected simple polygons are disconnected. In most cases it seems unreasonable that the average of two connected sets is a disconnected set. In this work we study the connectedness of the metric average of two simple polygons, but a modification to an averaging operation that avoids cases of unwanted disconnectedness is yet to be developed.

Even though compact two-dimensional sets can be approximated by simple polygonal sets, this approximation is not optimal in many senses. We suggest as a next step in the research of algorithms for the computation of the metric average of 2D compact sets the development and implementation of an algorithm for the computation of the metric average of two-dimensional compact sets with boundaries consisting of spline curves. A natural extension of this work is to develop and implement an algorithm for the computation of the metric average of two polyhedra.

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תקציר

בעבודה זו אנו מציגים אלגוריתם שמשתמש בדיאגרמת וורונוי של קטעים לחישוב הממוצע המטרי בין שני פוליגונים פשוטים במישור. הרעיון לשימוש בדיאגרמת וורונוי של קטעים הוא של א.ליפובטצקי. אנו מתארים את מימוש האלגוריתם ומציגים אוסף של דוגמאות חישוביות. בהתבסס על האלגוריתם אנו דנים בקשירות הממוצע המטרי בין שני פוליגונים פשוטים. בנוסף אנו מצביעים על תכונה לא רצויה של הממוצע המטרי בין שני פוליגונים פשוטים ומציעים שינוי לפעולת המיצוע שמתגבר על קושי זה. בהמשך העבודה אנו מתארים כיצד ניתן להרחיב את פעולת האלגוריתם לקבוצות שהן אוספים של פוליגונים פשוטים עם חורים.

תודות

ברצוני להודות לפרופ' נירה דין על הדרכתה בעבודת המחקר וכתיבת התיזה.

אני רוצה להודות למשפחתי על האהבה והתמיכה, ובמיוחד לאשתי, עינת, שעומדת לצידי תמיד.

אוניברסיטת תל-אביב

הפקולטה למדעים מדויקים

ע"ש ריימונד ובברלי סאקלר

בית הספר למדעי המתמטיקה

אלגוריתם לחישוב הממוצע המטרי
בין שני פוליגונים פשוטים
והרחבות

חיבור זה הוגש כחלק מהדרישות לקבלת תואר

מוסמך במדעים (M.Sc.) באוניברסיטת תל-אביב

על ידי

שי קלס

העבודה הוכנה בהדרכתה של

פרופ' נירה דין

אלול תשס"ח