Preliminaries

Preparatory reading. These books are intended for high-school students who like math. All three books are great, my personal favorite is the first one.

(1) R. Courant, H. Robbins, I. Stewart, What is mathematics, Oxford, 1996 (or earlier editions).
(2) T. W. Korner, The pleasures of counting, Cambridge U. Press, 1996.
(3) K. M. Ball, Strange curves, counting rabbits, and other mathematical explorations, Princeton University Press, 2003.

Reading.

(1) V. A. Zorich, Mathematical analysis, vol.1, Springer, 2004.
(2) D. Maizler, Infinitesimal calculus (in Hebrew).
(3) R. Courant and F. John, Introduction to calculus and analysis, vol.1, Springer, 1989 (or earlier editions).

You may also look at notes of Jon Aaronson who teaches this course in parallel with me. You may find helpful informal discussions of various ideas related to this course (as well to the other undergraduate courses) at the web page of Timothy Gowers:

`www.dpmms.cam.ac.uk/~wtg10/mathsindex.html`

I suppose that the students attend in parallel with this course the course "Introduction to the set theory", or the course "Discrete Mathematics". The notes (in Hebrew) of Moshe Jarden might be useful:

`www.math.tau.ac.il/~jarden/Courses/set.pdf`

Additional reading.

(1) E. Hairer, G. Wanner, Analysis by its history, Springer, 1996.
(2) A. Browder, Mathematical analysis. An introduction. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1996.
(3) W. Rudin, Principles of mathematical analysis, McGraw-Hill, 1976 (or earlier edition).

The first book gives a very interesting and motivated exposition of the main ideas of this course given in the historical order. Browder's and Rudin's books are more advanced textbooks which I recommend to the students who want to learn more.

Problem books. For those of you who are interested to try to solve more difficult and interesting problems and exercises, I strongly recommend to look at two excellent collections of problems:

(1) B. M. Makarov, M. G. Goluzina, A. A. Lodkin, A. N. Podkorytov, Selected problems in real analysis, American Mathematical Society, 1992.
(2) G. Polya, G. Szegö, Problems and theorems in analysis (2 volumes) Springer, 1972 (there are earlier editions).

Basic notations.

Symbols from logic.

$\vee$ or
$\wedge$ and
$\neg$ negation
$\Longrightarrow$ yields
$\Longleftrightarrow$ is equivalent to

Example: $(x^2 - 3x + 2 = 0) \Longleftrightarrow ((x = 1) \vee (x = 2))$

Quantifiers:

$\exists$ exists
$\exists!$ exists and unique (warning: this notation isn't standard)
$\forall$ for every

Set-theoretic notations.

$\in$ belongs
$\notin$ does not belong
$\subset$ subset
$\emptyset$ empty set
$\cap$ intersection of sets
$\cup$ union of sets
$\#(X)$ cardinality of the set X
$X \setminus Y = \{x \in X : x \notin Y\}$ complement to Y in X

Example: $(X \subset Y) := \forall x \ ((x \in X) \Longrightarrow (x \in Y))$

We shall freely operate with these notions during the course. Usually, the sets we deal with are subsets of the set of real numbers $\mathbb{R}$.

Subsets of reals:

$\mathbb{N}$ natural numbers (positive integers)
$\mathbb{Z}$ integers
$\mathbb{Z}_+$ non-negative integers
$\mathbb{Q}$ rational numbers
$\mathbb{R}$ real numbers
$[a, b] := \{x \in \mathbb{R} : a \le x \le b\}$ closed interval (one point sets are closed intervals as well)
$(a, b) := \{x \in \mathbb{R} : a < x < b\}$ open interval
$(a, b]$ and $[a, b)$ semi-open intervals

Some abbreviations.

`iff` "if and only if"
`wlog` "without loss of generality"
`RHS, LHS` "right-hand side", "left-hand side"
`qed` " end of the proof"[1]. Often is replaced by the box like this one: $\square$
$:=$ according to the definition (the same as $\stackrel{\mathrm{def}}{=}$)

[1] "quod erat demonstrandum" (in Latin), "which was to be demonstrated"

Basic Greek letters.

α	alpha
β	beta
γ, Γ	gamma
δ, Δ	delta
ε	epsilon
ζ	zeta
η	eta
θ, Θ	theta
ι	iota
κ	kappa
λ, Λ	lambda
μ	mu
ν	nu
ξ, Ξ	xi
π, Π	pi
ρ	rho
σ, Σ	sigma
τ	tau
υ, Υ	upsilon
φ, Φ	phi
χ	chi
ψ, Ψ	psi
ω, Ω	omega

Exercise: Translate from the Greek the word $\mu\alpha\theta\eta\mu\alpha\tau\iota\kappa\alpha$.

1. Real Numbers

1.1. **Infinite decimal strings.** All of you have an idea what are the real numbers. For instance, we often think of the real numbers as strings of elements of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ preceded by a sign (we write only a minus sign, the absence of the sign means that the sign is positive). A finite string of elements of this set followed by a decimal point followed by an infinite string of elements of this set. If the string starts with zeroes, they can be removed: $0142.35000... = 142.35$, if the string has an infinite sequence of nines, the last element which differs from nine should be increased by one, and then the nines should be replaced by the zeroes: $13.4999999... = 13.5000... = 13.5$. We call such strings *finite*.

Then we can define what is the sum, the product and the quotient of two such strings, and we can compare the strings. It is not completely obvious, but you've certainly learnt this in the high-school how to do this for finite strings:

Exercise 1.1.1. Write down the "algorithms" for addition, multiplication and comparison of two *finite* decimal strings.

One may prefer to operate with strings which consist of zeroes and ones only. In other civilizations, people used to operate with expansions with a different base, say $\{0, 1, 2, 3, 4, 5, 6, ..., 59\}$ (this base goes back to Babylon). Do they deal with *the same* set $\mathbb{R}$ of real numbers? How to formalize this question? and how to answer it?

1.2. **The axioms.** We know that it is possible to add and multiply real numbers. So let us write down the customary rules:

Axioms of addition $+$.

($+_1$) $\exists$ the null element 0 such that $\forall x \in \mathbb{R}$: $x + 0 = 0 + x = x$;
($+_2$) $\forall x \in \mathbb{R}$ $\exists$ an element $-x$ such that $x + (-x) = (-x) + x = 0$;
($+_3$) *associativity:* $\forall x, y, z \in \mathbb{R}$ $\; x + (y + z) = (x + y) + z$;
($+_4$) *commutativity:* $\forall x, y \in \mathbb{R}$ $\; x + y = y + x$.

In "scientific words" these axioms mean that $\mathbb{R}$ is an *abelian group*.

Axioms of multiplication $\cdot$.

($\cdot_1$) $\exists$ the unit element $1 \in \mathbb{R} \setminus \{0\}$ such that $\forall x \in \mathbb{R}$: $x \cdot 1 = 1 \cdot x = x$;
($\cdot_2$) $\forall x \in \mathbb{R} \setminus \{0\}$ $\exists$ the inverse element x^{-1} such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$;
($\cdot_3$) *associativity:* $\forall x, y, z \in \mathbb{R}$ $\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$
($\cdot_4$) *commutativity:* $\forall x, y \in \mathbb{R}$ $\; x \cdot y = y \cdot x$.

This group of the axioms means that the set $\mathbb{R} \setminus \{0\}$ with the multiplication is also an abelian group.

Relation between addition and multiplication is given by

Distributive axiom. $\forall x, y, z \in \mathbb{R}$ $\; (x + y) \cdot z = x \cdot z + y \cdot z$.

Exercise 1.2.1. Prove that $a \cdot 0 = 0$.

Any set K with two operations satisfying all these axioms is called a *field*. The fields are studied in the courses in algebra.

Exercise 1.2.2. Construct a finite field with more than two elements.

Axioms of order $\le$. Real numbers are equipped with another important structure: the order relation. Having two real numbers x and y we can always juxtapose them and tell whether they are equal or one of them is bigger than the other one. To make this formal, we need to check that the reals satisfy the third set of the axioms:

($\le_1$) $\forall x \in \mathbb{R}\ x \le x$;
($\le_2$) if $x \le y$ and $y \le x$, then $x = y$;
($\le_3$) if $x \le y$ and $y \le z$, then $x \le z$;
($\le_4$) $\forall x, y \in \mathbb{R}$ either $x \le y$ or $y \le x$.

These axioms say that $\mathbb{R}$ is a (linearly) ordered set. The next two axioms relate the order with addition and multiplication on $\mathbb{R}$:

($+, \le$) if $x \le y$, then $\forall z \in \mathbb{R}\ \ x + z \le y + z$;
($\cdot, \le$) if $x \ge 0$ and $y \ge 0$, then $x \cdot y \ge 0$.

Now, we can say that $\mathbb{R}$ is an *ordered field*.

Exercise 1.2.3. Let $x \ge y$. Prove that $x \cdot z \ge y \cdot z$ if $z > 0$ and $x \cdot z \le y \cdot z$ if $z < 0$.

Exercise 1.2.4. Let $x \ge y > 0$. Prove that $x^2 \ge y^2$.

The axioms introduce above still are not enough to start the course of analysis.

Completeness axiom: if X and Y are non-empty subsets of $\mathbb{R}$ such that

$$\forall x \in X \quad \forall y \in Y \qquad x \le y$$

then $\exists c \in \mathbb{R}$ such that

$$\forall x \in X \quad \forall y \in Y \qquad x \le c \le y\,.$$

Intuitively, this should hold for reals, however, it would take some time to check it for the infinite decimals. I will not do this verification in my lectures. Later, we will learn several equivalent forms of this axiom, then the verification will be much easier, see Exercise 2.1.8.

Why do we call all these rules the *axioms*? Let us say that a set F equipped with two operations (call them “addition” and multiplication”) and with an order relation is a *complete ordered field* if it satisfies all the axioms given above. We know (or rather believe) that the reals give us an example of a complete ordered field. This is a good point to turn things around (as we often do in math), and accept the following

Definition 1.2.5. *A field of real numbers* $\mathbb{R}$ *is a complete ordered field.*

I.e., from now on, we will allow ourselves to freely use the axioms introduced above.

When we start with an abstract system of axioms two questions arise: First, *whether there exists an object which satisfies them?* or maybe, the axioms from our system contradict each other? Second, assuming that such an object exists, *whether it is unique?* Imagine two different objects called "real numbers"! In our case, the answers to the both questions are positive. Since the proofs are too long for the first acquaintance with analysis, we'll skip them.

To prove existence, it suffices to check, for instance, that the infinite decimal strings satisfy these axioms. Note, that there are other constructions of the set of reals (like Dedekind cuts and Cauchy sequences of rationals). Luckily, all of them lead to the same object.

Suppose that we have two complete ordered fields, denote them $\mathbb{R}$ and $\mathbb{R}'$. How to say that they are equivalent? Some thought gives us the answer: we call $\mathbb{R}$ and $\mathbb{R}'$ *equivalent* if there exist a one-to-one correspondence f between $\mathbb{R}$ and $\mathbb{R}'$ which preserves the arithmetic operations and the order relation; i.e.

$$\begin{aligned} f(x+y) &= f(x) + f(y), \\ f(x \cdot y) &= f(x) \cdot f(y), \\ x \le y \quad \Longrightarrow \quad & f(x) \le f(y)\,. \end{aligned}$$

It's not very difficult to construct such a map f[2]. This construction leads to a theorem which says that *any two complete ordered field are equivalent.*

Natural and integer numbers. Naively, the set of natural numbers is the set of all real numbers of the form

$$1, 1+1, (1+1)+1, ((1+1)+1)+1, \dots .$$

A formal definitions is slightly more complicated.

Definition 1.2.6 (inductive sets)**.** A set $X \subset \mathbb{R}$ is called *inductive* if

$$\bigl(x \in X\bigr) \Longrightarrow \bigl(x+1 \in X\bigr)$$

For instance, the set of all reals is inductive.

Definition 1.2.7 (natural numbers)**.** The set of natural numbers $\mathbb{N}$ is the intersection of all inductive sets that contains the element 1.

In other words, a real number x is natural if it belongs to *each* inductive set that contains 1.

Claim 1.2.8. *The set of natural numbers is inductive.*

Proof: Suppose $n \in \mathbb{N}$. Let X be an arbitrary inductive subset of $\mathbb{R}$ that contains n. Since X is inductive, $n+1$ is also in X. Hence, $n+1$ belongs to each inductive subset of $\mathbb{R}$, whence, $n+1 \in \mathbb{N}$; i.e., the set $\mathbb{N}$ is inductive. $\square$

This definition provides a justification for the *principle of mathematical induction.* Suppose there is a proposition $P(n)$ whose truth depends on the

[2] I suggest to the students with curiosity to build such a map yourselves.

natural numbers. The principle states that if we can prove the truth of $P(1)$ ("the base"), and that assuming the truth of $P(n)$ we can prove the truth of $P(n+1)$, then $P(n)$ is true for all natural n.

Exercise 1.2.9. Prove that any natural number can be represented as a finite sum of ones: $1 + 1 + \ldots + 1$.

Example 1.2.10 (Bernoulli's inequality)**.** $\forall x > -1$ and $\forall n \in \mathbb{N}$

$$(1+x)^n \geq 1 + nx\,.$$

The equality sign is possible only when either $n = 1$ or $x = 0$.

Proof: Fix $x > -1$. For $n = 1$, the LHS and the RHS equal $1 + x$. Hence, we've checked the base of the induction.

Assume that we know that

$$(1+x)^n \geq 1 + nx\,.$$

Since $1+x$ is a positive number, we can multiply this inequality by $1+x$. We get

$$(1+x)^{n+1} \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2\,.$$

If $x \neq 0$, the RHS is bigger than $1 + (n+1)x$, and we are done. $\square$

Exercise 1.2.11. Prove that

$$\frac{1}{\sqrt[n]{1+m}} + \frac{1}{\sqrt[m]{1+n}} \geq 1\,.$$

Hint: Use Bernoulli's inequality.

Exercise 1.2.12. Suppose a_1, ..., a_n are non-negative reals such that $S = a_1 + \ldots + a_n < 1$. Prove that

$$1 + S \leq (1+a_1)\cdot \ldots \cdot (1+a_n) \leq \frac{1}{1-S}$$

and

$$1 - S \leq (1-a_1)\cdot \ldots \cdot (1-a_n) \leq \frac{1}{1+S}\,.$$

Exercise 1.2.13. Prove:

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\,, \qquad n \in \mathbb{N}\,.$$

Exercise 1.2.14. Prove that

$$2(\sqrt{n} - 1) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}\,.$$

Definition 1.2.15 (integers)**.**

$$\mathbb{Z} = \Big\{ x \in \mathbb{R} \colon \big(x \in \mathbb{N}\big) \bigvee \big(-x \in \mathbb{N}\big) \bigvee \big(x = 0\big) \Big\}\,.$$

Remark: It is purely a matter of agreement that we start the set of natural numbers with 1. In some textbooks the set $\mathbb{N}$ starts with 0.

In what follows, we denote the set of non-negative integers by $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Rational numbers.

Definition 1.2.16.

$$\mathbb{Q} = \left\{ x = \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\} .$$

Exercise 1.2.17. Whether the set of integers $\mathbb{Z}$ is a field? Whether the set of rationals $\mathbb{Q}$ is a field?

Exercise 1.2.18. Check that the rationals $\mathbb{Q}$ form an ordered field.

Exercise 1.2.19. Prove that the equation $s^2 = 2$ does not have a rational solution.

Exercise 1.2.20. Check that the field of rationals $\mathbb{Q}$ doesn't satisfy the completeness axiom.

1.3. Application: solution of equation $s^n = a$.

Exercise 1.3.1. Prove that this equation cannot have more than two real solutions.

Theorem 1.3.2. *For each $a > 0$ and each natural $n \in \mathbb{N}$, the equation $s^n = a$ has a positive solution s.*

Proof: Define the sets $X := \{x \in \mathbb{R} : x^n < a\}$ and $Y := \{y \in \mathbb{R} : y^n > a\}$, both sets are not empty (why?). The completeness axiom can be applied to these sets since

$$\forall x \in X, y \in Y \quad (x^n < a < y^n) \quad \Longrightarrow \quad (x < y) .$$

By the axiom,

$$\exists s \quad \forall x \in X, \forall y \in Y \quad x \leq s \leq y .$$

We claim that $s^n = a$.

First, observe that X contains a positive number so that s is positive as well. Indeed, take $t = 1 + 1/a$. Then $t^n \geq t > 1/a$, and $(1/t)^n < a$. Therefore, $1/t \in X$.

Now, assume that $s^n < a$. Our aim is to find another value s_1 which is bigger than s but still $s_1^n < a$. Then $s_1 \in X$, that is, X has an element which is (strictly) bigger than s. Hence, contradiction.

To find such s_1, we choose a small positive ϵ:

$$0 < \epsilon < \frac{a - s^n}{na}.$$

Then $a - s^n > \epsilon na$, and

$$s^n < (1 - n\epsilon)a \leq (1 - \epsilon)^n a .$$

In the second inequality we use the Bernoulli inequality in the form

$$1 - \epsilon n \leq (1 - \epsilon)^n , \qquad 0 < \epsilon \leq 1, \quad n \in \mathbb{N}$$

(this is legitimate since $\epsilon < 1$). That is,

$$\left(\frac{s}{1-\epsilon}\right)^n < a\,,$$

and by the definition of the set X the number $s/(1-\epsilon)$ must be in X. By the choice of s, $s > s/(1-\epsilon)$ which is impossible. Therefore, $s^n \geq a$.

A similar argument shows that $s^n \leq a$. Now, we start with assumption that $s^n > a$. Then we should find small positive ϵ such that

$$0 < \epsilon < \frac{s^n - a}{ns^n}\,.$$

We have $s^n - a > \epsilon n s^n$, and $a < (1-\epsilon n)s^n$. Using again Bernoulli's inequality, we get

$$a \leq (1-\epsilon)^n s^n = [(1-\epsilon)s]^n\,.$$

This means that $(1-\epsilon)s \in Y$ which again contradicts the choice of s. Therefore, $s^n = a$ proving the theorem. ☐

1.4. **The distance on $\mathbb{R}$.** We also know how to measure the distance between two real numbers. Set

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0 \end{cases}$$

The value $d(x,y) = |x-y|$ is *the distance* between x and y. It enjoys the following properties:

positivity: $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$;
symmetry: $d(x,y) = d(y,x)$;
triangle inequality: $d(x,y) \leq d(x,z) + d(z,y)$ with the equality sign iff the point z lies within the close segment with the end-points x and y.

The first two properties are obvious. Let's prove the triangle inequality.

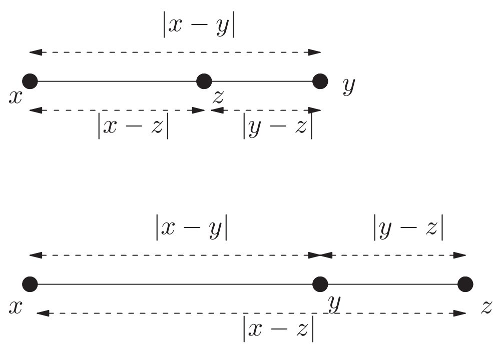

FIGURE 1. To the proof of triangle inequality

Let, say, $x < y$. If $z \in [x,y]$, then

$$d(x,y) = y - x = (y-z) - (z-x) = d(y,z) + d(x,z)\,.$$

If z does not belong to the interval $[x,y]$, say $z > y$, then

$$d(x,y) = y - x < z - x = d(x,z) < d(x,z) + d(y,z)\,.$$

Done! $\square$

Question: How the triangle inequality got its name?

There are other versions of the triangle inequality which we'll often use in this course:

Exercise 1.4.1. Prove the following inequalities:

$$|x + y| \leq |x| + |y| \,,$$
$$|x - y| \geq |\,|x| - |y|\,| \,,$$

and

$$|x_1 + \;...\; + x_n| \leq |x_1| + \;...\; + |x_n| \,.$$

In what follows, we apply the name "triangle inequality" to these inequalities as well.

2. Upper and lower bounds

2.1. **Maximum/minimum supremum/infimum.** The completeness axiom has a number of important corollaries which will be of frequent use during the whole course. We start with some definitions.

A subset $X \subset \mathbb{R}$ is *upper bounded* if $\exists c$ such that $\forall x \in X$, $x \leq c$. Any c with this property is called an *upper bound* (or a *majorant*) of X. A subset $X \subset \mathbb{R}$ is *lower bounded* if $\exists c$ such that $\forall x \in X$, $x \geq c$. Any c with this property is called a *lower bound* (or a *minorant*) of X. A set X is bounded if it is upper- and lower bounded.

Next, we define the *maximum* and *minimum* of a set X:

Definition 2.1.1 (maximum/minimum)**.**

$$(a = \max X) := (a \in X \ \wedge \ \forall x \in X \quad (x \leq a)) \,,$$

that is, a is a majorant of X and belongs to X. Similarly,

$$(a = \min X) := (a \in X \ \wedge \ \forall x \in X \quad (x \geq a)) \,,$$

that is, a is a minorant of X and belongs to X.

If a set is unbounded from above, then certainly it does not have a maximum. However, even if X is upper bounded, the maximum does not have to exists: for example consider an open interval $(0,1)$.

Example 2.1.2. The open interval $(0,1)$ has nor maximum neither minimum.

Proof: Suppose that c is a majorant of $(0,1)$. Then $c \geq 1$. Observe, that $(0,1) \cap [1,\infty) = \emptyset$, hence, c cannot belong to $(0,1)$. The proof that $(0,1)$ has no minimum is similar. □

Claim 2.1.3. If the maximum exists, then it is unique.

Proof: Suppose the set X has two different maxima: $a \neq b$. Then either $a < b$ or $b < a$. Assume, for instance, that $a < b$. Note that $b \in X$ since b is a maximum of X. Therefore, a does not majorize X. □

Let $X \subset \mathbb{R}$ be an upper bounded set. Consider the set of all upper bounds of X:

$$M_X \stackrel{\text{def}}{=} \{c \in \mathbb{R} \colon \ \forall x \in X \quad x \leq c\} \,.$$

This set is not empty and is lower bounded (why?).

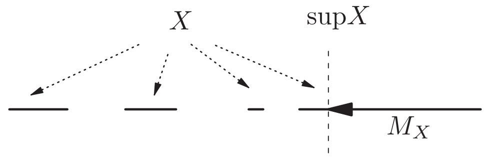

Figure 2. `Supremum of the set` X

Definition 2.1.4 (supremum)**.** The *supremum* of X is *the least upper bound of* X, that is the minimum of the set M_X:

$$\sup X := \min M_X \,.$$

An equivalent way to pronounce the same definition is

$$s = \sup X \qquad \text{iff} \qquad (\forall x \in X \quad x \le s) \wedge (\forall p < s \ \exists x' \in X \ p < x') \ .$$

We see from the previous exercise that if the supremum exists, then it is unique.

Examples: $\sup[-1,1] = \max[-1,1] = 1$, $\sup[-1,1) = 1$. In the second case the maximum does not exists.

Lemma 2.1.5 (existence of supremum)**.** *For every non-empty upper bounded set $X \subset \mathbb{R}$, the supremum exists.*

Proof: Consider the set M_X of all upper bounds of X. We have to show that this set has a minimum.

Since X is upper bounded, $M_X \neq \emptyset$. Condition of the completeness axiom is fulfilled for the sets X and M_X. Therefore,

$$\exists s \in \mathbb{R} \quad \forall x \in X \quad \forall c \in M_X \qquad x \le s \le c \,.$$

That is, s is an upper bound of X, and hence belongs to M_X. The same relation shows that s is a minorant of M_X. Therefore, $s = \min M_X$. □

Now, let $X \subset \mathbb{R}$ be a lower bounded set. The *infimum* of X is *the greatest lower bound of* X, that is

$$\inf X := \max\{c \in \mathbb{R} : \forall x \in X \quad x \ge c\} \,.$$

If the infimum exists, it is unique.

Here is an equivalent way to word the same definition:

$$s = \inf X \qquad \text{iff} \qquad (\forall x \in X \quad x \ge s) \wedge (\forall p > s \ \exists x' \in X \ x' < p) \ .$$

Exercise 2.1.6. Let $X \subset \mathbb{R}$ and let $-X := \{x \in \mathbb{R} : -x \in X\}$. Show $\inf X = \sup(-X)$. Deduce that every lower bounded set has an infimum.

It is interesting to note that existence of the supremum of an upper bounded set is *equivalent* to the completeness axiom:

Exercise 2.1.7. Let X and Y be non-empty subsets of $\mathbb{R}$ such that

$$\forall x \in X \quad \forall y \in Y \qquad x \le y \,.$$

Then the set X is bounded from above. Set $c = \sup X$. Check that $\forall x \in X$ $\forall y \in Y$ one has $x \le c \le y$.

The meaning of the following exercise is to verify that any upper bounded set of infinite decimals has a supremum. I.e., the infinite decimals satisfy the completeness axiom.

Exercise 2.1.8. For a non-negative decimal x, we denote by $l(x) = \min\{n \in \mathbb{Z}_+ \colon x \le 10^n\}$. In other words, this is the length of the part of the string left to the decimal point.

i. Let X be a set of non-negative infinite decimals. Check that X is bounded from above iff the set $\{l(x) \colon x \in X\}$ is bounded from above.

ii. Work out an "algorithm" that finds one by one the digits in the decimal expansion of $\sup X$.

2.2. **Some corollaries:** Most of the corollaries given below are evident if we define the reals using the infinite decimals. Here we deduce them from the axioms of the complete ordered field.

Claim 2.2.1. *Every bounded subset E of the set $\mathbb{N}$ of natural numbers has the maximum.*

Proof: Since E is upper bounded, there exists (a real) $s = \sup E$. By the definition of the supremum, there is an $n \in E$ such that $s - 1 < n \le s$. Suppose that there exists an $m \in E$ such that $m > n$. Then $m \ge n + 1 > s$. Contradiction!

Hence, $n = \max E$. □

Exercise 2.2.2. Check that any (non-empty) subset of $\mathbb{N}$ has the minimum.

Claim 2.2.3. *The set $\mathbb{N}$ is unbounded from above. The set of integers $\mathbb{Z}$ is unbounded from above and from below.*

Proof: If $\mathbb{N}$ is bounded, then according to the previous claim it has a maximal element n. Since $\mathbb{N}$ is an inductive set, $n + 1$ is also a natural number, and $n + 1 > n$. We obtain a natural number which is bigger than n. Hence, the contradiction. □

Claim 2.2.4 (Archimedes principle)**.** *For every $h > 0$ and every $x \in \mathbb{R}$ there exists a unique $k \in \mathbb{Z}$ such that $(k-1)h \le x < kh$.*

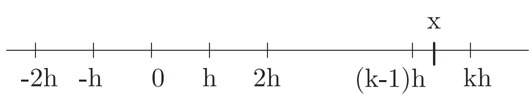

FIGURE 3. `Archimedes principle`

Proof: Assume $x/h \notin \mathbb{Z}$, otherwise there is nothing to prove. Consider a subset of the integers $\{n \in \mathbb{Z} : x/h < n\}$. This is a non-empty set which is lower bounded. Therefore, it has a minimum

$$k = \min\{n \in \mathbb{Z} \colon x/h < n\}$$

and this k satisfies $k - 1 \le x/h < k$. Done! □

Applying this principle with $h = 1$ we obtain the following:

$$\forall x \in \mathbb{R} \quad \exists! k \in \mathbb{Z} \qquad \text{such that} \quad k \le x < k + 1\,.$$

This number k is called an *integer part* of x and is denoted by $[x]$. Sometimes, the same function is called a *floor function* and is denoted by $\lfloor x \rfloor$. The fractional part of x is the number $\{x\} : x - [x]$. It is also defined uniquely and is always in the semi-open interval $[0, 1)$.

Exercise 2.2.5. Draw the graph of the function $f(x) = \{10x\}$.

Claim 2.2.6. *Whatever small is a positive ϵ, there is a natural number n such that $0 < 1/n < \epsilon$.*

Proof: otherwise, $\forall n \in \mathbb{N}\ 1/n \geq \epsilon$, or $n \leq 1/\epsilon$, that is the set of naturals $\mathbb{N}$ is upper bounded which is impossible. □

Claim 2.2.7. *Let $h \geq 0$ and $\forall n \in \mathbb{N} \quad h \leq 1/n$. Then $h = 0$.*

Proof: is the same as in in the previous claim: if $h > 0$, then $\forall n \in \mathbb{N}\ n \leq 1/h$ and as above we arrive at the contradiction. □

Claim 2.2.8. *Every open interval contains rationals:*

$$\forall (a, b) \subset \mathbb{R} \quad \exists r \in \mathbb{Q} \cap (a, b) \,.$$

Proof: Choose $n \in \mathbb{N}$ such that $0 < 1/n < b - a$. Then choose $m \in \mathbb{Z}$ such that $\dfrac{m-1}{n} \leq a < \dfrac{m}{n}$. Set $r = \dfrac{m}{n}$. By construction, $r > a$.

If $r \geq b$, then $\dfrac{m-1}{n} < a < b \leq \dfrac{m}{n}$, and $b - a < \dfrac{1}{n}$ which contradicts the choice of n. □

What about irrational numbers? Try to prove yourself that every open interval contains at least one irrational number or wait till the next lecture.

It is worth mentioning that one really needs the completeness axiom for derivation of these corollaries.

Consider a set of rational functions, that is functions represented as quotients of two polynomials: $r(x) = p(x)/q(x)$ (there could be points x where r is not defined. Two functions $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ are equal if $p_1q_2 - p_2q_1$ is a zero polynomial (that is, identically equals zero). Show that these functions form a field with usual addition and multiplication (that is, check the axioms). Now, introduce an order: let r_1 and r_2 be two rational functions. We say that $r_1 < r_2$ if there is an $x > 0$ such that $r_1(t) < r_2(t)$ for all $t \in (0, x)$.

Exercise* 2.2.9. Show that this is an ordered field (i.e., check the axioms).

The integers in this field are rational functions which identically equal an integer number. For example, the integer 7 is represented by a rational function $r = (7q)/q$ where q is an arbitrary polynomial.

Exercise* 2.2.10. Check that the rational function $r = 1/x$ is a majorant for the set of all integers in that field. In other words, the integers are bounded therein.

3. Three basic lemmas: Cantor, Heine-Borel, Bolzano-Weierstrass

In this lecture we prove three fundamental lemmas. The most of the proofs in the rest of the course rely upon them.

3.1. **The nested intervals principle.**

Lemma 3.1.1 (Cantor). *Any nested sequence of closed intervals $I_1 \supset I_2 \supset \ldots \supset I_n \supset I_{n+1} \supset \ldots$ has a non-empty intersection:*

$$\bigcap_{n\ge 1} I_n \neq \emptyset\,.$$

In other words, $\exists c \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$ $c \in I_n$.

Proof: Let $I_n = [a_n, b_n]$. Clearly, $\forall m, n$ we have $a_m \le b_n$ (otherwise, $I_m \cap I_n = [a_m, b_m] \cap [a_n, b_n] = \emptyset$). Consider the sets

$$A := \{a_m : m \in \mathbb{N}\}\,, \qquad B := \{b_n : n \in \mathbb{N}\}\,.$$

Any element from the set B is an upper bound for the set A, that is the completeness axiom is applicable. It says:

$$\exists c \in \mathbb{R}\colon \forall m, n \in \mathbb{N} \quad a_m \le c \le b_n\,.$$

In particular,

$$a_n \le c \le b_n, \quad \forall n \in \mathbb{N}\,,$$

proving the lemma. □

Clearly, the lemma fails if the nester intervals are open. For instance, $\cap_n (0, 1/n) = \emptyset$.

Question 3.1.2. Where in the proof of Cantor's lemma we used that the nested intervals are closed?

Exercise 3.1.3. *Whether the lemma holds true for semi-open nested intervals?*

Exercise 3.1.4. *In the assumptions of the Cantor lemma, $\bigcap_n I_n$ is always a closed interval.*

Sometimes, the following complement to the Cantor lemma is useful: *if, additionally, in the assumptions of the lemma, the lengths of the intervals I_n $|I_n| = b_n - a_n$ are getting closer and closer to zero (formally, $\forall \epsilon > 0$ $\exists k$ such that $|I_k| (= b_k - a_k) < \epsilon$,) then the intersection of I_j is a singleton:*

$$\bigcap_{j\ge 1} I_j = \{c\}\,.$$

Indeed, if there are two different points c_1 and c_2 in the intersection of I_j's (and, say, $c_1 < c_2$), then

$$a_n \le c_1 < c_2 \le b_n, \quad \forall n \in \mathbb{N},$$

whence $|I_n| = b_n - a_n \ge c_2 - c_1$ which contradicts to the assumption.

3.2. **The finite subcovering principle.** To proceed further, we need several new definitions. Let Y be a subset of $\mathbb{R}$, and let $\mathcal{S} = \{X\}$ be a collection of subsets of $\mathbb{R}$. We say that $\mathcal{S}$ *covers* Y, if

$$Y \subset \bigcup_{X \in \mathcal{S}} X\,.$$

In other words, for every point x in Y there is a set X from the collection $\mathcal{S}$ such that $x \in X$.

Examples:

1. Trivial coverings: let Y be an arbitrary subset of $\mathbb{R}$. Consider $\mathcal{S}_1 := \{\mathbb{R}\}$, that is, $\mathcal{S}_1$ consists of the one set $\mathbb{R}$. We get a covering. Another example is $\mathcal{S}_2 := \{y\}_{y\in\mathbb{R}}$, here $\mathcal{S}_2$ consists of all one-point sets, again we get a covering.

2. Let $Y = (0,1)$ and $\mathcal{S} = \{X_1, X_2\}$, where $X_1 = [-1, 1/2]$ and $X_2 = [1/3, 2]$.

3. Let $Y = [0,1]$, $\mathcal{S} = \{I_x\}_{x\in[0,1]}$, where $I_x = (x - 1/4, x + 1/4)$.

Lemma 3.2.1 (Heine-Borel)**.** *For any system of open intervals $\mathcal{S} = \{I\}$ which covers a closed interval J there is a finite subsystem which still covers J.*

In this case, we say that there exists a finite subcovering. Before going to the proof, we suggest to analyze the third example above and to choose a finite subcovering in that case.

Proof: We use a "bisection method". Assume that the lemma is wrong. Then we construct inductively an infinite nested sequence of closed sub-intervals J_n of J such that $\forall n$ the intervals J_n cannot be covered by any finite subcollection of $\mathcal{S}$, and $|J_n| = 2^{-n}|J|$.

Start with $J_0 = J$ and dissect it onto two equal closed subintervals. Since J_0 has no finite subcovering, one of these two parts also has no finite subcovering. Call this part J_1. Then $J_1 \subset J_0$, $|J_1| = 2^{-1}|J|$ and J_1 has no finite subcovering. Then we continue this dissection procedure.

According to the Cantor lemma (and its complement), the closed intervals J_n have one point intersection: $\bigcap_n J_n = \{c\}$. The point c belongs to J and therefore is covered by an open interval $I = (a,b)$ from the collection $\mathcal{S}$, that is $a < c < b$. Take $\epsilon = \min(b - c, c - a)$. We know that for some n the length of J_n (which is $2^{-n}|J|$) is less than ϵ, and that $c \in J_n$. Therefore, $J_n \subset (a,b) = I$. Hence, J_n has a finite subcovering from our sub-collection, in fact a subcovering by one open interval I. We arrive at the contradiction which proves the lemma. $\square$

Exercise 3.2.2. Try to change assumptions of this lemma. Whether the result persists if the intervals in the covering are closed? What about coverings of an open interval by closed ones? or by open ones? Consider all three remaining cases.

3.3. **The accumulation principle.** We start with some definitions. Let x be a real number. Any open interval $I \ni x$ is called a *vicinity* (or *neighbourhood*) of x. The set $I \setminus \{x\}$ is called a *punctured vicinity* of x.

Let $X \subset \mathbb{R}$. A point p is called an *accumulation point* of X if any vicinity of p contains infinitely many points from X. Equivalently, any punctured vicinity of p contains at least one point of X.

Exercise 3.3.1. Proof equivalence of these definitions.

Exercise 3.3.2. Find accumulation points of the following sets:

$$\{1/n\}_{n\in\mathbb{N}}, \quad [a,b), \quad (-2,-1)\cup(1,2), \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}\setminus\mathbb{Q}, \quad \mathbb{R}.$$

Lemma 3.3.3 (Bolzano-Weierstrass)**.** *Each infinite bounded set $X \subset \mathbb{R}$ has an accumulation point.*

Proof: Let $X \subset [a,b] =: J$. Assume the assertion is wrong, that is each point $x \in J$ has a neighbourhood $U(x)$ which has a finitely many points in the intersection with X. The open intervals $\{U(x)\}_{x\in J}$ obviously cover J and by the Borel lemma we can chose a finite subcovering. That is,

$$X \subset J \subset \bigcup_{k=1}^{N} U(x_k)\,,$$

and therefore the set X is finite:

$$\#(X) \le \sum_{k=1}^{N} \#(\, X \cap U(x_k)\,) < \infty\,.$$

This contradicts the assumption and proves the lemma. $\square$

Exercise 3.3.4. Starting with the Bolzano-Weierstrass lemma, derive the existence of the supremum for every upper bounded subset of $\mathbb{R}$.

The meaning of this exercise is simple: the four principles (completeness, existence of the supremum, Borel's covering lemma, and Bolzano-Weierstrass' lemma) appear to be *equivalent* to each other.

Exercise 3.3.5. All real points are coloured in two colours: black and white, and the both colours were used. Prove that there are points of different colours at the distance less than 0.001.

3.4. **Appendix: Countable and uncountable subsets of $\mathbb{R}$.** Here we touch very briefly the notions of countable and uncountable sets. You will learn more in the courses "Introduction to the set theory" or in "Discrete Mathematics". First, recall some terminology. A map $f\colon X \to Y$ is

injective if

$$\forall x_1, x_2 \in X \quad x_1 \ne x_2 \quad \Longrightarrow \quad f(x_1) \ne f(x_2)\,;$$

i.e., injective maps define one-to-one correspondence between X and its image $f(X) \subset Y$.

surjective if

$$\forall y \in Y \quad \exists x \in X \qquad f(x) = Y\,;$$

i.e., surjective maps map X into the whole Y. In this case, we say that f maps X *onto* Y.

bijective if it is injective and surjective; that is, bijective maps define one-to-one correspondence between the sets X and Y.

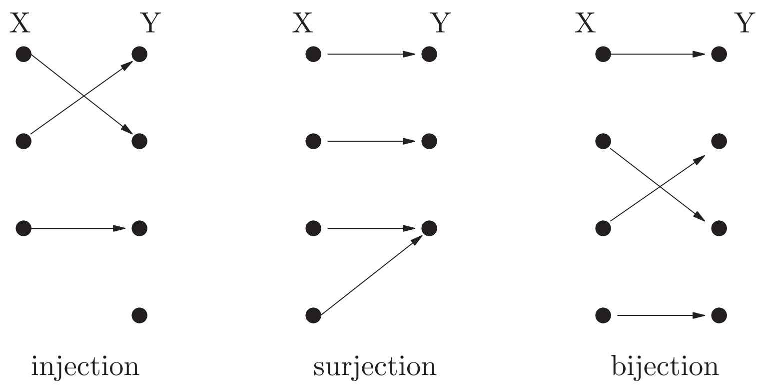

FIGURE 4. Injective, surjective, and bijective maps

Definition 3.4.1. A set X is countable if there is a bijection between this set and the set $\mathbb{N}$ of positive integers.

Lemma 3.4.2.

i) *Any infinite subset of a countable set is countable.*

ii) *Any finite or countable union of countable sets is countable.*

Proof: It suffices to prove i) in the special case when X_1 is an infinite subset of $\mathbb{N}$. Indeed, let X be our countable set, and X_1 be its infinite subset. Let $\varphi\colon X \to \mathbb{N}$ be a bijection, and $E_1 = \varphi(X_1) \subset \mathbb{N}$. We'll build the bijection $\theta\colon E_1 \to \mathbb{N}$. Then the composition $\varphi \circ \theta$ gives us the bijection between X_1 and $\mathbb{N}$.

Let $e_1 = \min E_1$. The set E_1 is infinite, hence the set $E_2 = E_1 \setminus \{e_1\}$ is also infinite. We set $e_2 = \min E_2$ (note that $e_2 > e_1$) and then consider the infinite set $E_3 = E_2 \setminus \{e_2\}$ of $\mathbb{N}$, etc. On the n-th step, we start with the infinite set $E_n \subset \mathbb{N}$, let $e_n = \min E_n$ (such that $e_n > e_{n-1} > \dots > e_1$) and define the new infinite set E_{n+1}.

In this way, we get a map $\theta\colon \mathbb{N} \to E_1$ such that $\theta(n) = e_n$. Since $e_i \neq e_j$ for $i \neq j$, this is an injective. Let's check that it maps $\mathbb{N}$ *onto* E_1. Consider an arbitrary element $e \in E_1 \setminus \theta(\mathbb{N})$. The set $\big\{n \in \mathbb{N}\colon n \le e\big\}$ is finite, hence its subset $\big\{n \in E_1\colon n \le e\big\}$ is also finite. Let k be the cardinality of this set. Then $e = e_k$, i.e., $e \in \theta(\mathbb{N})$. This proves the first statement.

The proof of the second statement is based on the following

Claim 3.4.3. *The set of ordered pairs of positive integer numbers*

$$\mathbb{N} \times \mathbb{N} \stackrel{\text{def}}{=} \big\{(m,n)\colon m,n \in \mathbb{N}\big\}$$

is countable.

22	30	39	49	60	72	85	
16	23	31	40	50	61	73	
11	17	24	32	41	51	62	
7	12	18	25	33	42	52	
4	8	13	19	26	34	43	
2	5	9	14	20	27	35	
1	3	6	10	15	21	28	

FIGURE 5. `Cantor's board`

The proof of this claim follows by inspection of the infinite Cantor board (Figure 5) that explains how to build a bijection between the sets $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$. $\square$

Now, let $\mathbb{N}_1 \subset \mathbb{N}$, and let $X = \bigcup_{m\in\mathbb{N}_1} X_m$ be a finite or countable union of countable sets. Let $X_m = \{x_{m,1}, x_{m,2}, \dots x_{m,n}, \dots\}$. Then $\psi\colon (m,n) \mapsto x_{m,n}$ defines a bijection between X and a subset of $\mathbb{N}\times\mathbb{N}$. The first statement and Claim yield that X is countable. $\square$

Corollary 3.4.4. *The set of rational numbers is countable.*

Proof: Consider the countable sets

$$Q_m \overset{\text{def}}{=} \left\{r = \frac{n}{m} : n \in \mathbb{Z}\right\}, \qquad m \in \mathbb{N}.$$

(For instance, $Q_7 = \left\{\dots, -\frac{2}{7}, -\frac{1}{7}, 0, \frac{1}{7}, \frac{2}{7}, \dots\right\}$). Then

$$\mathbb{Q} = \bigcup_{m\in\mathbb{N}} Q_m$$

is a countable union of countable sets. Hence, it is countable. $\square$

Exercise 3.4.5. Write down an explicit formula for the bijection between the sets $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$.

Theorem 3.4.6 (Cantor)**.** *Any interval of positive length contains uncountable many points.*

Proof: Since any interval of positive length contains a closed subinterval of positive length, it suffices to prove the statement for closed intervals. Suppose that the statement is not correct, i.e., there is a closed interval I_1 of positive length which contains countably many points: $I_1 = \{x_1, x_2, \dots, x_n, \dots\}$. Choose a closed subinterval $I_2 \subset I_1$ of positive length that does not contain the point x_1. Then choose a closed subinterval of positive length $I_3 \subset I_2$ that does not contain the point x_2, etc.

At the n-th step, having a closes interval of positive length I_n, we choose its closed subinterval $I_{n+1} \subset I_n$ of positive length that does not contain the point x_{n+1}. By Cantor's lemma, the intersection $\bigcap\limits_j I_j$ is not empty. Take any point $c \in \bigcap\limits_j I_j$. By construction, $c \in I_1$, but c differs from any of the points $x_1, x_2, ..., x_n, ...$ Contradiction! $\square$

Exercise 3.4.7. Check the following claims:

i) The set of all irrational numbers is uncountable.

ii) The set of all subsets of a countable set is uncountable.

iii) The set of all sequences $\{\epsilon_1, \epsilon_2, ..., \epsilon_n, ...\}$ with $\epsilon_n \in \{0, 1\}$ is uncountable.

Exercise 3.4.8.

i Prove that it is possible to draw uncountably many disjoint figures 5 on the plane but only countably many disjoint figures 8.

ii* Prove that it is possible to draw only countably many disjoint letters T on the plane.

4. Sequences and their limits

4.1. The infinite sequence is a function defined on the set $\mathbb{N}$ of natural numbers, $f\colon \mathbb{N} \to \mathbb{R}$. Such a function f can be written as a infinite string $\{f(1), f(2), f(3), \ldots, f(n), \ldots\}$. For historical reasons, in this case the argument is usually written as a subscript: $\{f_1, f_2, f_3, \ldots, f_n, \ldots\}$. A standard notation for such a string is $\{f_n\}_{n\in\mathbb{N}}$. The value f_n is called the n-th term of the sequence.

Examples:

Arithmetic progression

$$\{1, 2, 3, 4, 5, 6, \ldots\},$$

or more generally

$$\{a, a+d, a+2d, a+3d, a+4d, a+5d, \ldots\}.$$

Geometric progression

$$\{q^0, q^1, q^2, q^3, q^4, q^5, \ldots\}$$

Definition 4.1.1 (convergence)**.** *A sequence $\{x_n\}$ converges to the limit a if*

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \textit{such that} \quad \forall n \geq N \quad |x_n - a| < \epsilon\,.$$

In other words, whatever small ϵ is, only finitely many terms of the sequence do not belong to the interval $(a-\epsilon, a+\epsilon)$. If the sequence $\{x_n\}$ converges to

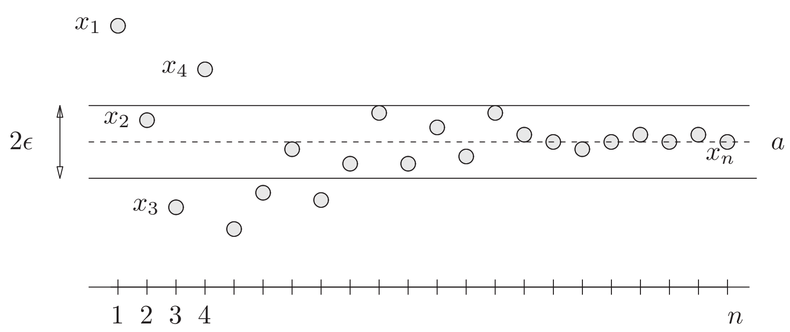

Figure 6. Convergent sequence

the limit a, we write

$$a = \lim_{n\to\infty} x_n\,,$$

or $x_n \to a$. If a sequence is not convergent, it is called *divergent*.

Examples:

$\{1/n\}$, the sequence converges to zero;
$\{(n+1)/n\}$, the sequence converges to one;
$\left\{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots.\right\}$, the sequence is divergent;
$\{1 + (-1)^n/n\}$, the sequence converges to one;
$\{\sin n/n\}$, the sequence converges to zero;
$\{q^n\}$, the sequence converges to zero if $|q| < 1$, converges to one if $q = 1$, and is divergent in the other cases.

4.2. Fundamental properties of the limits.

(a) *If the limit exists, it is unique.*

Proof: Let a and b be limits of a sequence $\{x_n\}$. We have to prove that $a = b$. Given positive ϵ, we can find $N \in \mathbb{N}$ such that simultaneously $|x_N - a| < \epsilon$ and $|x_N - b| < \epsilon$. Therefore,

$$|a - b| = |(a - x_N) + (x_N - b)| \le |x_N - a| + |x_N - b| < 2\epsilon\,.$$

Since this holds for *an arbitrary positive* ϵ, we conclude that $a = b$, completing the proof. $\square$

(b) *If a sequence converges, then it is bounded.*

Proof: Let a be a limit of a sequence $\{x_n\}$. Using the definition of convergence with $\epsilon = 1$, we find $N \in \mathbb{N}$ such that $|x_n - a| < 1$ for all $n \ge N$. Therefore, for these n's, $|x_n| < |a| + 1$. Hence $\{x_n\}$ is bounded:

$$|x_n| \le M := \max(|x_1|, |x_2|, \dots, |x_{N-1}|, |a| + 1)\,, \qquad \forall n \in \mathbb{N}\,.$$

$\square$

Note that the bounded sequence $\{(-1)^n\}$ diverges.

(c) *Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that the set $\{n \in \mathbb{N} : x_n \ne y_n\}$ is finite, and let $\{x_n\}$ converges to a. Then $\{y_n\}$ converges to a as well.*

In other words, the limit depends only on a *tail* of the sequence. We leave this as an exercise.

Exercise 4.2.1. Prove that every convergent sequence has either the maximal term, or the minimal term, or the both ones. Provide examples for each of the three cases.

Exercise 4.2.2. Let a sequence $\{x_n\}$ converge to zero, and let a sequence $\{y\}$ be obtained from $\{x_n\}$ by a permutation of its terms, then $\{y_n\}$ converges to zero as well.

With sequences we can do the same operations as with functions: for example, we can add and multiply them termwise.

Theorem 4.2.3. *Let $a = \lim x_n$ and $b = \lim y_n$. Then*

(i) $\lim(x_n \pm y_n) = a \pm b$;
(ii) $\lim(x_n \cdot y_n) = a \cdot b$;
(iii) *if* $b \ne 0$, *then* $\lim(x_n/y_n) = a/b$.

Proof:

(i) Given $\epsilon > 0$, we choose N_1 such that $|x_n - a| < \epsilon$ for all $n \ge N_1$ and choose N_2 such that $|y_n - b| < \epsilon$ for all $n \ge N_2$. Thus, for $n \ge N := \max(N_1, N_2)$, both inequalities hold. Therefore,

$$|(x_n \pm y_n) - (a \pm b)| \le |x_n - a| + |y_n - b| < 2\epsilon\,,$$

proving the claim.

(ii) Since $\{x_n\}$ is convergent, it is bounded. Take $M = \sup |x_n|$. Given $\epsilon > 0$, choose values N_1 and N_2 such that for all $n \geq N_1$ we have $|x_n - a| < \epsilon$, and for all $n \geq N_2$ we have $|y_n - b| < \epsilon$. Then

$$\begin{aligned} |x_n \cdot y_n - a \cdot b| &= |x_n \cdot (y_n - b) + (x_n - a) \cdot b| \\ &\leq (\sup |x_n|) \cdot |y_n - b| + |b| \cdot |x_n - a| < M \cdot \epsilon + |b| \cdot \epsilon = (M + |b|)\epsilon \, . \end{aligned}$$

(iii) We start with a warning some terms of the sequence $\{y_n\}$ can vanish. A good news is that a number of vanishing terms of this sequence is always finite. So that, the sequence $\{x_n/y_n\}$ is well-defined for sufficiently large indices n.

Now, keeping in mind that (ii) has been proved already, we conclude that it suffices to prove (iii) only in a special case when $x_n = 1$ for all $n \in \mathbb{N}$. We have to estimate the quantity

$$\left| \frac{1}{y_n} - \frac{1}{b} \right| = \frac{|y_n - b|}{|y_n| \cdot |b|} \, .$$

Since the sequence $\{y_n\}$ has a non-zero limit, we can choose $N_1 \in \mathbb{N}$ such that $|y_n| \geq \delta (> 0)$ for all $n \geq N_1$. Then, given $\epsilon > 0$, we choose $N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$ $|y_n - b| < \epsilon$. Therefore, $\forall n \geq N := \max(N_1, N_2)$

$$\left| \frac{1}{y_n} - \frac{1}{b} \right| < \frac{\epsilon}{\delta |b|} \, ,$$

completing the proof of the theorem. □

Exercise 4.2.4. Prove:

1. Let $a = \lim x_n$, $b = \lim y_n$ and $a < b$. Then $x_n < y_n$ for all sufficiently large indices n.

2. Let $a = \lim x_n$, $b = \lim y_n$ and $x_n \leq y_n$ for all sufficiently large indices n. Then $a \leq b$.

Theorem 4.2.5 (Two policemen, a.k.a. the sandwich)**.** *Let*

$$x_n \leq c_n \leq y_n \, , \qquad n \in \mathbb{N},$$

and let the sequences $\{x_n\}$ and $\{y_n\}$ converge to the same limit a. Then the sequence $\{c_n\}$ also converges to a.

Question: Explain, how the theorem got these names.

Proof: Given $\epsilon > 0$, choose the naturals N_1 and N_2 such that

$$\forall n \geq N_1 \qquad a - \epsilon < x_n \, ,$$

and

$$\forall n \geq N_2 \qquad y_n < a + \epsilon \, .$$

Then for any $n \geq N := \max(N_1, N_2)$

$$a - \epsilon < c_n < a + \epsilon \, ,$$

proving the convergence of $\{c_n\}$ to a. □

Definition 4.2.6 (monotonic sequence)**.** A sequence $\{x_n\}$ does not decrease if

$$x_1 \leq x_2 \leq \, ... \, \leq x_n \leq \, ... \, .$$

A sequence $\{x_n\}$ does not increases if

$$x_1 \geq x_2 \geq \, ... \, \geq x_n \geq \, ... \, .$$

If the strong inequalities hold, we'll say correspondingly that the sequence increases/decreases. In any of these cases, a sequence is called *monotonic*.

The next result is fundamental:

Theorem 4.2.7. *Any upper bounded non-decreasing sequence* $\{x_n\}$ *converges, and*

$$\lim x_n = \sup x_n \, .$$

Proof: Take $a := \sup x_n$. According to the definition of the supremum, $x_n \leq a$ for each $n \in \mathbb{N}$, and given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $x_N > a - \epsilon$. By monotonicity,

$$\forall n \geq N \qquad x_n \geq x_N > a - \epsilon \, .$$

Therefore, for all sufficiently large indices n, $a - \epsilon < x_n \leq a$, proving the theorem. □

This result is *equivalent* to the existence of the supremum of any upper bounded subset of the reals (and therefore, to all other equivalent forms of this statement we already know).

5. Convergent sequences

5.1. Examples.

5.1.1. Fix $q > 1$ and consider a sequence with terms

$$x_n = \frac{n}{q^n}\,.$$

We shall prove that it converges to zero.

First, check that the sequence eventually (that is, for large enough n) decreases. Indeed,

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{n \cdot q}\,.$$

If n is sufficiently large, the left hand side is less than one since $\lim(n+1)/n = 1$ and $q > 1$. That is, for large n, $x_{n+1} < x_n$.

Therefore, by the theorem from the previous lecture, the sequence $\{x_n\}$ converges to a non-negative limit a. Let us show that $a = 0$. We have

$$a = \lim x_{n+1} = \lim\left(\frac{n+1}{qn} \cdot x_n\right) = \frac{1}{q}\,\underbrace{\lim \frac{n+1}{n}}_{=1}\,\lim x_n = \frac{a}{q}\,.$$

Comparing the right and left hand sides, we conclude that $a = 0$. $\square$

Corollary 5.1.1. $\lim \sqrt[n]{n} = 1$.

Indeed, taking into account the limit we've just computed, given $\epsilon > 0$ we can take N so large that $\forall n \geq N$

$$1 < n < (1+\epsilon)^n\,.$$

Then

$$1 < \sqrt[n]{n} < 1 + \epsilon\,,$$

proving the convergence to one. $\square$

Exercise 5.1.2. Let $M \in \mathbb{N}$, $a > 0$, and $q > 1$. Prove that

$$\lim \frac{n^M}{q^n} = 0 \qquad \text{and} \qquad \lim \sqrt[n]{a} = 1\,.$$

5.1.2. For each positive q,

$$\lim_{n\to\infty} \frac{q^n}{n!} = 0\,.$$

We use a similar argument: first show that the sequence $x_n = q^n/n!$ eventually decays:

$$\frac{x_{n+1}}{x_n} = \frac{q^{n+1}}{q^n} \cdot \frac{n!}{(n+1)!} = \frac{q}{n+1} < 1\,,$$

if n is sufficiently large. Therefore, the sequence converges to a limit a. We check that a vanishes:

$$a = \lim x_{n+1} = \lim \frac{q}{n+1} \cdot x_n = 0 \cdot a = 0\,.$$

$\square$

In the following example the sequence is defined *recurrently*.

5.1.3. Take $x_0 = 1$, $x_n = \sqrt{2 + x_{n-1}}$. We show that the sequence $\{x_n\}$ converges to 2. Less formally,

$$\sqrt{2 + \sqrt{2 + \,...\, \sqrt{2 + ...}}} = 2\,.$$

First, using induction by n, we check that the sequence $\{x_n\}$ increases, and that $x_n < 2$ for all n. The base $n = 1$ of the induction is evident. Assume that the claims are verified for n, check that they hold for $n + 1$. Since $x_n < 2$, we have $x_{n+1} = \sqrt{2 + x_n} > \sqrt{x_n^2} = x_n$, and $x_{n+1} = \sqrt{2 + x_n} < \sqrt{4} = 2$, proving the claim for $n + 1$.

We conclude that $\{x_n\}$ is an increasing upper bounded sequence, so that, it has a limit which we call a. Then

$$a^2 = \lim_{n\to\infty} x_{n+1}^2 = 2 \lim_{n\to\infty} x_n = 2a\,,$$

so that $a = 2$. □

5.1.4.

$$\lim_{n\to\infty} \frac{1 \cdot 3 \cdot 5 \cdot \,...\, \cdot (2n-1)}{2 \cdot 4 \cdot 6 \ldots \cdot 2n} = 0\,.$$

This follows from the following chain:

$$\begin{aligned}\left(\frac{1 \cdot 3 \cdot 5 \cdot \,...\, \cdot (2n-1)}{2 \cdot 4 \cdot 6 \ldots \cdot 2n}\right)^2 &= \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \,...\, \cdot \frac{(2n-3)(2n-1)}{(2n-2)^2} \cdot \frac{2n-1}{2n} \cdot \frac{1}{2n} \\ &< \frac{1}{2n}\,.\end{aligned}$$

so that

$$\frac{1 \cdot 3 \cdot 5 \cdot \,...\, \cdot (2n-1)}{2 \cdot 4 \cdot 6 \ldots \cdot 2n} < \frac{1}{\sqrt{2n}}\,, \tag{5.1.3}$$

and the statement follows. □

It's worth to mention that the estimate (5.1.3) is not bad. In reality,

$$\lim_{n\to\infty} \sqrt{n}\, \frac{1 \cdot 3 \cdot 5 \cdot \,...\, \cdot (2n-1)}{2 \cdot 4 \cdot 6 \ldots \cdot 2n} = \frac{1}{\sqrt{2\pi}}\,.$$

This follows from the Wallis formula which, hopefully, you will learn in the second semester.

5.2. Two theorems. Now we prove two rather useful results. They assert that if $\{x_n\}$ is a convergent sequence, then sequences of arithmetic and geometric means must converge to the same limit.

Theorem 5.2.1. *Let* $\lim x_n = a$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = a.$$

Proof: Without loss of generality, we assume that $a = 0$, otherwise we just replace x_n by $x_n - a$. Put $M = \sup |x_n|$.[3] Given $\epsilon > 0$, find sufficiently large N such that $|x_k| < \epsilon$ for all $k \geq N$. Then

$$\left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| \leq \frac{1}{n} \sum_{k=1}^{n} |x_k| = \frac{1}{n} \sum_{k=1}^{N} |x_k| + \frac{1}{n} \sum_{k=N+1}^{n} |x_k| \leq \frac{N \cdot M}{n} + \epsilon < 2\epsilon \,,$$

provided that $n \geq \frac{N \cdot M}{\epsilon}$. This proves the theorem. □

Exercise 5.2.2. Prove or disprove the following statement: *If a sequence*

$$\frac{1}{n} \sum_{k=1}^{n} x_k$$

converges, then the sequence $\{x_k\}$ *converges as well.*

Exercise 5.2.3. If a sequence $\{x_n\}$ is such that $\lim(x_{n+1} - x_n) = c$, then

$$\lim \frac{x_n}{n} = c$$

as well.

Theorem 5.2.4. *Let x_n be a positive sequence such that* $\lim x_n = a$. *Then*

$$\lim_{n \to \infty} \sqrt[n]{x_1 x_2 \dots x_n} = a \,.$$

Proof: The idea of the proof is the same as in the previous theorem. First consider the case when the limit $a \neq 0$. Then without loss of generality, we assume that $a = 1$, otherwise we just replace x_n by x_n/a. Put $M = \sup |x_n|$, and $m = \inf |x_n|$. Observe that $m > 0$ (why?). Given $\epsilon > 0$, we have $1 - \epsilon < x_n < 1 + \epsilon$ for all sufficiently large $n > N$. Then

$$x_1 \cdot \ \dots \ \cdot x_n < M^N (1+\epsilon)^{n-N} = \big(M/\epsilon\big)^N (1+\epsilon)^n$$

and

$$\{x_1 \cdot \ \dots \ \cdot x_n\}^{1/n} < Q^{1/n}(1+\epsilon)$$

with $Q = (M/\epsilon)^N$. Since $Q^{1/n} \to 1$ as $n \to \infty$, we can choose N_1 (depending on ϵ and M) such that, for $n > N_1$, we have $Q^{1/n} < 1 + \epsilon$. Whence,

$$\{x_1 \cdot \ \dots \ \cdot x_n\}^{1/n} < (1+\epsilon)^2$$

for $n > \max(N, N_1)$. Similarly

$$\{x_1 \cdot \ \dots \ \cdot x_n\}^{1/n} \geq (1-\epsilon)^2$$

(check this!). If $\epsilon < 1$, these two estimates yield

$$-2\epsilon < (1-\epsilon)^2 - 1 \leq \{x_1 \cdot x_2 \cdot \ \dots \ \cdot x_n\}^{1/n} - 1 \leq (1+\epsilon)^2 - 1 < 3\epsilon \,,$$

completing the proof.

The case $a = 0$ is similar, and we leave it as an exercise. □

[3] More formally, $M = \sup\{|x_n| \colon n \in \mathbb{N}\}$.

Corollary 5.2.5. *Let* $t_n > 0$ *and*

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = c.$$

Then $\lim\{t_n\}^{1/n} = c$ *as well.*

Proof: we reduce this statement to Theorem 5.2.4. Put

$$x_1 := t_1, \qquad x_n = \frac{t_n}{t_{n-1}}\,.$$

Then $t_n = x_1 \cdot x_2 \cdot\, ...\, \cdot x_n$ and the statement follows from Theorem 5.2.4. $\square$

5.3. More examples.

5.3.1. Take in the previous corollary $t_n = \binom{2n}{n}$ (the binomial coefficient "choose n from $2n$"). The corollary is applicable since

$$\frac{t_{n+1}}{t_n} = \frac{(2n+2)!}{(\,(n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{(2n+1)(2n+2)}{(n+1)^2}\,,$$

tends to 4 when $t \to \infty$. We obtain

$$\lim_{n\to\infty} \sqrt[n]{\binom{2n}{n}} = 4\,.$$

Exercise 5.3.1. For a (fixed) natural k, find

$$\lim_{n\to\infty} \sqrt[n]{\binom{kn}{n}}\,.$$

The next two limits are quite famous.

5.3.2. Let $x_0 > 0$ and

$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \qquad a > 0\,.$$

Then the sequence $\{x_n\}$ converges to $\sqrt{a}$.

This is an iterative Newton method of finding square roots.

If we know that the sequence $\{x_n\}$ is convergent, then it is quite easy to guess that the limit is $\sqrt{a}$. Indeed, denote the limit c. Then using the recurrence from the definition of $\{x_n\}$, we get an equation

$$c = \frac{1}{2}\left(c + \frac{a}{c}\right).$$

That is, $c^2 = a$ and $c = \sqrt{a}$.

This argument is not accurate since we have not checked that $c > 0$. Anyway, below we will give a rigorous proof that x_n converges to a.

Proof: in order to simplify recursion, let us replace x_n by

$$\xi_n := \frac{x_n - \sqrt{a}}{\sqrt{a}}\,.$$

Then $x_n = \sqrt{a}(1+\xi_n)$. Let us find a recursion for ξ_n: substituting the previous formula into recursion for x_n, we get

$$\sqrt{a}(1+\xi_{n+1}) = \frac{1}{2}\left(\sqrt{a}(1+\xi_n) + \frac{a}{\sqrt{a}(1+\xi_n)}\right).$$

Whence (after some simplifications)

$$\xi_{n+1} = \frac{\xi_n^2}{2(1+\xi_n)}.$$

Next, observe that ξ_n are positive for any $n \in \mathbb{N}$. Indeed, $1+\xi_0 = \frac{x_0}{\sqrt{a}} > 0$, so that $\xi_1 > 0$. Then $\xi_2 > 0$ etc. Therefore,

$$\xi_{n+1} < \frac{\xi_n^2}{2\xi_n} = \frac{\xi_n}{2} < \ldots < \frac{\xi_1}{2^n}.$$

That is, ξ_n converges to zero and x_n converges to $\sqrt{a}$. $\square$

The proof above also gives a convergence of the Newton algorithm with the rate of geometric progression:

$$|x_n - \sqrt{a}| < \frac{\text{Const}}{2^n}.$$

In fact, the convergence even faster. This explain a remarkable efficiency of Newton's method.

Exercise 5.3.2. Try to give a better estimate of $|x_n - \sqrt{a}|$. Using Newton method (and calculator, if needed) find $\sqrt{111}$ with error of order 10^{-6}. How many iterations were you needed for that?

5.3.3. The sequence

$$x_n := \left(1 + \frac{1}{n}\right)^n$$

converges to a limit. To prove this, we define another sequence

$$y_n := \left(1 + \frac{1}{n}\right)^{n+1}.$$

We'll show that the sequence $\{y_n\}$ decays. Then since it is lower bounded ($y_n > 1$) it is convergent. Since

$$x_n = y_n \cdot \frac{n}{n+1}$$

and the second factor on the right hand side converges to one, x_n converges to the same limit as y_n.

To check that $\{y_n\}$ decays, we use Bernoulli's inequality. We have

$$\begin{aligned}
\frac{y_{n-1}}{y_n} = \frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} &= \frac{n^{2n+1}}{(n-1)^n(n+1)^{n+1}} \\
&= \frac{n^{2n}}{(n^2-1)^n} \cdot \frac{n}{n+1} = \left(1+\frac{1}{n^2-1}\right)^n \cdot \frac{n}{n+1} \\
&\geq \left(1+\frac{n}{n^2-1}\right) \cdot \frac{n}{n+1} > \left(1+\frac{1}{n}\right) \cdot \frac{n}{n+1} = 1\,,
\end{aligned}$$

completing the argument. □

The limit of this sequence is denoted by e. This is one of the most important constants. It's easy to see that $2 \leq e < 3$. Indeed, by Bernoulli's inequality

$$x_n = \left(1+\frac{1}{n}\right)^n \geq 1 + n\,\frac{1}{n} = 2\,.$$

To get the upper bound, note that

$$y_5 = \left(1+\frac{1}{5}\right)^6 = \left(\frac{6}{5}\right)^6 = \frac{46656}{15625} < 3\,.$$

Since the sequence y_n decays, its limit is less than 3. The approximate value is $e \approx 2.18281828459...$. Later, we'll find another representation for this constant:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ ...\ + \frac{1}{n!}\right)$$

which is more convenient for numerical computation of e. We will also prove that e is an irrational number.

6. Cauchy's sequences. Upper and lower limits. Extended convergence

In this lecture, we continue our study of convergent sequences.

6.1. Cauchy's sequences. Suppose, we need to check that some sequence converges but we have no clue about its limiting value. The definition of the limit will not help us too much: it is not an easy task to verify it without a priori knowledge about the limit. It would be useful to have an equivalent definition of convergence which does not mention the limiting value at all.

Definition 6.1.1 (Cauchy's sequence). *A sequence $\{x_n\}$ is called Cauchy's sequence, if*

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \qquad \textit{such that} \qquad \forall m, n \geq N \quad |x_n - x_m| < \epsilon\,. \tag{C}$$

Theorem 6.1.2 (Cauchy). *A sequence $\{x_n\}$ is convergent if and only if it is Cauchy's sequence.*

Proof: In one direction the result is clear: if the sequence $\{x_n\}$ converges to a limit a, then according to the definition of the limit,

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \qquad \text{such that} \qquad \forall m, n \geq N$$

$$|x_n - a| < \epsilon\,, \qquad |x_m - a| < \epsilon\,,$$

and therefore

$$|x_n - x_m| = |(x_n - a) + (a - x_m)| < 2\epsilon\,,$$

proving that $\{x_n\}$ is Cauchy's sequence.

In the other direction, first, let us observe that the sequence $\{x_n\}$ is bounded: choose $N \in \mathbb{N}$ such that

$$x_N - 1 < x_m < x_N + 1$$

for all $m \geq N$. Then the bound for $|x_n|$ is

$$\sup_n |x_n| \leq \max\{|x_1|, |x_2|, ..., |x_{N-1}|, |x_N| + 1\}\,.$$

Now, introduce the sequences

$$\underline{x}_n = \inf_{m \geq n} x_m\,, \qquad \overline{x}_n = \sup_{m \geq n} x_m\,.$$

The values $\underline{x}_n$, and $\overline{x}_n$ are finite since the sequence $\{x_n\}$ is bounded. Compare $\underline{x}_n$ with $\underline{x}_{n+1}$: in the definition of $\underline{x}_{n+1}$ we take an infimum over a smaller set, therefore, $\underline{x}_{n+1} \geq \underline{x}_n$. Similarly, $\overline{x}_{n+1} \leq \overline{x}_n$. Besides, we always have $\underline{x}_n \leq \overline{x}_n$. Summarizing,

$$... \leq \underline{x}_n \leq \underline{x}_{n+1} \leq \ ... \ \leq \bar{x}_{n+1} \leq \bar{x}_n \leq \ ...\,,$$

and we get a sequence of closed nested intervals $[\underline{x}_n, \bar{x}_n]$. By Cantor's lemma, the intersection of these intervals is not empty, so we choose

$$c \in \bigcap_{n \geq 1} [\underline{x}_n, \overline{x}_n]$$

as a candidate for $\lim x_n$. We claim that the sequence $\{x_n\}$ converges to c.

Note that the values c and x_n both belong to the interval $[\underline{x}_n, \overline{x}_n]$. Hence

$$|c - x_n| \le \overline{x}_n - \underline{x}_n \,.$$

In order to estimate the difference on the left hand side, fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ according to (C). Let $n \ge N$. Then for some $m \ge n$

$$\overline{x}_n \left(= \sup_{k \ge n} x_k\right) \le x_m + \epsilon \le x_n + 2\epsilon,$$

and similarly

$$\underline{x}_n \ge x_n - 2\epsilon \,.$$

Hence $\overline{x}_n - \underline{x}_n \le (x_n + 2\epsilon) - (x_n - 2\epsilon) = 4\epsilon$, and $|c - x_n| \le 4\epsilon$ completing the proof. $\square$

Example 6.1.3. Consider the sequence

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots \,.$$

Then

$$S_{2n} - S_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2} \,.$$

Hence the sequence $\{S_n\}$ is not Cauchy's sequence and therefore is divergent.

Of course, one can check divergence of this sequence without appeal to the Cauchy criterion. The property $S_{2n} - S_n \ge \frac{1}{2}$ we've established shows that the sequence S_n is unbounded.

6.2. Upper and lower limits. In the proof of the Cauchy theorem, for a given sequence $\{x_n\}$, we defined two sequences $\{\underline{x}_n\}$ and $\{\overline{x}_n\}$. Sometimes, they are called the *lower and upper envelopes of the sequence* $\{x_n\}$. If the sequence $\{x_n\}$ was not upper bounded, then its upper envelope is identically $+\infty$, if the sequence is not lower bounded, then its lower envelope is identically $-\infty$.

Note that if the sequence $\{x_n\}$ does not decrease, then $\underline{x}_n = x_n$, and if the sequence $\{x_n\}$ does not increase, then $\overline{x}_n = x_n$.

Example 6.2.1.

(i) If $x_n = \frac{1}{n}$, then $\overline{x}_n = \frac{1}{n}$ while $\underline{x}_n = 0$.

(ii) If $x_n = (-1)^n$, then $\underline{x}_n = -1$ while $\overline{x}_n = 1$.

(iii) If $x_n = \frac{(-1)^n}{n}$, then

$$\{\underline{x}_n\} = \{-1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{5}, \ldots\}, \qquad \{\overline{x}_n\} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \ldots\} \,.$$

In the course of the proof of Cauchy's theorem, we observed that

(i) the sequence $\underline{x}_n$ does not decrease;
(ii) the sequence $\overline{x}_n$ does not increase;
(iii) $\forall m, n \;\; \underline{x}_n \le \overline{x}_m$

In the case when the sequence $\{x_n\}$ is not bounded this requires an obvious agreement about inequalities which involve the symbols $\pm\infty$.

In particular, we see that the both envelopes are monotonic sequences, and therefore they converge when they are bounded. Now, we look more carefully at their limits.

Definition 6.2.2 (limsup, liminf)**.** *If the sequence $\{x_n\}$ is bounded, then its upper limit (or limit superior) is*

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \overline{x}_n = \lim_{n\to\infty} \sup_{m\geq n} x_m \, .$$

If the sequence $\{x_n\}$ is not upper bounded, we say that its upper limit equals $+\infty$.

If the sequence $\{x_n\}$ is lower bounded, then its lower limit is

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \underline{x}_n = \lim_{n\to\infty} \inf_{m\geq n} x_m \, .$$

If the sequence $\{x_n\}$ is not lower bounded, we say that its lower limit equals $-\infty$.

We see that always $\liminf x_n \leq \limsup x_n$.

Deciphering the definition of the upper limit, we see that $\limsup x_n = L$ if and only if the following two conditions are fulfilled:

(a) $\forall \epsilon > 0 \;\; \exists N \in \mathbb{N}$ such that $\forall n \geq N \; x_n < L + \epsilon$;

(b) $\forall \epsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n > N$ such that $x_n > L - \epsilon$.

Indeed, condition (a) says that $\forall n \geq N \;\; \overline{x}_n < L + \epsilon$; i.e., that $\lim \overline{x}_n \leq L$, while condition (b) says that $\forall n \geq N \;\; \overline{x}_n \geq L$; i.e., that $\lim \overline{x}_n \geq L$.

Exercise 6.2.3. Formulate and prove the similar criterium for $\liminf x_n$.

Theorem 6.2.4. *A sequence $\{x_n\}$ converges to the limit a if and only if*

$$\liminf x_n = \limsup x_n = a \, . \tag{L}$$

In other words, the sequence $\{x_n\}$ converges to the limit a if and only if the envelopes $\{\overline{x}_n\}$ and $\{\underline{x}_n\}$ converge to the same limit a.

Proof: In one direction, since $\underline{x}_n \leq x_n \leq \overline{x}_n$, then (L) combined with the two policemen theorem give us convergence of $\{x_n\}$.

In the other direction, if $\{x_n\}$ converges to the limit a, then we fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\forall m \geq N$ we have $|x_m - a| < \epsilon$. If $n \geq N$, then for some $m \geq n$ we have

$$a - \epsilon < x_n \leq \overline{x}_n \leq x_m + \epsilon < a + 2\epsilon \, ,$$

therefore $\limsup x_n = \lim \overline{x}_n = a$, and similarly $\liminf x_n = a$ proving (L). $\square$

Note that we use more or less the same argument as in the proof of Cauchy's theorem.

Exercise 6.2.5. Check that

$$\limsup(-x_n) = -\liminf x_n\,;$$

and if $0 < a \le x_n \le b < \infty$,

$$\limsup 1/x_n = 1/\liminf x_n\,.$$

Prove the inequalities

$$\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n\,,$$

$$\limsup(x_n \cdot y_n) \le \limsup x_n \cdot \limsup y_n\,,$$

(in the second inequality, we assume that $x_n, y_n > 0$). Show that, if one of the sequences $\{x_n\}$ or $\{y_n\}$ converges, then there is an equality sign in these inequalities.

Exercise 6.2.6. Let $0 < a \le x_n \le b < +\infty$. Show that

$$\limsup x_n \cdot \limsup \frac{1}{x_n} \ge 1\,.$$

Show that the equality sign is attained there if and only if the sequence $\{x_n\}$ is convergent.

Exercise 6.2.7. Let a_n be positive numbers such that

$$A_n = \sum_{k=1}^{n} a_k \to \infty, \quad n \to \infty\,.$$

For any sequence $\{t_n\}$ set

$$\widetilde{t}_n = \frac{1}{A_n}\sum_{k=1}^{n} a_k t_k\,.$$

Then

$$\liminf t_n \le \liminf \widetilde{t}_n \le \limsup \widetilde{t}_n \le \limsup t_n\,.$$

In particular, if $t_n \to L$, then $\widetilde{t}_n \to L$. This extends Theorem 5.2.1 which corresponds to the case $a_n = 1$.

6.3. Convergence in wide sense.

Definition 6.3.1 (convergence to ∞)**.** *The sequence x_n converges to ∞, if*

$$\forall M < \infty \quad \exists N \in \mathbb{N} \qquad \textit{such that} \qquad \forall n \ge N \quad |x_n| \ge M\,.$$

Of course, this just means that the sequence $\{1/x_n\}$ converges to zero and nothing else.

Definition 6.3.2 (convergence to $\pm\infty$)**.** *The sequence $\{x_n\}$ converges to $+\infty$ if*

$$\forall M < \infty \quad \exists N \in \mathbb{N} \qquad \textit{such that} \qquad \forall n \ge N \quad x_n \ge M\,,$$

and that a sequence $\{x_n\}$ converges to $-\infty$ if

$$\forall M > -\infty \quad \exists N \in \mathbb{N} \qquad \textit{such that} \qquad \forall n \ge N \quad x_n \le M\,,$$

Exercise 6.3.3. Give 3 examples of sequences $\{x_n\}$ satisfying each of the following properties:

(i) $\{x_n\}$ converges to $+\infty$;
(ii) $\{x_n\}$ converges to $-\infty$;
(iii) $\{x_n\}$ converges to ∞ but converges neither to $+\infty$ nor to $-\infty$;
(iv) $\{x_n\}$ is divergent in the wide sense.

(There should be 12 examples all together.)

Exercise 6.3.4. Extend Theorem 6.2.4 to the wide convergence.

Exercise 6.3.5 (Stoltz' lemma)**.** Suppose the sequence $\{y_n\}$ increases and $\lim y_n = +\infty$. If there exists the limit

$$\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L\,,$$

then

$$\lim \frac{x_n}{y_n} = L\,.$$

Here, L is a real number or $\pm\infty$.
Hint: use Exercise 6.2.7 with

$$a_k = y_k - y_{k-1}, \qquad t_k = \frac{x_k - x_{k-1}}{y_k - y_{k-1}}$$

(for convenience, we set $x_0 = y_0 = 0$).

Exercise 6.3.6. Show that for each $p \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n} k^p = \frac{1}{p+1}\,.$$

Hint: use Stoltz' lemma.

Exercise* 6.3.7. Let $x_n \leq \frac{1}{2}(x_{n-1} + x_{n-2})$. Show that the sequence $\{x_n\}$ is convergent (either to a finite number or to $-\infty$.

7. Subsequences and partial limits.

7.1. Subsequences. Let $\{x_n\}$ be a sequence, we want to define its *subsequence.* In plain words, we write down the sequence $\{x_n\}$ as a string, and then drop out some elements from this string taking care that an infinite number of elements remain. What remains is called *a subsequence.* More formally, we take an increasing sequence $\{n_k\}$ of natural numbers ($n_1 < n_2 <...< n_k <...$) and form a new function $k \mapsto x_{n_k}$ defined on $\mathbb{N}$.

Exercise 7.1.1. Prove that any sequence contains a monotonic subsequence.

Exercise 7.1.2. Show that a monotonic sequence converges if it contains a convergent subsequence.

Our first result is a version of the Bolzano-Weierstrass lemma 3.3.3.

Lemma 7.1.3 (Bolzano-Weierstrass)**.** *Each bounded sequence has a convergent subsequence.*

Proof: Let E be the set of all values attended by the sequence $\{x_n\}$. Consider two cases:
(a) The set E is finite. The we can choose an infinite number of elements in our sequence which have the same value:

$$x_{n_1} = x_{n_2} = \ ... \ = x_{n_k} = \ ... = x \in E\,, \qquad n_1 < n_2 < \ ... \ < n_k < \ ... \,.$$

We get a subsequence $\{x_{n_k}\}$ converging to x.
(b) Now, assume that the set E is infinite. According to the Bolzano-Weierstrass lemma about accumulation points, E has an accumulation point x. Choose $n_1 \in \mathbb{N}$ such that $|x_{n_1} - x| < 1$. Then choose $n_2 > n_1$ such that $|x_{n_2} - x| < \frac{1}{2}$, etc. At the k-th step, choose $n_k > n_{k-1}$ such that $|x_{n_k} - x| < \frac{1}{k}$. Clearly, the subsequence $\{x_{n_k}\}$ converges to x. □

Another proof of this lemma follows from the first exercise above combined with a theorem about convergence of monotonic bounded sequences we proved earlier.

It is not difficult to formulate and to prove a version of this lemma for the extended convergence:

Lemma 7.1.4 (Bolzano-Weierstrass for extended convergence)**.** *Each sequence has a subsequence convergent in the wide sense.*

Exercise 7.1.5. Prove this lemma.

7.2. Partial limits. If a subsequence $\{x_{n_k}\}$ is convergent, then its limit is called a *partial limit* of $\{x_n\}$. It's not difficult to verify that if the original sequence $\{x_n\}$ converges to the limit a, then any of its subsequences also converges to a. Define the limit set $PL(\{x_n\})$ of *all* partial limits of the sequence $\{x_n\}$.

Theorem 7.2.1. *Let $\{x_n\}$ be a bounded sequence. Then*

$$\limsup x_n = \max\{c \colon c \in PL(\{x_n\})\}\,,$$

and

$$\liminf x_n = \min\big\{c\colon c \in PL(\{x_n\})\,\big\}\,.$$

Proof: We'll prove only the first of these two relations, the proof of the second one is similar. In fact, we have to prove two statements: (α) any partial limit of $\{x_n\}$ does not exceed $\limsup x_n$ and (β) $\limsup x_n \in PL(\{x_n\})$.

Let us recall what we already know about the value $L = \limsup x_n$:

(a) $\forall \epsilon > 0 \;\; \exists N \in \mathbb{N}$ such that $\forall n \geq N \;\; x_n < L + \epsilon$;

(b) $\forall \epsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n > N$ such that $x_n > L - \epsilon$.

A minute reflection shows that (α) follows from (a) and then (β) follows from (a) and (b) (check this formally!) completing the proof. □

In the previous lecture we proved that the sequence $\{x_n\}$ converges to a limit a if and only if

$$\liminf x_n = \limsup x_n = a\,.$$

Combining this with the theorem above, we obtain

Corollary 7.2.2. *A sequence $\{x_n\}$ converges if and only if the set of its limit set is a singleton: $PL(\{x_n\}) = \{a\}$. In this case, $a = \lim x_n$.*

Exercise 7.2.3. Find $\limsup x_n$, $\liminf x_n$, $\sup x_n$, $\inf x_n$, and the set $\mathrm{PL}(\{x_n\})$ of all partial limits for the sequences

$$x_n = \cos^n \frac{n\pi}{4} \qquad \text{and} \qquad x_n = n^{(-1)^n n}.$$

Exercise 7.2.4. Construct a sequence whose set of partial limits coincides with the *closed* interval $[0,1]$.

Exercise 7.2.5. **(a)** Show that there is no sequence $\{x_n\}$ with $\mathrm{PL}(\{x_n\}) = (0,1)$.

(b) Show that there is no sequence $\{x_n\}$ with $\mathrm{PL}(\{x_n\}) = \{1, \frac{1}{2}, ..., \frac{1}{n}, ...\}$.

(c) Show that any accumulation point of the set $\mathrm{PL}(\{x_n\})$ must belong to $\mathrm{PL}(\{x_n\})$ as well.

Exercise 7.2.6. Suppose the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to the same limit. Show that the sequence $\{x_n\}$ converge.

Exercise 7.2.7. Let $\{x_n\}$ be a sequence such that $\forall n \geq 1 \;\; |x_{n+1} - x_n| \leq \frac{1}{2^n}$. Can this sequence be unbounded? Can this sequence be divergent? The same questions for $|x_{n+1} - x_n| \leq \frac{1}{n}$.

Problem 7.2.8. Let $\{x_n\}$ be a bounded sequence such that

$$\lim(x_n - x_{n-1}) = 0.$$

Show that the set $\mathrm{PL}(\{x_n\}$ *coincides* with the (closed) interval

$$[\liminf x_n, \limsup x_n].$$

Problem* 7.2.9 (Fekete's lemma)**.** Let a sequence $\{x_n\}$ satisfy $0 \le x_{m+n} \le x_m + x_n$, $\forall m, n \in \mathbb{N}$ (such sequences are called *subadditive*). Show that there exists the limit

$$\lim_{n\to\infty} \frac{x_n}{n} = \inf_{n\ge 1} \frac{x_n}{n}.$$

7.2.1. *Appendix: The continued fraction of the golden mean and the Fibonacci numbers.* Let

$$x_{n+1} = 1 + \frac{1}{x\,n}\,, \qquad x_0 = 1\,.$$

We shall show that $\lim x_n = \frac{\sqrt{5}+1}{2}$. (This number is called the *golden mean.*) In other words,

$$1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}} = \frac{\sqrt{5}+1}{2}\,.$$

The expression on the left hand side is an example of a *continued fraction.*

First, let us write down several the beginning of the sequence $\{x_n\}$:

$$x_0 = \frac{1}{1}, \quad x_1 = 1 + \frac{1}{1} = \frac{2}{1}, \quad x_2 = 1 + \frac{1}{2} = \frac{3}{2}, \quad x_3 = 1 + \frac{2}{3} = \frac{5}{3},$$

$$x_4 = 1 + \frac{3}{5} = \frac{8}{5}, \quad x_5 = 1 + \frac{5}{8} = \frac{13}{8}, \quad x_6 = 1 + \frac{8}{13} = \frac{21}{13}, \dots .$$

Let $x_n = \frac{p_n}{q_n}$, p_n and q_n are mutually prime natural numbers. Then by induction

$$p_n = p_{n-1} + p_{n-2}, \qquad p_0 = 1, \quad p_1 = 2,$$

$$q_n = q_{n-1} + q_{n-2}, \qquad q_0 = q_1 = 1.$$

We see that p_n and q_n are famous *Fibonacci numbers.* We conclude from these formulas that

$$q_n p_{n-1} - q_{n-1} p_n = -(q_{n-1} p_{n-2} - q_{n-2} p_{n-1}) = \dots = (-1)^n (q_1 p_0 - q_0 p_1) = (-1)^n \quad (A)$$

and that

$$q_n p_{n-2} - q_{n-2} p_n = q_{n-1} p_{n-2} - q_{n-2} p_{n-1} = (-1)^{n-1}\,. \qquad (B)$$

From (A) we get

$$x_{n-1} - x_n = \frac{(-1)^n}{q_n q_{n-1}}\,, \qquad (C)$$

from (B) we get

$$x_{n-2} - x_n = \frac{(-1)^{n-1}}{q_n q_{n-2}}\,. \qquad (D)$$

Looking at (D), we conclude by induction that the subsequence $\{x_{2n}\}$ increases (and is < 2), while the subsequence $\{x_{2n+1}\}$ decreases (and is > 1). Therefore, the both subsequences converges. Further, the increasing sequence of natural numbers $\{q_n\}$ tends to $+\infty$, so looking at (C), we conclude that the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have the same limit α. From the initial recursion we see that α is a positive solution to the equation $\alpha = 1 + \frac{1}{\alpha}$, that is $\alpha = \frac{1+\sqrt{5}}{2}$.

Problem 7.2.10. Show that

$$1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}} = \sqrt{2}\,.$$

If you want to learn more about fascinated continued fractions, read section 1.6 of the book by Hairer and Wanner mentioned in the introduction.

8. Infinite series

8.1. Let $\{a_j\}$ be a sequence of real numbers, the sum $a_n + a_{n+1} + ... + a_m$ is denoted by

$$\sum_{j=n}^{m} a_j = \sum_{n\le j\le m} a_j = \sum_{n}^{m} a_j \,.$$

Our goal is to prescribe a meaning for the sum of all terms of the sequence $\{a_j\}$; i.e. to the expression

$$\sum_{j=1}^{\infty} a_j = a_1 + a_2 + ... + a_n + ... \qquad (*)$$

called *(an infinite) series.* Numbers a_j are called the *terms.*

Define a sequence of partial sums $S_n = \sum_{j=1}^{n} a_j$.

Definition 8.1.1. *The series $\sum_1^\infty a_j$ is called convergent if the sequence S_n of partial sums converges. In this case, the limiting value $S = \lim S_n$ is called the sum of the series: $\sum_1^\infty a_j = S$.*

Dealing with series, usually it is not very difficult to check convergence or divergence, to find the value of the sum is a much more delicate problem which we almost will not touch here. We start with several simple observations and examples.

1. *Convergence or divergence of the series depends on its tail only;* i.e. if two series have the same terms a_j for $j \ge j_0$ then they converge or diverge simultaneously.

2. *If the series $(*)$ converges, then* $\lim a_n = 0$. Indeed, $a_n = S_{n+1} - S_n$ and therefore

$$\lim a_n = \lim (S_{n+1} - S_n) = \lim S_{n+1} - \lim S_n = S - S = 0 \,.$$

8.2. Examples.

8.2.1. *Geometric series.* Let $a_j = q^{j-1}$. Then

$$S_n = \frac{1-q^n}{1-q} \,,$$

and if $|q| < 1$ the series converges to $\frac{1}{1-q}$. In the case $|q| \ge 1$ the series is divergent.

8.2.2. *Harmonic series.* Let $a_j = \frac{1}{j}$. Then, as we know, $\lim S_n = +\infty$ and therefore the series is divergent. Later in this course, we will show that there exists the limit

$$\lim_{n\to\infty} (S_n - \log n) = \gamma \,,$$

called the *Euler constant.*

8.2.3. Let $a_j = (-1)^j$. Then $S_n = 0$ if n is even, and $S_n = 1$ if n is odd. Therefore, the series diverges.

8.2.4. Let

$$a_j = \frac{1}{(\alpha+j)(\alpha+j+1)}\,.$$

Observe that

$$a_j = \frac{1}{\alpha+j} - \frac{1}{\alpha+j+1},$$

so that

$$S_n = \sum_{j=1}^{n}\left[\frac{1}{\alpha+j} - \frac{1}{\alpha+j+1}\right] = \frac{1}{\alpha+1} - \frac{1}{\alpha+n+1}$$

(such sums with cancelation of all intermediate terms are called sometimes *telescopic*). We see that the series converges to the value $\frac{1}{\alpha+1} = \lim S_n$.

8.2.5. Let

$$a_j = \frac{(-1)^{j-1}}{j}.$$

In this case, we consider separately partial sums with even and odd indices. We have

$$S_{2n} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \,...\, + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)\,.$$

Therefore, the sequence S_{2n} increases. It is bounded from above by 1:

$$S_{2n} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \,....\, < 1\,.$$

Hence, $\{S_{2n}\}$ converges to the limit S. Further the sequence $\{S_{2n+1}\}$ converges to the same limit:

$$\lim S_{2n+1} = \lim\left(S_{2n} + \frac{(-1)^{2n}}{2n+1}\right) = \lim S_{2n} = S\,.$$

Therefore, the whole sequence S_n converges. As we have seen $S_{2n} \uparrow S$, it is not difficult to see that $S_{2n+1} \downarrow S$ (check this!).

The sum of this series is $S = \log 2$, we'll be able to explain this later.

Exercise 8.2.1 (Leibniz)**.** Consider the series $\sum(-1)^k a_k$ with $a_k \downarrow 0$. Prove that the series converges to a value S and that the error of the n-th partial sum $S_n = \sum_{k=1}^{n}(-1)^k a_k$ does not exceed the first neglected term:

$$|S - S_n| \le a_{n+1}\,.$$

Hint: repeat the argument from Example 8.2.5.

A warning. Many operations we used to do with finite sums generally speaking are illegal with infinite convergent sums. Let us return to Example 8.2.5. We have

$$2S = \frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} ... = \frac{2}{1} - \frac{1}{1} + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + ...$$

Consider separately the terms with even and odd denominators. The terms with even denominators are negative:

$$-\frac{1}{2}, \quad -\frac{1}{4}, \quad -\frac{1}{5}, ...$$

There are two terms with any odd denominator, one term is positive, another one is negative, and the difference is positive:

$$\frac{2}{1} - \frac{1}{1} = \frac{1}{1}, \quad \frac{2}{3} - \frac{1}{3} = \frac{1}{3}, \quad \frac{2}{5} - \frac{1}{5} = \frac{1}{5}, ...$$

Collecting the terms together in such a way that the denominators increase, we get

$$2S = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... = S\,.$$

Therefore, $S = 0$. On the other hand, this is definitely impossible, since the sequence S_{2n} increases to S, and $S_2 = \frac{1}{2}$, so that $S > \frac{1}{2}$. Find what was illegal in our actions.

Exercise* 8.2.2 (Riemann)**.** Given $s \in \mathbb{R}$, there is a rearrangement of the sequence $\left\{(-1)^k/k\right\}$ such that the corresponding series converges to the value s.

The same holds for any Leibniz series $\sum (-1)^k a_k$ with $a_k \downarrow 0$ and $\sum a_k = +\infty$.

8.3. Some results. There are two common tools to investigate convergence of series. The theorem on convergence of upper bounded increasing sequences immediately gives us

Theorem 8.3.1. *The series with positive terms converges if and only if the sequence of its partial sums is upper bounded.*

The Cauchy's criterion for convergence of sequences immediately gives us

Theorem 8.3.2 (Cauchy's criterion for the series convergence)**.** *The series* $(*)$ *converges if and only if* $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ *such that* $\forall m \geq n \geq N$

$$|a_n + a_{n+1} + ... + a_m| < \epsilon\,.$$

Now, we turn to the applications of these criteria.

Corollary 8.3.3. *Let* $0 < a_j \leq b_j$, $j \geq j_0$. *If the series* $\sum b_j$ *converges, then the series* $\sum a_j$ *also converges. If the series* $\sum a_j$ *diverges, then the series* $\sum b_j$ *also diverges.*

This follows from Theorem 1. Sometimes, another form of the same result is useful:

Corollary 8.3.4. *If a_j and b_j are positive and*

$$0 < \liminf \frac{a_j}{b_j} \leq \limsup \frac{a_j}{b_j} < \infty\,,$$

then the series $\sum a_j$ and $\sum b_j$ converge or diverge simultaneously.

Usually, in applications of this corollary there exists the limit

$$\lim_{j\to\infty} \frac{a_j}{b_j} = L\,,$$

and we need only to check that $0 < L < +\infty$.

Example 8.3.5. The series

$$\sum_{j=1}^{\infty} \frac{1}{j^2}$$

converges. This we see by comparison with the convergent series

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)}.$$

In this case, the quotient of the terms tends to 1.

Example 8.3.6. The series

$$\sum_{j=1}^{\infty} \frac{\sqrt{j+1}}{j^{3/2}}$$

diverges. This we see by comparison with the divergent harmonic series $\sum_{j=1}^{\infty} \frac{1}{j}$.

8.4. Absolutely convergent series.

Definition 8.4.1 (absolute convergence)**.** The series $\sum a_j$ is called *absolutely convergent* if the series $\sum |a_j|$ converges.

Claim 8.4.2. *If the series converges absolutely, then it converges in the usual sense.*

This follows at once from the Cauchy criterion. In the opposite direction the result is wrong: the series $\sum \frac{(-1)^j}{j}$ converges but not absolutely.

In what follows we consider only series with positive terms a_j. The simplest was to check the convergence of such series is to compare them with the geometric series.

Claim 8.4.3 (Cauchy's root test)**.** *Set*

$$\alpha := \limsup a_j^{1/j}\,.$$

If $\alpha < 1$, then the series $\sum a_j$ converges. If $\alpha > 1$, then the series diverges.

Proof: Let $\alpha < 1$. Choose α': $\alpha < \alpha' < 1$. Then according to the definition of the upper limit, $a_j < {\alpha'}^j$, $j \geq j_0$, and by Corollary 8.3.3 the series converges.

If $\alpha > 1$, then choose α' such that $1 < \alpha' < \alpha$, and by the definition of $\limsup$ we see that there are arbitrary large indices j such that $a_j \geq {\alpha'}^j > 1$. Therefore, the sequence a_j *does not* tend to zero[4], and the series $\sum a_j$ diverges. $\Box$

Exercise 8.4.4 (D'Alembert's "ratio test"). Suppose $a_j > 0$ and there exists the limit

$$\beta = \lim_{j\to\infty} \frac{a_{j+1}}{a_j} \,.$$

If $\beta < 1$, then the series converges, if $\beta > 1$, the series diverges.

Hint: use Corollary 5.2.5.

Example 8.4.5. The series

$$\sum_{j\geq 2} \frac{1}{(\log j)^j}$$

converges by application of the Cauchy test.

Example 8.4.6. The series

$$\sum_{j\geq 1} \frac{x^j}{j!}$$

(absolutely) converges for any real x by application of the d'Alambert test.

Example 8.4.7. The series

$$\sum_{j\geq 1} \frac{x^j}{j^s}$$

converges for $x < 1$ and diverges for $x > 1$. This can be obtain easily by application of any of the two tests, and the answer does not depend on the choice of real s. In the remaining case $x = 1$ the answer depends on s. As we already know, the series diverges for $s = 1$ and therefore for all $s \leq 1$. A bit later, we'll see that the series converges for all $s > 1$.

The both tests do not lead to any conclusion in the "boundary" case when α or β equal 1. In this case, the following theorem is very useful:

Theorem 8.4.8 (Cauchy's compression). *Let a_j be a non-increasing sequence of positive numbers. Then the series $\sum_{j\geq 1} a_j$ converges and diverges simultaneously with the series $\sum_{k\geq 0} 2^k a_{2^k}$.*

Proof: Let s_n be a partial sum $\sum_{j=1}^n a_j$, let $A_k = 2^k a_{2^k}$, and let S_n be a partial sum $S_n = \sum_{k=0}^n A_k$. Since the terms a_j do not increase, for each $k \geq 0$ we have

$$\frac{1}{2} A_{k+1} = 2^k a_{2^{k+1}} \leq a_{2^k+1} + a_{2^k+2} + \,\dots\, + a_{2^{k+1}} \leq 2^k a_{2^k} = A_k \,.$$

[4]Moreover, $\limsup a_j = +\infty$.

Summing up these inequalities from $k = 0$ till $k = n$, we get

$$\frac{1}{2} S_{n+1} \le s_{2^{n+1}} \le S_n \,.$$

This means that the increasing sequence of partial sums $\{s_n\}$ is bounded from above if and only if the increasing sequence of partial sums $\{S_n\}$ is bounded from above. Therefore, the sequences s_n and S_n converge and diverge simultaneously. □

The theorem is useful since the new series $\sum_{k\ge 1} 2^k a_{2^k}$ usually has "better convergence" than the original one.

Example 8.4.9. The series

$$\sum_{n\ge 1} \frac{1}{n^s}$$

converges if and only if $s > 1$. Indeed, in this case the new series from Cauchy's theorem is

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^{ks}} = \sum_{k=1}^{\infty} 2^{k(1-s)} \,.$$

If $s > 1$, we get a convergent geometric series, if $s \le 1$ the terms do not tend to zero and the series diverges.

Exercise 8.4.10. Check convergence or divergence of the series $\sum_{n\ge 1} a_n$ when

$$a_n = 2^n n! n^{-n}, \qquad a_n = 3^n n! n^{-n}, \qquad a_n = \frac{1}{\log n!} \quad (n \ge 2),$$

$$a_n = n^n e^{-n^{1.001}}, \qquad a_n = \frac{n^{\log n}}{(\log n)^n}, \qquad a_n = \frac{(n!)^2}{(2n)!},$$

$$a_n = \left(\sqrt{n+1} - \sqrt{n-1}\right)^{\alpha}, \qquad a_n = \frac{\sqrt{n+1} - \sqrt{n-1}}{n^{\alpha}} \quad (\alpha \in \mathbb{R}),$$

$$a_n = \frac{1}{n \log^a n}, \qquad a_n = \frac{1}{n \log^a n \log\log^b n} \quad (a, b \in \mathbb{R})$$

Exercise 8.4.11. Suppose that $a_n \downarrow 0$, and $\sum a_n = +\infty$. Prove that

$$\sum \min(a_n, 1/n) = +\infty \,.$$

Hint: Use Cauchy's compression.

There are many interesting problems about the infinite series with positive terms. For instance,

Problem 8.4.12. Let $a_n \ge 0$ and the series $\sum a_n$ diverges.
(i) Show that the series $\sum \frac{a_n}{1 + a_n}$ also diverges.
(ii*) Let $S_n = a_1 + \ldots + a_n$. Show that

$$(a) \quad \sum_{n\ge 1} \frac{a_n}{S_n} = +\infty; \qquad (b) \quad \sum_{n\ge 1} \frac{a_n}{S_n^{1+\epsilon}} < \infty \quad \text{for each } \epsilon > 0\,.$$

9. Limits of functions. Basic properties

9.1. Cauchy's definition of limit. Denote by $U_\delta^*(a) = \{x\colon 0 < |x-a| < \delta\}$ the punctured δ-neighbourhood of a.

Definition 9.1.1 (the limit according to Cauchy)**.** Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$, and let a be an accumulation point of E. We say that f *has a limit* L *when* x *tends to* a *along* E: $\lim\limits_{E \ni x \to a} f(x) = L$, if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \forall x \in U_\delta^*(a) \bigcap E \qquad |f(x) - L| < \epsilon\,.$$

Usually, we deal with the case when the set E contains some punctured neighbourhood of a. Then we just say that f *has a limit* L *at the point* a: $\lim\limits_{x\to a} f(x) = L$, or $f(x) \to L$ for $x \to a$.

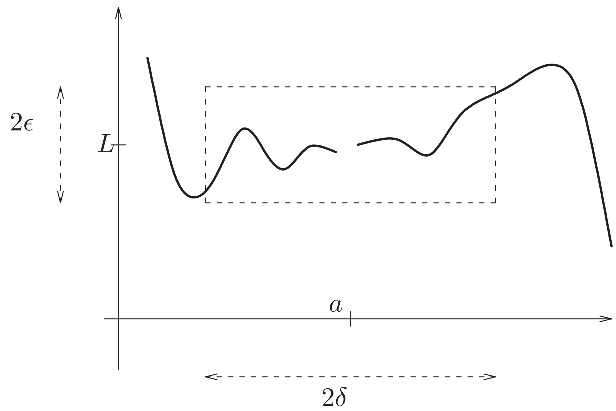

Figure 7. To the definition of the limit

Remarks:

i. Existence of the limit and its value do not depend on the value of the function $f(x)$ at the point $x = a$, moreover, the function f does not need to be defined at a at all. For example, the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = 2x + 1$, has the limit $\lim\limits_{x\to 0} f(x) = 1$. If we consider the function $f_1(x) : \mathbb{R} \to \mathbb{R}$ which equals $f(x)$ for $x \neq 0$ and equals C at the origin, then its limit at the origin is the same for any C:

$$\lim_{x\to 0} f_1(x) = \lim_{x\to 0} f(x) = 1\,.$$

ii. If $E_1 \subset E$, a is an accumulation of E_1 (and therefore of E) and the limit $\lim\limits_{E\ni x\to a} f(x)$ exists, then the limit of f along E_1 also exists and has the same value.

Example 9.1.2.

$$\lim_{x\to 0} x \sin\frac{1}{x} = 0\,.$$

More generally,

Claim 9.1.3. *If* $\lim_{x\to a} f(x) = 0$*, and a function* g *is bounded in a punctured neighbourhood* $U^*(a)$ *of* a*, then* $\lim_{x\to a} f(x)g(x) = 0$*.*

Proof: Indeed, set $M = \sup\{|g(x)| : x \in U^*(a)\}$, fix $\epsilon > 0$ and choose $\delta > 0$ such that

$$|f(x)| < \frac{\epsilon}{M} \qquad \text{for} \qquad x \in U^*_\delta(a)\,.$$

We may always assume that $U^*_\delta(a) \subset U^*(a)$, otherwise we make δ smaller. Then

$$|f(x)g(x)| < \frac{\epsilon}{M} \cdot M = \epsilon\,, \qquad x \in U^*_\delta(a)\,,$$

and we are done. $\square$

In the example above, $f(x) = x$ and $g(x) = \sin \frac{1}{x}$.

Agreement. If $E = (a, b)$ $(b > a)$, then we use notations

$$\lim_{x\downarrow a} f(x) = \lim_{x\to a+0} f(x) \overset{\text{def}}{=} \lim_{E\ni x\to a} f(x)$$

(this is called the limit from above, or the right limit). If $E = (b, a)$ $(b < a)$, then we write

$$\lim_{x\uparrow a} f(x) = \lim_{x\to a-0} f(x) \overset{\text{def}}{=} \lim_{E\ni x\to a} f(x)$$

(this is called the limit from below, or the left limit).

Example 9.1.4. $f(x) = \mathrm{sgn}(x)$. In this case the limit at the origin does not exist, however

$$\lim_{x\uparrow 0} \mathrm{sgn}(x) = -1, \qquad \lim_{x\downarrow 0} \mathrm{sgn}(x) = +1\,.$$

Exercise 9.1.5. Suppose that the limits from above and from below exist and are equal. Then the usual limit exists as well and has the same value.

Example 9.1.6. Let m and n be positive integers. Then

$$\lim_{x\to 1} \frac{x^m - 1}{x^n - 1} = \lim_{x\to 1} \frac{1 + x + ... + x^{m-1}}{1 + x + ... + x^{n-1}} = \frac{m}{n}\,.$$

As a corollary, we obtain the value for another limit:

$$\lim_{x\to 1} \frac{x^{1/m} - 1}{x^{1/n} - 1} = \frac{n}{m}\,.$$

Indeed, we introduce a new variable $x = t^{mn}$, then $t \to 1$ for $x \to 1$ (`why?`), and

$$\lim_{x\to 1} \frac{x^{1/m} - 1}{x^{1/n} - 1} = \lim_{t\to 1} \frac{t^n - 1}{t^m - 1} = \frac{n}{m}\,.$$

9.2. Heine's definition of limit. The next theorem shows the limit of functions can be defined using only the notion of limits of sequences.

Theorem 9.2.1. *The following two conditions are equivalent:*

(A) $\lim\limits_{E\ni x\to a} f(x) = L\,,$

and

(B) *for any sequence* $\{x_n\} \subset E\backslash\{a\}$ [5] *convergent to* a*, the sequence* $\{f(x_n)\}$ *converges to* L*.*

Proof: Implication $(A) \Rightarrow (B)$ follows by straightforward inspection. We shall prove that (B) implies (A). Assume that (B) holds but (A) fails, that is

$$\exists\epsilon > 0 \quad \forall\delta > 0 \quad \exists x \in U^*_\delta(a) \qquad |f(x_n) - L| \ge \epsilon\,.$$

Choosing here $\delta = \frac{1}{n}$ we get

$$\forall n \in \mathbb{N} \quad \exists x_n \quad \text{such that } 0 < |x_n - a| < \frac{1}{n} \quad \text{and} \qquad |f(x) - L| \ge \epsilon\,.$$

We see that $f(x_n)$ does not converge to L and therefore we arrived at the contradiction. □

Remark 9.2.2. In the theorem, we can replace (B) by a seemingly weaker condition

(B') for any sequence $\{x_n\} \subset E \setminus \{a\}$ convergent to a the sequence $\{f(x_n)\}$ converges.

This already yields (B): assume that (B) fails but (B') holds, i.e., there are two sequences $\{x'_n\}, \{x''_n\} \subset E \setminus \{a\}$, both are convergent to a, such that $\lim f(x'_n) = L'$ and $\lim f(x''_n) = L''$, where $L' \ne L''$. Take $x_n = x'_m$ for $n = 2m$ and $x_n = x''_m$ for $n = 2m + 1$. Then $x_n \to a$ but the sequence $f(x_n)$ has two limit points L' and L'', and therefore it does not converge. We arrive at the contradiction which proves (B).

Example 9.2.3. Consider the Dirichlet function $\mathcal{D} : \mathbb{R} \to \mathbb{R}$ which equals 0 at irrational x and 1 at rational x. Then $\mathcal{D}$ does not have a limit at *any* real point a. Indeed, take two sequences $\{x_n\} \subset \mathbb{Q}$ and $\{y_n\} \subset \mathbb{R} \setminus \mathbb{Q}$ converging to a. Then $\mathcal{D}(x_n) = 1$ for all n, hence $\lim \mathcal{D}(x_n) = 1$. Similarly, $\lim \mathcal{D}(y_n) = 0$.

Exercise 9.2.4. Show that

$$\mathcal{D}(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(2\pi x m!)\,.$$

Theorem 9.2.1 will allow us to transfer all the properties of the limit of sequences we've already known to the limits of functions.

Corollary 9.2.5 (Cauchy's criterion)**.** *The limit* $\lim\limits_{E\ni x\to a} f(x)$ *exists if and only if*

$$\forall\epsilon > 0 \quad \exists\delta > 0 \quad \textit{such that} \qquad |f(x') - f(x'')| < \epsilon\,, \tag{C}$$

provided $x', x'' \in E$ *and* $0 < |x' - a| < \delta$*,* $0 < |x'' - a| < \delta$*.*

[5] more accurately, $x\colon \mathbb{N} \to E \setminus \{a\}$ or $\{x_n : n \in \mathbb{N}\} \subset E \setminus \{a\}$

Here is a logic of the proof:

$$\begin{aligned}\exists \lim_{E\ni x\to a} f(x) \quad &\Rightarrow \quad (C)\\ &\Rightarrow \quad \forall\{x_n\}\subset E\setminus\{a\} \text{ convergent to } a,\ \{f(x_n)\} \text{ is Cauchy's sequence}\\ &\qquad\qquad\qquad\qquad \Rightarrow \quad (B')\Rightarrow \quad \exists \lim_{E\ni x\to a} f(x)\,.\end{aligned}$$

We leave the rest as an exercise.

Exercise 9.2.6. Prove that $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ does not exist.

9.3. The first remarkable limit: $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$. Since the function $\frac{\sin x}{x}$ is even, it suffices to consider the case when $x\downarrow 0$. First, we prove the inequality

$$(*)\qquad\qquad \sin x < x < \tan x$$

valid for $0<x<\frac{\pi}{2}$. For that, consider the circle of radius one centered at O and two points A and B on that circle such that the angle $\angle AOB$ equals x radians. Let C be the intersection point of the tangent to the circle at A and the line containing the radius OB. Then

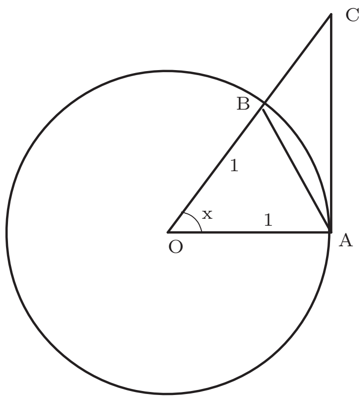

FIGURE 8. The triangles AOB and AOC

$$\triangle AOB\subset \text{sector} AOB\subset \triangle AOC\,,$$

so that

$$\text{Area}(\triangle AOB) < \text{Area}(\text{sector} AOB) < \text{Area}(\triangle AOC)\,.$$

Computing the areas, we get

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}\,,$$

that is $(*)$.

Dividing $(*)$ by $\sin x$, we obtain

$$1 > \frac{\sin x}{x} > \cos x\,,$$

or
$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x\,.$$
But
$$1 - \cos x = 2\sin^2\frac{x}{2} < 2\left(\frac{x}{2}\right)^2 = \frac{x^2}{2}$$
(we have used the first inequality from $(*)$). So that
$$0 < 1 - \frac{\sin x}{x} < \frac{x^2}{2}\,.$$
This yields the limit in the box. Done! □

Corollary 9.3.1.
$$\lim_{n\to\infty}\left\{\cos\frac{t}{2}\cdot\cos\frac{t}{2^2}\cdot\cos\frac{t}{2^3}\cdot\ \ldots\ \cdot\cos\frac{t}{2^n}\right\} = \frac{\sin t}{t}\,.$$

Proof: Indeed,
$$\sin t = 2\cos\frac{t}{2}\sin\frac{t}{2} = 2^2\cos\frac{t}{2}\cos\frac{t}{2^2}\sin\frac{t}{2^2}$$
$$= \ldots = 2^n\cos\frac{t}{2}\cos\frac{t}{2^2}\ldots\cos\frac{t}{2^n}\sin\frac{t}{2^n}\,,$$
so the product of cosines equals
$$\frac{\sin t}{2^n\sin\frac{t}{2^n}} = \frac{\sin t}{t}\cdot\frac{\frac{t}{2^n}}{\sin\frac{t}{2^n}}\,.$$
Notice, that the second factor converges to 1 since $\frac{t}{2^n}$ converges to 0. □

Exercise 9.3.2 (Vieta)**.** Prove that
$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\ldots$$
(the product on the RHS is infinite).
Hint: Let $t = 2/\pi$ in the previous corollary. Using induction, check that $\cos\frac{\pi}{2^{n+1}} = \frac{\sqrt{2+\sqrt{2+\ \ldots + \sqrt{2}}}}{2}$, $n\in\mathbb{N}$, with n square roots on the RHS.

9.4. Limits at infinity and infinite limits. We extend the definition of limit to two cases: first, we allow the point a to be $\pm\infty$. Second, we allow the limit to be $\pm\infty$.

Definition 9.4.1. Let f be a function defined for $x > x_0$. We say that $\lim\limits_{x\to+\infty} f(x) = L$ if
$$\forall\epsilon > 0\quad\exists M\quad\forall x > M\qquad |f(x) - L| < \epsilon\,.$$
If f is defined for $x < x_0$ we say that $\lim\limits_{x\to-\infty} f(x) = L$ if
$$\forall\epsilon > 0\quad\exists M\quad\forall x < M\qquad |f(x) - L| < \epsilon\,.$$

Exercise 9.4.2. Check that $\lim_{x\to+\infty} f(x) = \lim_{y\downarrow 0} f\left(\frac{1}{y}\right)$.

Example 9.4.3.

$$\lim_{x\to+\infty} \arctan x = \frac{\pi}{2}\,, \qquad \lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}\,.$$

Consider the first case. Fix $\epsilon > 0$ and choose $M = \tan(\frac{\pi}{2}-\epsilon)$. If $x > \tan(\frac{\pi}{2}-\epsilon)$, then $\arctan x > \frac{\pi}{2} - \epsilon$, and since $\arctan x$ is always less than 1, we are done. The second case is similar to the first one. $\Box$.

Definition 9.4.4. We say that $\lim_{E\ni x\to a} f(x) = +\infty$, if

$$\forall M > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \forall x \in U^*_\delta(a) \qquad f(x) > M\,.$$

Similarly, we say that $\lim_{E\ni x\to a} f(x) = -\infty$ if

$$\forall M > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \forall x \in U^*_\delta(a) \qquad f(x) < -M\,.$$

In both cases, $\lim_{E\ni x\to a} \frac{1}{f(x)} = 0$.

Example 9.4.5.

i

$$\lim_{x\downarrow 0} \frac{1}{\sin x} = +\infty, \qquad \lim_{x\uparrow 0} \frac{1}{\sin x} = -\infty.$$

ii.

$$\lim_{x\to\pm\infty} x^3 = \pm\infty.$$

9.5. Limits of monotonic functions. Set $\sup_E f = \sup\{f(x) : x \in E\}$ if f is bounded from above on E, and $= +\infty$ otherwise, and set $\inf_E f = \inf\{f(x) : x \in E\}$ if f is bounded from below and $= -\infty$ otherwise.

Theorem 9.5.1. *Suppose $f\colon (a,b) \to \mathbb{R}$ does not decrease. Then the limits*

$$\lim_{x\uparrow b} f(x) = \sup_{(a,b)} f\,, \tag{1}$$

and

$$\lim_{x\downarrow a} f(x) = \inf_{(a,b)} f \tag{2}$$

exist.

Proof: We shall prove the first relation, proof of the second one is similar.

First, assume that f is bounded from above on (a,b), then $\sup_{(a,b)} f < +\infty$. We fix $\epsilon > 0$ and use of the definition of the supremum. We find $x_0 < b$ such that $f(x_0) > \sup_{(a,b)} f - \epsilon$. Since f does not decrease on the interval (a,b), we have $f(x) \ge f(x_0)$ for $x \ge x_0$, so that

$$\sup_{(a,b)} f - \epsilon < f(x) \le \sup_{(a,b)} f\,, \qquad x_0 \le x < b\,.$$

This proves (1) in the case when f is bounded from above.

Now, let f be unbounded from above. Then for any M we find x_0 such that $f(x_0) > M$, hence $f(x) > M$ for $x_0 \le x < b$, and $\lim_{x\uparrow b} f(x) = +\infty$. $\square$

9.6. Limits and arithmetic operations. Set $(f+g)(x) = f(x)\cdot g(x)$, $(f\cdot g)(x) = f(x)\cdot g(x)$, and $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Theorem 9.6.1. *Let the functions f and g be defined on a set $E\setminus\{a\}$ where $\{a\}$ is an accumulation point of E. Suppose that*

$$\lim_{E\ni x\to a} f(x) = A, \qquad \textit{and} \qquad \lim_{E\ni x\to a} g(x) = B\,.$$

Then there exists the limits:

a) $\lim_{E\ni x\to a} (f+g)(x) = A+B$,

b) $\lim_{E\ni x\to a} (f\cdot g)(x) = A\cdot B$,

c) *if $B\ne 0$ and $g(x)\ne 0$ for $x\in E$, then*

$$\lim_{E\ni x\to a} \frac{f}{g}(x) = \frac{A}{B}\,.$$

This theorem can be checked using the definition of the limit, it also follows at once from the corresponding properties of the limits of sequences, so we shall not prove it here.

Example 9.6.2. Let $P(x) = a_p x^p + ...$ and $Q(x) = b_q x^q + ...$ be polynomials of degrees p and q. Then

$$\begin{aligned}\lim_{x\to+\infty} \frac{P(x)}{Q(x)} &= \lim_{x\to+\infty} \frac{a_p x^p + a_{p-1}x^{p-1} + ... + a_0}{b_q x^q + b_{q-1}x^{q-1} + ... + b_0} \\ &= \lim_{x\to+\infty} x^{p-q}\cdot \frac{a_p + a_{p-1}x^{-1} + ... + a_0 x^{-p}}{b_q + b_{q-1}x^{-1} + ... + b_0 x^{-q}}\,.\end{aligned}$$

The latter limit equals 0 if $p<q$, equals $+\infty$ if $p>q$ and a_p and b_q have the same signs, and $-\infty$ if they are of different signs, and equals the quotient $\frac{a_p}{b_q}$ of the leading coefficients if the polynomials have the same degrees $p=q$.

Exercise 9.6.3. Find the following limits:

$$\lim_{x\downarrow 0} x\left[\frac{1}{x}\right], \qquad \lim_{x\uparrow 0} x\left[\frac{1}{x}\right], \qquad \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}, \qquad \lim_{x\to 0} x\cos\frac{1}{x},$$

$$\lim_{x\to+\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right), \qquad \lim_{x\to\pi}\frac{\sin x}{\pi - x}, \qquad \lim_{x\to 0}\frac{x}{\tan x},$$

$$\lim_{x\to\pm\infty}\frac{x+\sin x}{x-\sin x}, \qquad \lim_{x\downarrow 0}\frac{\sin x}{x^2}, \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2}, \qquad \lim_{x\to 0}\frac{\sin 5x - \sin 3x}{x},$$

$$\lim_{n\to\infty} \sin\pi\sqrt{n^2+1}, \qquad \lim_{n\to\infty}\sin\pi\left(n^3+1\right)^{1/3}, \qquad \lim_{n\to\infty}\underbrace{\sin\sin\,...\,\sin}_{n\ \text{times}} x\,.$$

10. The exponential function and the logarithm

10.1. The function $t \mapsto a^t$. Fix $a > 0$. First, we recall the definition of the function $t \mapsto a^t$ for $t \in \mathbb{Z}$ that you've known from the high-school, then then we extend it to the set of all rationals $\mathbb{Q}$, and then to the whole real axis. The discussion will be brief.

10.1.1. $t \in \mathbb{Z}$. We set $a^0 = 1$, $a^t = \underbrace{a \cdot a \cdot \ldots \cdot a}_{t \text{ times}}$, and $a^{-t} = \dfrac{1}{a^t}$ for $t \in \mathbb{N}$. This function has the following properties

- **(a)** $a^m \cdot a^n = a^{m+n}$;
- **(b)** $(a^m)^n = a^{mn}$;
- **(c)** $a^n \cdot b^n = (ab)^n$;
- **(d)** for $n > 0$, $a^n < b^n$ if and only if $a < b$;
- **(e)** let $n < m$, then $a^n < a^m$ provided $a > 1$, and $a^n > a^m$ provided $a < 1$.

10.1.2. $t \in \mathbb{Q}$. Suppose $t = \dfrac{m}{n}$. Then we denote by $x = a^t$ a unique positive solution to the equation $x^n = a^m$. Note that with this definition

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

(`why?`).

First of all, we need to check that this definition is *correct*; i.e., that if we use a different representation $t = \dfrac{m'}{n'}$ then the answer will be the same. Let

$$x = a^{\frac{m}{n}}, \qquad y = a^{\frac{m'}{n'}},$$

then

$$x^{nn'} = a^{mn'}, \qquad y^{nn'} = a^{m'n}.$$

Since $\dfrac{m'}{n'} = \dfrac{m}{n}$, we have $m'n = mn'$; i.e., $x^{nn'} = y^{nn'}$. Since the positive nn'-th root is unique, we get $x = y$. $\square$

Notice that the properties (a)–(e) formulated above hold true for the extension $t \mapsto a^t$, $t \in \mathbb{Q}$. We check only (**a**) and leave the rest as an exercise.

Claim 10.1.1. *For $t_1, t_2 \in \mathbb{Q}$, $a^{t_1+t_2} = a^{t_1} \cdot a^{t_2}$.*

Proof: Suppose

$$x_1 = a^{\frac{m_1}{n_1}}, \qquad x_2 = a^{\frac{m_2}{n_2}}.$$

We need to check that

$$x_1 \cdot x_2 = a^{\frac{m_1}{n_1} + \frac{m_2}{n_2}}.$$

We have

$$x_1^{n_1 n_2} = a^{m_1 n_2}, \qquad x_2^{n_1 n_2} = a^{m_2 n_1},$$

whence

$$(x_1 \cdot x_2)^{n_1 n_2} = a^{m_1 n_2} \cdot a^{m_2 n_1} = a^{m_1 n_2 + m_2 n_1}$$

(note that in the last equation, we've used the property (**a**) for integer t's). That is

$$x_1 \cdot x_2 = a^{\frac{m_1 n_2 + m_2 n_1}{n_1 n_2}} = a^{\frac{m_1}{n_1} + \frac{m_2}{n_2}},$$

completing the proof. □

We need one more property of the exponential function:

(f) $\lim\limits_{\mathbb{Q} \ni r \to t} a^r = a^t$, $t \in \mathbb{Q}$.

*Proof of (**f**)*: First, we prove (**f**) in a special case when $t = 0$; i.e, we prove that $\lim\limits_{\mathbb{Q} \ni r \to 0} a^r = 1$. We prove it in the case $a > 1$, the case $a < 1$ is similar.

We use Heine's definition of the limit. Let $\{r_n\}$ be a sequence of rationals converging to 0. We fix an arbitrarily small $\epsilon > 0$ and choose $k \in \mathbb{N}$ such that

$$1 - \epsilon < a^{-1/k} < a^{1/k} < 1 + \epsilon$$

(`why this is possible?`). Then we choose $N \in \mathbb{N}$ such that for $n \geq N$,

$$-\frac{1}{k} < r_n < \frac{1}{k}.$$

Then we have

$$1 - \epsilon < a^{-1/k} \overset{(\mathbf{e})}{<} a^{r_n} \overset{(\mathbf{e})}{<} a^{1/k} < 1 + \epsilon,$$

proving the claim in the case $t = 0$.

Now, consider the general case. We have

$$\lim_{\mathbb{Q} \ni r \to t} a^r \cdot a^{-t} = \lim_{\mathbb{Q} \ni r \to t} a^{r-t} = \lim_{\mathbb{Q} \ni s \to 0} a^s = 1,$$

hence, the claim. □

10.1.3. $t \in \mathbb{R}$. Assume again that $a > 1$. Given $t \in \mathbb{R}$, consider the numbers

$$s = \sup\{a^r : r \in \mathbb{Q}, \quad r < t\}, \qquad i = \inf\{a^r : r \in \mathbb{Q}, \quad r > t\}.$$

It is not difficult to see that these two numbers must coincide. First note that $s \leq i$ (`why?`). Then, given $k \in \mathbb{N}$, choose the rationals r and q such that $r < t < q$ and $q - r < \frac{1}{k}$. Then

$$0 \leq i - s < a^q - a^r = a^r(a^{q-r} - 1) < s(a^{1/k} - 1).$$

Letting $k \to \infty$, we get $s = i$. □

Definition 10.1.2. For $a > 1$ and for each $t \in \mathbb{R}$, we set $a^t = s = i$. If $a < 1$, then we set $a^t = \left(\frac{1}{a}\right)^{-t}$.

An equivalent definition says

$$a^t \overset{def}{=} \lim_{\mathbb{Q} \ni r \to t} a^r.$$

Exercise 10.1.3. Show that the limit on the right hand side exists, and prove the equivalence of these definitions.

This extends the function $t \mapsto a^t$ to the whole real axis preserving the properties (a)–(f):

(a) $a^t \cdot a^s = a^{t+s}$;
(b) $(a^t)^s = a^{ts}$;
(c) $a^t \cdot b^t = (ab)^t$;
(d) for $t > 0$, $a^t < b^t$ if and only if $a < b$, for $t < 0$, $a^t < b^t$ if and only if $a > b$.
(e) let $t < s$, then $a^t < a^s$ provided $a > 1$, and $a^t > a^s$ provided $a < 1$;
(f) $\lim_{s\to t} a^s = a^t$.

Exercise 10.1.4. Check the properties (a)–(f).

Next, we'll need one more property of the exponential function:

Claim 10.1.5. *The function $t \mapsto a^t$ maps $\mathbb{R}$ onto $\mathbb{R}_+$.*

I.e., for each positive y, there is $t \in \mathbb{R}$ such that $a^t = y$. Note, that due do monotonicity claimed in (e), if such a t exists then it must be unique.

Proof: Suppose that $a > 1$. Fix $y > 0$ and consider the sets

$$A_< = \{t \in \mathbb{R} \colon a^t < y\} \qquad \text{and} \qquad A_> = \{t \in \mathbb{R} \colon a^t > y\}\,.$$

The both sets are not empty, for instance, if we take a big enough $n \in \mathbb{N}$, then $1/n \in A_<$ and $n \in A_>$. By (e), for each $t_1 \in A_<$ and $t_2 \in A_>$, we have $t_1 < t_2$. Therefore, by the completeness axiom, there exists $t \in \mathbb{R}$ such that $t_1 \le t \le t_2$ for each $t_1 \in A_<$ and each $t_2 \in A_>$. Let us show that $a^t = y$.

Suppose that $a^t < y$. Since $a^{t+1/n} \to a^t$ when $n \to \infty$, we can choose big enough n such that $t + \frac{1}{n} \in A_<$. This contradicts to our assumption that the point t *separates* the sets $A_<$ and $A_>$. Similarly, the assumption $a^t > y$ also leads to the contradiction. Thus, $a^t = y$, completing the proof. □

The claim we've just proven allows us to define the inverse function to a^t which is called the *logarithmic function* $\log_a \colon \mathbb{R}_+ \mapsto \mathbb{R}$.

10.2. The logarithmic function $\log_a x$**.** This function is defined as inverse to the function $t \mapsto a^t$, that is $\log_a(a^t) = a^{\log_a t} = t$. It follows from the definition that $\log_a 1 = 0$ and $\log_a a = 1$. Now we list the basic properties of the logarithmic function:

(i) $\log_a(xy) = \log_a x + \log_a y$;
(ii) $\log_a(x^y) = y \log_a x$.
(iii) if $x < y$, then $\log_a x < \log_a y$ provided $a > 1$, and $\log_a x > \log_a y$ provided $a < 1$;
(iv) $\lim\limits_{x\to y} \log_a x = \log_a y$;

Exercise 10.2.1. Check the properties (i)–(iv) of the logarithmic functions.

Another important property is

(v)

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Indeed, if $u = \log_b x$ and $v = \log_b a$, then $b^u = x$ and $b^v = a$. Now, we need to express the value $t = \log_a x$, that is the solution of the equation $a^t = x$ through u and v. We have $b^{vt} = a^t = x = b^u$, hence $vt = u$ and $t = \frac{u}{v}$ as we needed. $\square$

In particular, we see that

$$\log_a x = \frac{1}{\log_x a}\,.$$

If the basis a equals e, then we simply write $\log x = \log_e x$. Such logarithms are called the *natural* ones. The reason why the base e is important will be clear later (the base $a = 2$ is also very useful). It is worth to remember the special case of (v):

$$\log_a x = \frac{\log x}{\log a}$$

which allows to convert any logarithms to the natural ones.

Having the logarithms, we can define the power function $x \mapsto x^\alpha$ for $x > 0$ by

$$x^\alpha = e^{\alpha \log x}\,.$$

If $\alpha \in \mathbb{Z}$ this definition coincide with the one we know from the high-school (`why?`). If $\alpha > 0$ the function $x \mapsto x^\alpha$ increases, if $\alpha < 0$, then this function decreases.

It is important to remember that the exponential function grows at infinity faster than the power function:

Claim 10.2.2. *For $a > 1$ and $p < \infty$,*

$$(*) \qquad \lim_{x\to+\infty} \frac{x^p}{a^x} = 0.$$

Proof: The relation $(*)$ easily follows from its special case for the sequences. We know that $n^p/a^n \to 0$, as $\mathbb{N} \ni n \to \infty$. Therefore, we can fix sufficiently small $\epsilon > 0$ and choose big enough N such that $\forall n > N$

$$\frac{n^{[p]+1}}{a^n} < \epsilon\,.$$

Then for $n = [x]$ (x is large enough) we have

$$0 < \frac{x^p}{a^x} < \frac{(n+1)^{[p]+1}}{a^{n+1}} \cdot a < a\epsilon\,.$$

Done! $\square$

Corollary 10.2.3.

i. Setting in $(*)$ $a^t = x^\alpha$, we see that the logarithmic function grows slower than any power function:

$$\lim_{x\to+\infty} \frac{\log_a x}{x^\alpha} = \frac{1}{\alpha} \lim_{t\to+\infty} \frac{t}{a^t} = 0\,.$$

Here $\alpha > 0$, of course.

ii. Making the change of variables $s = \frac{1}{x}$, we arrive at another important limit:

$$\lim_{s \to 0} s^{\alpha} |\log_a s| = 0\,.$$

Here again $\alpha > 0$.

Example 10.2.4.

i.

$$\lim_{x \downarrow 0} x^x = \lim_{x \downarrow 0} e^{x \log x} = e^0 = 1.$$

ii.

$$\lim_{x \downarrow 0} x^{x^x} = \lim_{x \downarrow 0} e^{x^x \log x} = 0\,.$$

Now, the exponent tends to $-\infty$, hence the limit equals 0.

11. The second remarkable limit. The symbols "o small" and "$\sim$"

11.1. $\lim\limits_{x\to\pm\infty}\left(1+\dfrac{1}{x}\right)^x = e$.

Proof: We already know the special case:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\,,$$

which is a definition of the number e. Now, let $x \to +\infty$, and let $n = [x]$ be the integer part of x. Then

$$\begin{aligned}\left(1+\frac{1}{n+1}\right)^{n+1}\frac{n+1}{n+2} &= \left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x \\ &< \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n \frac{n+1}{n}\,,\end{aligned}$$

and the result follows.

Now, consider the second case: $x \to -\infty$. First, observe that

$$\lim_{x\to+\infty}\left(1-\frac{1}{x^2}\right)^x = 1\,.$$

Indeed, fix $\epsilon > 0$, then for $x \geq 1$ and $n = [x]$ we get

$$1 > \left(1-\frac{1}{x^2}\right)^x > \left(1-\frac{1}{n^2}\right)^{n+1} \geq 1 - \frac{n+1}{n^2} > 1-\epsilon$$

if x is sufficiently large. Next, observe that

$$\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x = \lim_{x\to+\infty}\frac{\left(1-\frac{1}{x^2}\right)^x}{\left(1+\frac{1}{x}\right)^x} = \frac{\lim_{x\to+\infty}\left(1-\frac{1}{x^2}\right)^x}{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x} = \frac{1}{e}\,.$$

Thus,

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^{-x} = (1/e)^{-1} = e\,.$$

Done! □

Corollary 11.1.1.

$$\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$$

and

$$\lim_{t\to 0}\frac{\log(1+t)}{t} = 1\,.$$

Proof: To get the first limit put $x = 1/t$ in the 2nd remarkable limit. The second relation follows from the first one: if $y = (1+t)^{1/t} \to 1$, then $\log y \to 0$, and $\log y$ is nothing but $\frac{1}{t}\log(1+t)$.

11.2. Infinitesimally small values and the symbols o and $\sim$. Here we develop a useful formalism which in many cases make the formulas simpler.

Definition 11.2.1. Let $E \subset \mathbb{R}$, and a be an accumulation point of E. The function $\alpha : E \to \mathbb{R}$ is called *infinitesimally small* at a, if

$$\lim_{E \ni x \to a} \alpha(x) = 0.$$

Let us make several trivial comments. If α and β are infinitesimally small at a, then their sum $\alpha + \beta$ is infinitesimally small as well. If α is infinitesimally small at a and β is bounded, then the product $\alpha \cdot \beta$ is infinitesimally small as well. At last, relation $f(x) = L + \alpha(x)$ where α is infinitesimally small at a is equivalent to $\lim_{x \to a} f(x) = L$.

Another notation for infinitesimally small values is $o(1)$ ("o small"). This notation is quite useful.

Definition 11.2.2. Let $f, g : E \to \mathbb{R}$, and let a be an accumulation point of E. We say that

$$f(x) = o(g(x)), \qquad x \to a, \quad x \in E,$$

if $f(x) = \alpha(x) g(x)$, where α is infinitesimally small at a.

For instance,

$$\begin{aligned} x^2 &= o(x), \qquad x \to 0, \\ x &= o(x^2), \qquad x \to \pm\infty, \\ \frac{1}{x} &= o\left(\frac{1}{x^2}\right), \qquad x \to 0, \end{aligned}$$

and

$$\frac{1}{x^2} = o\left(\frac{1}{x}\right), \qquad x \to \pm\infty.$$

Definition 11.2.3. We say that the functions f and g are *equivalent* at a:

$$f \sim g, \qquad x \to a, \quad x \in E,$$

if

$$\lim_{E \ni x \to a} \frac{f(x)}{g(x)} = 1.$$

Another way to express the same is to write

$$f(x) = g(x) + o(g(x)) = (1 + o(1)) g(x), \qquad x \to a, \quad x \in E.$$

Examples:

(i) if $P_{n-1}(x)$ is a polynomial of degree $\le n-1$, then $x^n + P_{n-1}(x) \sim x^n$ for $x \to \pm\infty$.

The next relations hold for $x \to 0$:

(ii) $x^2 + x \sim x$;

(iii) $\sin x \sim x$;

(iv) $\log(1+x) \sim x$;
(v) $e^x - 1 \sim x$;
(vi) $(1+x)^a - 1 \sim ax$.

Let us prove the last two relations: in (v) we introduce a new variable $x = \log(1+t)$, then (v) reduces to (iv). In (vi) we use both (iv) and (v):

$$\begin{aligned}\lim_{x\to 0} \frac{(1+x)^a - 1}{x} &= \lim_{x\to 0} \frac{e^{a\log(1+x)} - 1}{x} \\ &= \lim_{x\to 0} \frac{e^{a\log(1+x)} - 1}{a\log(1+x)} \cdot \frac{a\log(1+x)}{x} \\ &= \lim_{y\to 0} \frac{e^y - 1}{y} \cdot a \lim_{x\to 0} \frac{\log(1+x)}{x} = a\,.\end{aligned}$$

Exercise 11.2.4. Show that $\sqrt{x + \sqrt{x + \sqrt{x}}} \sim x^{\frac{1}{8}}$ for $x \to 0$, and is $\sim \sqrt{x}$ for $x \to +\infty$.

Exercise 11.2.5. Find the limits

$$\lim_{x\to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n} \right), \qquad \lim_{x\to 0} \left(\frac{1+\tan x}{1+\sin x} \right)^{1/x^3}, \qquad \lim_{x\to 0} \frac{\log\cos\alpha x}{\log\cos\beta x} \quad (\beta \neq 0)\,,$$

$$\lim_{x\to\infty} \left(\frac{x^2+1}{x^2-1} \right)^{x^2}, \qquad \lim_{x\to+\infty} (e^x - 1)^{1/x}, \qquad \lim_{x\to 1} x^{\frac{1}{x-1}},$$

$$\lim \left(\frac{a^t + b^t}{2} \right)^{1/t} \qquad (t \to +\infty, t \to -\infty, t \to 0)\,.$$

Let

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty\,.$$

If $g(x) = o(f(x))$ for $x \to +\infty$, then we say that f *grows faster at* $+\infty$ *than* g (or, equivalently, that g *grows slower at* $+\infty$ *than* f). For example, for each $\alpha > 0$, and $p < \infty$, x^α grows faster than $\log^p x$, and for each $a > 1$, a^x grows faster than x^α.

Exercise* 11.2.6. Prove that for any sequence of functions

$$f_1(x),\ f_2(x),\ \ldots f_n(x), \ldots \qquad x_0 < x < +\infty,$$

such that

$$\lim_{x\to+\infty} f_n(x) = +\infty\,, \qquad \forall n \in \mathbb{N}\,,$$

it is possible to construct other two functions $\varphi(x)$ and $\psi(x)$ such that φ *grows to* $+\infty$ *faster than any of* f_n (i.e., for each n, $\limsup\limits_{x\to+\infty}(\varphi/f_n)(x) = +\infty$) and ψ *grows to* $+\infty$ *slower than any of* f_n (i.e., for each n, $\liminf\limits_{x\to+\infty}(\psi/f_n)(x) = 0$).

12. Continuous functions, I

12.1. Continuity.

Definition 12.1.1. The function f defined in a neighbourhood of a point a is called *continuous* at a if

$$f(a) = \lim_{x \to a} f(x).$$

In other words, $\forall \epsilon > 0$ exists $\delta > 0$ such that $\forall x \in U_\delta(a)$

$$|f(x) - f(a)| < \epsilon.$$

Here, as usual, $U_\delta(a) = \{t : |t - a| < \delta\}$ is a δ-neighbourhood of a.

If a function f is continuous at any point it is defined, we say that this function is continuous everywhere.

The function f can be defined only on a set E and $a \in E$. If a is an accumulation point of E then we say that f is *continuous at a along E* if

$$f(a) = \lim_{E \ni x \to a} f(x).$$

If a is an isolated point of E, then we also say that also f is continuous at a.

Examples:

i. The constant function $f(x) = \text{const}$ is continuous everywhere.

ii. The identity function $f(x) = x$ is continuous everywhere.

iii. The function $f(x) = \sin x$ is continuous everywhere. Indeed, if $|x - a| < \epsilon$, then we get

$$\begin{aligned} |\sin x - \sin a| &= \left| 2\cos\frac{x+a}{2} \sin\frac{x-a}{2} \right| \\ &\le 2\left|\sin\frac{x-a}{2}\right| \le 2\left|\frac{x-a}{2}\right| = |x - a| < \epsilon\,. \end{aligned}$$

Similarly, the cosine function is continuous.

iv. The exponential function $x \mapsto a^x$ and the logarithmic function $x \mapsto \log x$ are continuous everywhere they are defined. This follows from the properties of these functions established in the previous lecture.

v. The function $f : [0, +\infty) \to [0, \infty)$ defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$ is continuous at every point of $[0, +\infty)$.

12.2. Points of discontinuity. There are various reasons for a function f to be discontinuous at a point a. We give here a brief classification of possible cases. In what follows, we'll use notations

$$f(a - 0) = \lim_{x \uparrow a} f(x), \qquad f(a + 0) = \lim_{x \downarrow a} f(x)\,.$$

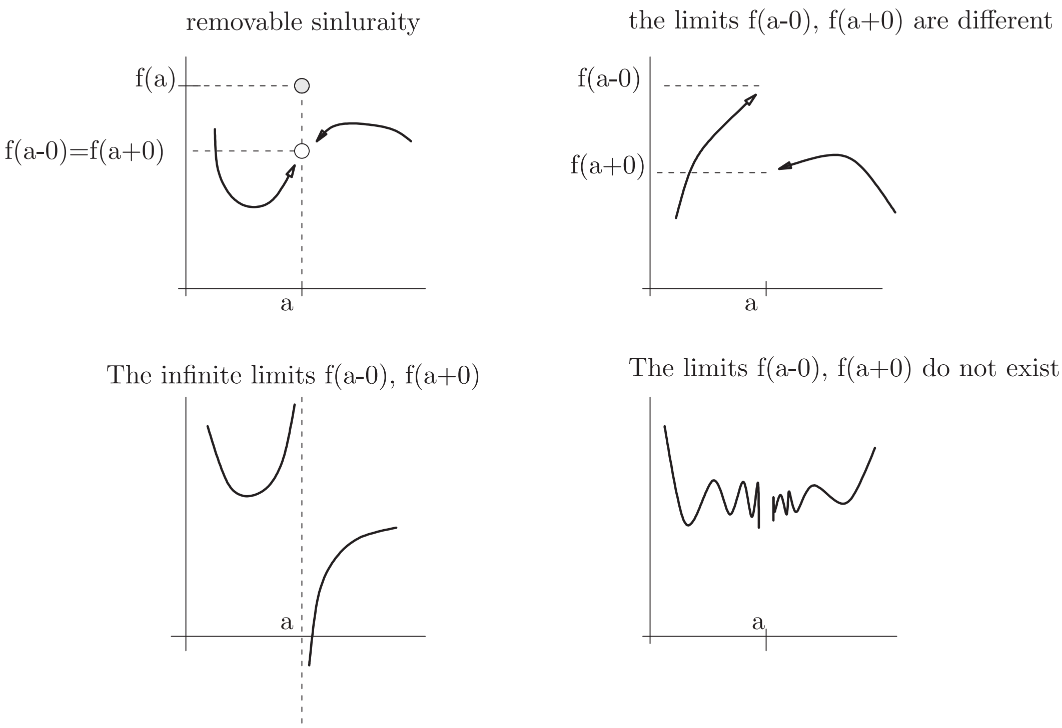

FIGURE 9. Possible discontinuities at a

Removable singularity. We say that the function f has a *removable singularity* at the point a if the limits from above and from below at this point exist and have the same value: $f(a-0) = f(a+0)$. In this case, we can always define (or re-define) the function f at this point by the common value of these limits making the function continuous.

Examples:

i. Let $f(x) = x$ for $x \neq 0$ and $f(0) = 10$. This function is clearly discontinuous at the origin. However, re-defining f at the origin by prescribing it the zero value, we obtain a continuous function at the origin.

ii. Let $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$. Again setting $f(0) = 0$, we get a continuous function.

iii. Let $f(x) = \frac{\sin x}{x}$ for $x \neq 0$. Setting $f(0) = 1$, we get a continuous function.

iv. Consider the Riemann function

$$\mathcal{R}(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\},\ (m,n) = 1 \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ or } x = 0. \end{cases}$$

Here (m,n) is the greatest common divisor of m and n; i.e., $(m,n) = 1$ means that m and n are mutually primes. We show that $\mathcal{R}$ has a limit at any point $a \in \mathbb{R}$ and

$$\lim_{x \to a} \mathcal{R}(x) = 0\,. \tag{R}$$

We fix a and an arbitrary large natural number N. The set

$$Q_N = \Big\{ r = \frac{m}{n} \colon m \in \mathbb{Z}, n \in \mathbb{N},\ (m,n) = 1,\ n \le N \Big\}$$

does not have finite accumulation points (`why?`). Hence, we can find a punctured neighbourhood $U^*(a)$ such that it contains *no* rational numbers represented in the form $r = \frac{m}{n}$ with mutually primes m and n with $n \le N$. This is possible since the set

This means that

$$\forall x \in U^*(a) \qquad 0 \le \mathcal{R}(x) < \frac{1}{N}\,,$$

that is (R) holds. Relation (R) yields that Riemann's function is continuous at any irrational point and at the origin, and is discontinuous at any rational point except of $x = 0$. □

Problem* 12.2.1. Whether there exists a function $f : \mathbb{R} \to \mathbb{R}$ continuous at all rational points and discontinuous at all irrational points?

Different one-sided limits. Another simple singularity appears when the function f has different one-sided limits at the point a, i.e., $f(a-0$ and $f(a+0$ exist but do not equal. For instance, if a discontinuity point of a monotonic function is not removable, then it must be of that kind.

Examples:

i. $f(x) = \mathrm{sgn} x$, $a = 0$.

ii. $f(x) = \tan x$, $a = \frac{\pi}{2}$.

Exercise 12.2.2. Give an example of the function $f \colon \mathbb{R} \to \mathbb{R}$ which is continuous at $\mathbb{R} \setminus \mathbb{Z}$ and discontinuous at all integer points.

Exercise 12.2.3. The function $f(x) = \sin \frac{1}{x}$ has no limits from the left and the right at the origin.

Problem 12.2.4. The discontinuity set of an arbitrary monotonic function is at most countable.

12.3. Local properties of continuous functions. Everywhere below we assume that the function $f \colon E \to \mathbb{R}$ is continuous at a. We list some simple *local* properties of f:

Local boundedness. There exists a neighbourhood $U(a)$ of a such that f is bounded in $E \cap U(a)$.

Local conservation of the sign. If $f(a) \ne 0$, then there exists a neighbourhood $U(a)$ of a where f has the same sign as at a:

$$\mathrm{sgn} f(x) = \mathrm{sgn} f(a)\,, \qquad \forall x \in E \cap U(a)\,.$$

Arithmetic of continuous functions. If $g\colon E \to \mathbb{R}$ is continuous at a, then the functions $f+g$ and $f\cdot g$ are also continuous at a. If $g(x) \neq 0$ in a neighbourhood of a, then the quotient $\frac{f}{g}$ is also continuous at a.

Exercise 12.3.1. Prove these three properties.

Using these properties, we see for example, that every polynomial is a continuous function on $\mathbb{R}$ and any rational function (that is the function of the form $R = \frac{P}{Q}$ where P and Q are polynomials) is continuous everywhere except of the zeroes of the denominator.

Continuity of the composition. If $f\colon E \to V$ is continuous at a, and $g\colon V \to \mathbb{R}$ is continuous at $b = f(a)$, then the composition $(g \circ f)(x)$ is continuous at a.

Proof: Indeed, fix $\epsilon > 0$ and choose $\delta > 0$ such that

$$|g(y) - g(b)| < \epsilon$$

provided $|y - b| < \delta$. Then having this δ choose an $\eta > 0$ such that

$$|f(x) - f(a)| < \delta$$

provided $|x - a| < \eta$. With this choice

$$|g(f(x)) - g(f(a))| = |g(y) - g(b)| < \epsilon\,.$$

Done! $\square$

The last property implies continuity of the power function $x \mapsto x^\alpha = e^{\alpha \log x}$ on $(0, +\infty)$ for $\alpha < 0$ and on $[0, +\infty)$ for $\alpha > 0$. Using this fact, we prove now that

$$e^\lambda = \lim_{x\to\infty} \left(1 + \frac{\lambda}{x}\right)^x$$

for each $\lambda \in \mathbb{R}$. Indeed, we may assume that $\lambda \neq 0$ (if $\lambda = 0$ the formula is trivial). Then we introduce a new variable $t = \frac{x}{\lambda}$ which goes to ∞ with x. We have

$$\lim_{x\to\infty} \left(1 + \frac{\lambda}{x}\right)^x = \lim_{t\to\infty} \left[\left(1 + \frac{1}{t}\right)^t\right]^\lambda = \left[\lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^t\right]^\lambda = e^\lambda\,.$$

The limit was interchanged with the brackets using continuity of the power function, the limit of the expression in the brackets equal e, as we know from the previous lecture.

Exercise 12.3.2. Suppose that the functions $f, g\colon E \to \mathbb{R}$ are continuous at a. Show that the functions $\max(f, g)(x)$ and $\min(f, g)(x)$ are also continuous at a. Deduce that if f is continuous at a, then $|f|$ is continuous at a as well.

Exercise* 12.3.3 (Cauchy's functional equation)**.** Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that, for each $x, y \in \mathbb{R}$, $f(x+y) = f(x) + f(y)$. Then $f(x) = kx$ for some $k \in \mathbb{R}$.

I.e., the linear functions are the only continuous solutions of the functional equation $f(x + y) = f(x) + f(y)$.

Hint: First, using induction, check that $f(nx) = nf(x)$ for any $n \in \mathbb{Z}$. Then check that $f(\frac{m}{n}x) = \frac{m}{n}f(x)$. Then use the continuity of f.

Exercise* 12.3.4. Prove the same under a weaker assumption that f is bounded from above in a neighbourhood of the origin.

Exercise* 12.3.5.

a. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function that does not vanish identically and such that, for each $x, y \in \mathbb{R}$, one has $f(x+y) = f(x)f(y)$. Then $f(x) = e^{kx}$ for some $k \in \mathbb{R}$.

b. Formulate and prove a similar characterization of the logarithmic function $f(x) = k \log x$, and the power function $f(x) = x^k$ (in the both cases, $k \in \mathbb{R}$).

13. Continuous functions, II

13.1. Global properties of continuous functions. In what follows we denote by $C(E)$ the collection of all continuous functions on the set $E \subset \mathbb{R}$.

Theorem 13.1.1. *Let $f \in C[a,b]$ and let the values of the function f at the end-points have different signs: $f(a)f(b) < 0$. Then there exists an intermediate point $c \in (a,b)$ where the function f vanishes.*

Our intuitive understanding of the word "continuous" suggests that the result is correct: the graph of continuous function should be a "continuous curve" and we cannot connect a point above the x-axis with a point below x-axis by a continuous line which does not intersects the x-axis.

Proof: We construct inductively a sequence of nested intervals $I_n = [a_n, b_n]$, $I_0 \supset I_1 \supset \ldots \supset I_n \supset \ldots$ such that $|I_n| = 2^{-n}|I_0|$, and $f(a_n)f(b_n) < 0$.

Set $a_0 = a$, $b_0 = b$, and $I_0 = [a_0, b_0]$. As we know, at the end-points of I_0 the function f has different signs: $f(a_0)f(b_0) < 0$. Having the interval I_n, we consider its middle point ξ and check the sign of $f(\xi)$. If $f(\xi) = 0$, then the theorem is proven and there is no need in the further construction. If $f(\xi) \neq 0$, then either $f(a_n)$ or $f(b_n)$ has the opposite sign with $f(\xi)$. If $f(a_n)f(\xi) < 0$, then we set $a_{n+1} = a_n$, $b_{n+1} = \xi$, otherwise we set $a_{n+1} = \xi$, $b_{n+1} = b_n$. In any case, we get a new interval I_{n+1} with the same properties.

By Cantor's lemma the intersection of the intervals I_n is a singleton set:

$$\{c\} = \bigcap_{n \geq 1} I_n \,.$$

We claim that the function f vanishes at c. By construction,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c.$$

By continuity of f

$$f^2(c) = \lim_{n\to\infty} f(a_n)f(b_n) \leq 0\,,$$

so that $f(c) = 0$. We are done. □

The proof of this theorem is constructive, and it can be easily turned to a simple and effective numerical algorithm (called sometimes *bisection method*) for finding roots of equations.

The result can be put in a more general form:

Theorem 13.1.2 (Intermediate Value Property). *Let $f \in C[a,b]$, and let $f(a) = A$, $f(b) = B$, where $A \neq B$. Then for any intermediate value C between A and B (that is $A < C < B$ or $B < C < A$) there exists $c \in (a,b)$ such that $f(c) = C$.*

Proof: Consider a new function $f_1(x) = f(x) - C$. Its values at the end-points have different signs, so applying Theorem 1 we find a point $c \in (a,b)$ such that $f_1(c) = 0$, or $f(c) = C$. □

Corollary 13.1.3. *For each polynomial P of odd degree there exists a point $\xi \in \mathbb{R}$ such that $P(\xi) = 0$.*

Proof: Let $P(x) = a_{2N-1}x^{2N-1} + \ldots$ be a polynomial of degree $2N-1$, i.e., $a_{2N-1} \neq 0$. Suppose, for instance, that $a_{2N-1} > 0$. Then $\lim_{x\to\pm\infty} P(x) = \pm\infty$. Therefore, we can find a sufficiently big positive M such that $P(M) > 0$ and $P(-M) < 0$. The rest follows from continuity of P and from the IVP-property. $\square$

Corollary 13.1.4. *If $f \in C(a,b)$ then the image $f(a,b)$ is an interval (maybe, infinite, semi-infinite, or a singleton).*

Proof: Take any two points $y_1 < y_2$ in $f(a,b)$. We need to check that $(y_1, y_2) \subset f(a,b)$. Since $y_1, y_2 \in f(a,b)$, there are points $\xi_1, \xi_2 \in (a,b)$ such that $f(\xi_i) = y_i$, $i = 1,2$. Suppose, for instance, that $xi_1 < xi_2$. Then by the IVP-property, for any $y \in (y_1, y_2)$, there is $\xi \in (\xi_1, \xi_2)$ such that $f(\xi) = y$; i.e., $(y_1, y_2) \subset f(a,b)$. $\square$

Exercise 13.1.5. A point ξ is said to be a *fixed point* of the function f if $f(\xi) = \xi$.
i. Prove that any continuous function that maps the interval $[0,1]$ into itself has a fixed point. In other words, if $f \in C[0,1]$ and $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, then there exists a point $\xi \in [0,1]$ such that $f(\xi) = \xi$.
ii. Let the function f be defined on $[a,b]$ and satisfy there

$$|f(x) - f(y)| \leq K|x-y|, \qquad \forall x, y \in [a,b]$$

with some $K < 1$. Show that f has a *unique* fixed point at the interval $[a,b]$.

Exercise 13.1.6. Let P be a polygon in the plane. Prove that there is a vertical line which splits P onto two polygons of equal area.

Exercise 13.1.7. Let $a_1, a_2, a_3 > 0$, $\lambda_1 < \lambda_2 < \lambda_3$. Show that equation

$$\frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \frac{a_3}{x-\lambda_3} = 0$$

has exactly 2 real solutions.

Exercise 13.1.8. Let $f \in C[0,1]$, and $f(0) = f(1)$. Show that there exists $a \in [0, \frac{1}{2}]$ such that $f(a) = f(a + \frac{1}{2})$.

Theorem 13.1.9 (Weierstrass)**.** *If $f \in C[a,b]$, then f is bounded on $[a,b]$ and attains there its maximum and minimum values.*

Proof: First, we prove the boundedness of f. In the previous lecture we proved local boundedness of continuous functions. Therefore, for each $x \in [a,b]$ there exists a neighbourhood $U(x)$ and a constant C_x such that

$$|f(y)| \leq C_x\,, \qquad y \in U(x)\,.$$

The neighbourhoods $\{U(x)\}_{x\in[a,b]}$ form a covering of $[a,b]$. Hence, using the Borel covering lemma we can find a finite sub-covering

$$[a,b] \subset \bigcup_{k=1}^{N} U(x_k)$$

Then

$$|f(x)| \le \max\{C_{x_1}, ..., C_{x_k}\}, \qquad x \in [a,b],$$

that is. f is bounded on $[a,b]$.

Now we show that f achieves its maximum and minimum values. We'll show this only for the maximum value. The other case is similar. Let

$$M = \sup_{[a,b]} f.$$

By the definition of the supremum, there is a sequence $\{x_n\} \subset [a,b]$ such that

$$\lim_{n\to\infty} f(x_n) = M.$$

Since the sequence $\{x_n\}$ is bounded we can find a convergent subsequence

$$\{x_{n_i}\} \to x^* \in [a,b].$$

Then by continuity of f

$$f(x^*) = \lim_{i\to\infty} f(x_{n_i}) = M.$$

We are done. □

Remark 13.1.10. The both conclusions of the Weierstrass theorem may fail if f is continuous on an open interval (or on the whole real axis).

For instance, the function $f(x) = 1/x$ is continuous on the interval $(0,1)$ but is unbounded there. The function $f(x) = x$ is bounded on the same interval but has no maximal and minimal values on that interval.

Combining the Weierstrass theorem and the IVP of continuous functions, we get

Corollary 13.1.11. *If $f \in C[a,b]$, then the image $f[a,b]$ is a closed interval.*

Exercise 13.1.12.

i. Give an example of a bounded continuous function on $\mathbb{R}$ which has no maximum and minimum.

ii. Prove, that if $f \in C(\mathbb{R})$ is a positive function and $\lim_{x\to\infty} f(x) = 0$, then f attains its maximum value.

Exercise 13.1.13. Show that if $f \in C[a,b]$, then the image of $[a,b]$ under f is a segment (closed interval).

13.2. Uniform continuity.

Definition 13.2.1. The function $f : E \to \mathbb{R}$ is called *uniformly continuous on E* if $\forall\epsilon > 0\ \exists\delta > 0$ such that the inequality

$$(\alpha) \qquad |f(x) - f(y)| < \epsilon$$

holds $\forall x, y \in E$ provided that $|x - y| < \delta$.

It is instructive to compare this definition with the definition of continuity *everywhere on* E. The latter says that $\forall x \in E\ \forall \epsilon > 0\ \exists \delta > 0$ (*depending on* x *and* ϵ) such that (α) holds provided that $|x - y| < \delta$. Here, δ depends on a point x. The uniform continuity guarantees the choice of δ which works everywhere on E, which is, at least formally, a stronger property than continuity everywhere.

In order to show that a continuous function f is not uniformly continuous, one has to find two sequences of points $\{x_n\}$ and $\{y_n\}$ in the domain of f such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \text{const}$.

Examples:

i. Consider the function $f(x) = \sin \frac{1}{x}$ on the set $E = (0, 1]$. The function is continuous (as a composition of two continuous functions) but not uniformly continuous. Indeed, consider two sequences of points: $x_n = (2\pi n)^{-1}$ and $y_n = [\,(2\pi + \frac{1}{2})n\,]^{-1}$. Clearly, $|x_n - y_n| \to 0$ but $f(x_n) = 0$, $f(y_n) = 1$.

ii. The identity function $f(x) = x$ is uniformly continuous everywhere on $\mathbb{R}$.

iii. The square function $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly. Suppose $x_n = \sqrt{n+1}$ and $y_n = \sqrt{n}$. Then

$$|x_n - y_n| = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

but $f(x_n) - f(y_n) = 1$.

iv. The function $f(x) = \sqrt{x}$ is continuous on $\{x \geq 0\}$. This follows from inequality

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}\,, \qquad x, y \geq 0\,.$$

To prove this inequality, we suppose that $y = x + h$ with $h > 0$. Then

$$\sqrt{y} - \sqrt{x} = \frac{h}{\sqrt{x+h} + \sqrt{x}} \leq \sqrt{h} = \sqrt{y - x}\,.$$

v. The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0.1]$. Indeed, consider the sequences $x_n = \frac{1}{2n}$ and $y_n = \frac{1}{2n+1}$, the difference between them converges to zero, but $f(y_n) - f(x_n) = 1$.

vi. The function $f(x) = \sin x^2$ is not uniformly continuous on $\mathbb{R}$. Choose $x_n = \sqrt{\frac{\pi}{2}(n+1)}$, $y_n = \sqrt{\frac{\pi}{2} n}$, then $|x_n - y_n| \to 0$ but $f(x_n) - f(y_n) = 1$.

Theorem 13.2.2 (Cantor). *If* $f \in C[a, b]$, *then* f *is uniformly continuous on* $[a, b]$.

Proof: Since f is continuous everywhere on $[a, b]$, for each point $t \in [a, b]$ and each $\epsilon > 0$, we find $\delta = \delta_{t,\epsilon}$ such that

$$|f(x) - f(y)| < \epsilon, \qquad \forall x, y \in U_\delta(t) = \{\xi \colon |\xi - t| < \tfrac{1}{2}\delta\}\,.$$

These neighbourhoods cover the segment $[a,b]$ and we can choose a finite subcovering

$$[a,b] \subset \bigcup_{j=1}^{N} U_{\delta_j}(t_j), \qquad \delta_j = \delta_{t_j},$$

and set

$$\delta = \min(\delta_1, \dots, \delta_N).$$

Now, let $x, y \in [a,b]$ be two points such that $|x-y| < \frac{\delta}{2}$. Choose a point t_j such that $x \in U_{\delta_j}(t_j)$, then $|x - t_j| < \frac{\delta_j}{2}$ and $|y - t_j| \le |y - x| + |x - t_j| < \delta_j$. By the choice of δ_j, we get

$$|f(x) - f(y)| \le |f(x) - f(t_j)| + |f(y) - f(t_j)| < 2\epsilon.$$

Done! $\square$

An alternative proof can be done using the Bolzano-Weierstrass lemma. Here is its sketch. Assume that f is not uniformly continuous on $[a,b]$, then, for some $\epsilon > 0$, one can find two sequences $\{x_n\}$ and $\{y_n\}$ such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \ge \epsilon$. Passing to the subsequences, we may assume that $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converge to $c \in [a,b]$. Then $|f(x_{n_k}) - f(y_{n_k})| \to 0$ and we arrive at the contradiction. Work out the details.

Exercise 13.2.3. If $f \in C[a,b]$, then the functions

$$m(x) = \inf_{a \le \xi \le x} f(\xi), \qquad \text{and} \qquad M(x) = \sup_{a \le \xi \le x} f(\xi)$$

are also continuous on $[a,b]$.

Exercise 13.2.4.

i. Let the function f be uniformly continuous on a bounded set E. Prove that f is bounded.

ii. Let $f \in C(a,b)$ where (a,b) is a finite interval. Prove that f is uniformly continuous on (a,b) if and only if there exist the limiting values $f(a+0)$ and $f(b-0)$.

iii. Let $f \in C(\mathbb{R})$ be bounded and monotonic. Prove that f is uniformly continuous.

Exercise 13.2.5. Check the uniform continuity of the following functions:

$$\log x, \quad x \in (0,1]; \qquad \frac{1}{\log x}, \quad x \in (0,1); \qquad x + \frac{x}{x+1}, \quad x \in [0,+\infty);$$

$$x \sin x; \qquad \sin x^2; \qquad \sin\sqrt{x} \quad (x \in \mathbb{R}).$$

Exercise 13.2.6. Let $f\colon E \to \mathbb{R}$, $E \subset \mathbb{R}$. Show that the function f is uniformly continuous on E if and only if

$$\omega_f(\delta) \stackrel{\text{def}}{=} \sup\{|f(x) - f(y)|\colon x, y \in E, |x-y| < \delta\} \to 0$$

for $\delta \to 0$.

13.3. Inverse functions. We start with a simple result (in fact, we've used it already):

Theorem 13.3.1. *Suppose the function $f : X \to \mathbb{R}$ is strongly monotonic, and $Y = fX$ is the range of f. Then there exists the inverse function $f^{-1} : Y \to X$ which is also strongly monotonic. It increases when f increases, and decreases when f decreases.*

The proof follows by a straightforward inspection and we skip it.

For continuous functions, strong monotonicity is also a necessary conditions for existence of the inverse function.

Theorem 13.3.2. *Let the function $f \in C[a, b]$ have an inverse function. Then f is strongly monotonic.*

Proof: First, observe that since f is invertible, for any $x, y \in [a, b]$, $f(x) \neq f(y)$.

Strongly monotonic functions have the following characteristic property: for each triple of points $x_1 < x_2 < x_3$ the value $f(x_2)$ must be belong to the open interval with the end-points at $f(x_1)$ and $f(x_3)$. Now, assume that the theorem is wrong and that there exists a triple $x_1 < x_2 < x_3$ such that, for example, $f(x_1) < f(x_3) < f(x_2)$ (the other cases are similar). Therefore, by the IVP-property there exists $\xi \in (x_1, x_2)$ such that $f(\xi) = f(x_3)$ which contradicts invertibility of f. $\square$

The next theorem says that for monotonic functions continuity is equivalent to the IVP-property.

Theorem 13.3.3. *Suppose $f : [a, b] \to \mathbb{R}$ is monotonic. Then f is continuous on $[a, b]$ if and only if the image $f[a, b]$ is a closed interval with the end-points at $f(a)$ and $f(b)$.*

Proof: If f is continuous, then by the IVP-property the image $f[a, b]$ contains any intermediate point between $f(a)$ and $f(b)$.

In the other direction, suppose $f[a, b]$ be a closed interval and suppose that f is discontinuous at $c \in [a, b]$. By monotonicity of f, the one-sided limits $f(c - 0)$ and $f(c + 0)$ exist, and at least one of open intervals

$$(f(c), f(c + 0)), \qquad (f(c - 0), f(c))$$

is not empty, let us call this interval I. The function f does not attain any value from this interval, on the other hand, $I \subset [f(a), f(b)]$. The contradiction proves the theorem. $\square$

Note that the theorem fails without monotonicity assumption:

Exercise 13.3.4. Consider the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \in \mathbb{R} \setminus \{0\} \\ 0 & x = 0\,. \end{cases}$$

This function is discontinuous at the origin. Check that for any closed interval $I \subset \mathbb{R}$ the image fI is an interval as well.

Combining these theorems, we obtain

Corollary 13.3.5. *Let $f \in C[a, b]$ be strongly monotonic. Then the inverse function f^{-1} is also continuous and strongly monotonic.*

Proof: Indeed, by Theorem 13.3.1, the inverse function f^{-1} is strongly monotonic. Suppose for instance, that f and hence f^{-1} are (strongly) increasing functions. Let $\alpha = f(a)$ and $\beta = f(b)$. Then by the IVP-property $f[a, b] = [\alpha, \beta]$; i.e., $f^{-1}[\alpha, \beta] = [a, b]$, and by Theorem 13.3.3 the function f^{-1} must be continuous. □

For example, the function $\arcsin x$ is continuous on $[-1, 1]$ and the function $\arctan x$ is continuous on $\mathbb{R}$.

In some sense, the continuity assumption in the last corollary is redundant:

Problem 13.3.6. Let $f \colon (a, b) \to \mathbb{R}$ be monotonic, and let the inverse f^{-1} be defined on a set E. Then f^{-1} is continuous on E.

Problem 13.3.7. Let $f \colon [0, 1] \to [0, 1]$ be a continuous increasing function. Then for each $x \in [0, 1]$ one of the following holds: either x is a fixed point of f (that is, $f(x) = x$), or the n-th iterate $f^n(x)$ converges to a fixed point of f when $n \to \infty$.

14. The derivative

14.1. Definition and some examples.

Definition 14.1.1 (The derivative)**.** f be a function defined in an open neighbourhood U of a point $x \in \mathbb{R}$. The function f is called *differentiable at x* if there exists the limit

$$f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = \lim_{\epsilon\to 0} \frac{f(x+\epsilon)-f(x)}{\epsilon}$$

called the *derivative of f at x*. The function f is differentiable on an open interval (a,b) if it is differentiable at every point $x \in (a,b)$.

Sometimes, we denote the differences by the symbols Δ:

$$\Delta x = y - x = \epsilon$$

and

$$\Delta f(x,\epsilon) = f(x+\epsilon) - f(x).$$

Notice that Δf is a function of two variables: x and $\Delta x = \epsilon$. In these notations

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta f(x,\Delta x)}{\Delta x} = \frac{df}{dx},$$

where df and dx are (in the meantime) symbolic notations called the *differentials* of f and of x.

If the function f is defined on the closed interval $[a,b]$, then we say that f is differentiable at the end-points a and b if there exist one-sided limits:

$$f'(a+0) = \lim_{y\downarrow a} \frac{f(y)-f(a)}{y-a}, \qquad f'(b-0) = \lim_{y\uparrow b} \frac{f(y)-f(b)}{y-b}.$$

It follows immediately from the definition, that if f is differentiable at x, then it must be continuous at x, otherwise, the limit in the definition of the derivative is infinite.

Examples:

(i) Let $f(x)$ be the constant function. Then $f'(x) = 0$ everywhere. Soon, we'll see that this property characterizes the constant functions: they are the only functions with the zero derivative.

(ii) Let $f(x) = x^n$, $n \in \mathbb{N}$. Then

$$\Delta f(x,\epsilon) = (x+\epsilon)^n - x^n = nx^{n-1}\epsilon + o(\epsilon), \qquad \epsilon \to 0.$$

So that

$$f'(x) = \lim_{\epsilon\to 0} \frac{\Delta f(x,\epsilon)}{\epsilon} = \lim_{\epsilon\to 0} \left(nx^{n-1} + o(1)\right) = nx^{n-1}.$$

In particular, if the function $f(x)$ is linear, than its derivative is a constant function: $(ax+b)' = a$. We'll learn soon that the linear functions are the only functions with constant derivative.

(iii) Consider the sine-function $f(x) = \sin x$. Then

$$\Delta f(x,\epsilon) = \sin(x+\epsilon) - \sin x = 2\sin\frac{\epsilon}{2}\cos\left(x+\frac{\epsilon}{2}\right),$$

and

$$(\sin x)' = \lim_{\epsilon\to 0}\left(\frac{\sin(\epsilon/2)}{\epsilon/2}\right)\cos\left(x+\frac{\epsilon}{2}\right) = \cos x.$$

In a similar way, one finds the derivative of the cosine function

$$(\cos x)' = -\sin x.$$

(iv) Next, consider the exponential function $f(x) = a^x$. Now

$$\Delta f = a^{x+\epsilon} - a^x = a^x\left(a^\epsilon - 1\right) = a^x\left(e^{\epsilon\log a} - 1\right),$$

and

$$\lim_{\epsilon\to 0}\frac{\Delta f(x,\epsilon)}{\epsilon} = a^x\lim_{\epsilon\to 0}\frac{e^{\epsilon\log a}-1}{\epsilon} = a^x\log a\,\lim_{\delta\to 0}\frac{e^\delta - 1}{\delta} = a^x\log a.$$

Therefore,

$$(a^x)' = a^x\log a\,.$$

In particular,

$$(e^x)' = e^x.$$

This explains why in many situations it is simpler to work with the base e than with the other bases.

(v) Now, let $f(x) = x^\mu$, $x > 0$ and $\mu > 0$. Then

$$\begin{aligned}\Delta f(x,\epsilon) &= (x+\epsilon)^\mu - x^\mu \\ &= x^\mu\left\{\left(1+\frac{\epsilon}{x}\right)^\mu - 1\right\} \\ &= x^\mu\left\{1+\mu\frac{\epsilon}{x} + o(\epsilon) - 1\right\} \\ &= \mu x^{\mu-1}\epsilon + o(\epsilon)\,,\end{aligned}$$

and

$$(x^\mu)' = \mu x^{\mu-1}\,.$$

This computation extends example (ii).

(vi) Consider the logarithmic function $f(x) = \log_a|x|$ defined for $x\in\mathbb{R}\setminus\{0\}$. In this case,

$$\Delta f(x,\epsilon) = \log_a|x+\epsilon| - \log_a|x| = \log_a\left|1+\frac{\epsilon}{x}\right|.$$

If ϵ is sufficiently small: $|\epsilon| < |x|$, then the expression $1+\epsilon/x$ is positive and

$$\Delta f(x,\epsilon) = \log_a\left(1+\frac{\epsilon}{x}\right) = \frac{\log\left(1+\epsilon/x\right)}{\log a} = \frac{\epsilon}{x\log a} + o(\epsilon)\,.$$

Hence

$$(\log_a|x|)' = \frac{1}{x\log a}.$$

In particular,

$$(\log|x|)' = \frac{1}{x}.$$

(vii) At last, consider the function $f(x) = |x|$. It is easy to see directly from the definition that $f'(x) = \operatorname{sgn}(x)$ for $x \neq 0$ and that f has no derivative at the origin.

14.2. Some rules. In this section we show several simple rules which help us to compute derivatives.

Theorem 14.2.1. *Let the functions f and g be defined on an interval (a,b) and suppose they are differentiable at the point $x \in (a,b)$. Then*

(i) *the sum $f+g$ is differentiable at x and $(f+g)'(x) = f'(x) + g'(x)$;*

(ii) *the product $f \cdot g$ is differentiable at x and*

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

In particular, if c is a constant, then $(cf)'(x) = cf'(x)$.

(iii) *if $g(x) \neq 0$, then the quotient $\frac{f}{g}$ is differentiable at x and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Proof: The proof of (i) is obvious. Next,

$$\begin{aligned}(f \cdot g)(x+\epsilon) &- (f \cdot g)(x) \\ &= f(x+\epsilon)g(x+\epsilon) - f(x)g(x+\epsilon) + f(x)g(x+\epsilon) - f(x)g(x) \\ &= (f(x+\epsilon) - f(x))g(x+\epsilon) + f(x)(g(x+\epsilon) - g(x))\end{aligned}$$

which readily gives us (ii).

Having (ii), it suffices to prove (iii) in a special case when f equals identically 1:

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{g^2(x)}. \qquad (iv)$$

We have

$$\begin{aligned}\frac{1}{g(x+\epsilon)} - \frac{1}{g(x)} &= -\frac{g(x+\epsilon) - g(x)}{g(x+\epsilon)g(x)} \\ &= -\frac{g(x+\epsilon) - g(x)}{g^2(x)} \cdot \frac{g(x+\epsilon)}{g(x)},\end{aligned}$$

which yields (iv). This proves the theorem. □

Example 14.2.2. Consider the function $f(x) = \tan x = \frac{\sin x}{\cos x}$. We have

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

That is,

$$(\tan x)' = \frac{1}{\cos^2 x}.$$

Similarly,

$$(\cot x)' = -\frac{1}{\sin^2 x}.$$

Example 14.2.3. If

$$P(x) = \sum_{j=0}^{n} a_j x^j$$

is a polynomial of degree n, then

$$P'(x) = \sum_{i=0}^{n-1} (i+1) a_{i+1} x^i.$$

is a polynomial of degree $n-1$.

14.3. Derivative of the inverse function and of the composition.

Theorem 14.3.1. *Let the function $f\colon (a,b) \to \mathbb{R}$ be a continuous, strictly monotone function. Suppose f is differentiable at the point $x_0 \in (a,b)$ and $f'(x_0) \neq 0$. Then the inverse function $g = f^{-1}$ is differentiable at $y_0 = f(x_0)$ and*

$$g'(y_0) = \frac{1}{f'(x_0)}.$$

Symbolically, if $y = f(x)$, then $x = g(y)$ and

$$g'(y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

Proof: Let $x = g(y)$. If $y \to y_0$, then $g(y) \to g(y_0)$ (since the function g is continuous at y_0) or, what is the same, $x \to x_0$. Then we have

$$\begin{aligned} \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} &= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} \\ &= \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}, \end{aligned}$$

proving the theorem. □

Theorem 14.3.1 gives us the expression for $g'(y)$ in terms of the variable x, however, applying Theorem 14.3.1, we have to return to the variable y.

Examples:

i. Let $f(x) = \sin x$, $x \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$.

$$(\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}.$$

Similarly,

$$(\arccos y)' = -\frac{1}{\sqrt{1-y^2}}\,.$$

ii. Let $f(x) = \tan x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then

$$(\arctan y)' = \frac{1}{(\tan x)'} = \cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}\,.$$

Similarly,

$$(\mathrm{arccot} y)' = -\frac{1}{1+y^2}\,.$$

iii. Let $f(x) = a^x$. Then $g(y) = \log_a y$ and

$$(\log_a y)' = \frac{1}{a^x \log a} = \frac{1}{y \log a}\,.$$

(We've known already the answer in advance, of course).

Theorem 14.3.2 (The Chain Rule)**.** *Let the function $y = f(x)$ be differentiable at the point x_0 and let the function $z = g(y)$ be differentiable at the point $y_0 = f(x_0)$. Then the composition function $g \circ f$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0).$$

Symbolically,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

Proof: We have

$$\begin{aligned}\frac{(g\circ f)(x) - (g \circ f)(x_0)}{x - x_0} &= \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \\ &= \frac{g(y) - g(y_0)}{y - y_0} \cdot \frac{f(x) - f(x_0)}{x - x_0}.\end{aligned}$$

If $x \to x_0$, then $y \to y_0$ (since the function f is continuous at x_0), and we see that the last expression tends to $g'(y_0)f'(x_0)$ proving the theorem. □

The chain rule is easily extended to the composition of several functions: if $F = f_1 \circ f_2 \circ \ldots \circ f_n$, then

$$F' = f_1'(f_2 \circ \ldots \circ f_n) f_2'(f_3 \circ \ldots \circ f_n) \ldots f_n'.$$

This can be easily proved by induction with respect to n. In particular, if

$$F = f \circ f \circ \ \ldots \ \circ f = f^{\circ n}$$

is the n-th iterate of the function f, then

$$F' = f'(f^{\circ (n-1)}) f'(f^{\circ (n-2)}) \ldots f'(f) f'\,.$$

Examples:

i. The logarithmic derivative. Let $f(x) = \log g(x)$. Then
$$f'(x) = \frac{g'}{g}(x)\,.$$
For example, if $P(x) = c(x - x_1)...(x - x_n)$ is a polynomial of degree n, then
$$\frac{P'}{P}(x) = \frac{1}{x - x_1} + \,...\, + \frac{1}{x - x_n}\,.$$
ii. If $f(x) = e^{g(x)}$, then $f'(x) = g'(x)e^{g(x)}$.
iii. If $f(x) = u(x)^{v(x)}$, then
$$f' = \left(e^{v\log u}\right)' = e^{v\log u}(v\log u)' = u^v\left(v'\log u + v\frac{u'}{u}\right).$$
For example,
$$(x^x)' = x^x\left(\log x + x\,\frac{1}{x}\right) = x^x\left(\log x + 1\right).$$

15. Applications of the derivative

The differential calculus was systematically developed by Newton and Leibnitz, however Archimedes, Fermat, Barrow and many other great mathematicians already used it in some concrete situations. In this lecture we bring just a few of numerous applications without trying to make the arguments completely formal.

15.1. Local linear approximation. Given a function $f : (a, b) \to \mathbb{R}$ and a point $x_0 \in (a, b)$, we want to find a linear approximation to the function f which will be good in a small neighbourhood of the point x_0. More precisely, we are looking for the linear function $L(x) = c_0 + c_1(x - x_0)$ such that

$$f(x) = L(x) + o(x - x_0), \qquad x \to x_0.$$

In the limit $x \to x_0$, we obtain condition: $f(x_0) = L(x_0)$ (of course, if the function f is continuous at x_0, so let's assume that this is the case), that is $c_0 = f(x_0)$. Then

$$c_1 = \frac{f(x) - f(x_0)}{x - x_0} + o(1)\,,$$

and in the limit we obtain $c_1 = f'(x_0)$ (provided that f is differentiable at x_0). Therefore, the linear function L equals

$$L(x) = f(x_0) + (x - x_0)f'(x_0),$$

and we obtain

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + o(x - x_0), \qquad x \to x_0.$$

Sometimes, the approximate equality

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0)$$

can be used in order to find the numerical value of $f(x)$ if $f(x_0)$ is known. The closer x to x_0, the better approximation we get. Consider two examples:

If $f(x) = \log x$ and $x_0 = 1$, then we get an approximation for small values of t:

$$\log(1 + t) \approx t$$

which shows, for example, that $\log 1.02 \approx 0.02$ while my calculator gives $\log 1.02 = 0.0198026$.

If $f(x) = \sqrt{x}$ and $x_0 = 100$, then $f(x_0) = 10$, $f'(x_0) = \frac{1}{20}$, so we get

$$\sqrt{100 + t} \approx 10 + \frac{t}{20}.$$

For example, $\sqrt{101} \approx 10.05$, and my calculator gives $\sqrt{101} = 10.049876$.

Exercise 15.1.1. Without using the calculator, find the approximate values of $\tan 44^\circ$ and of $\frac{1}{0.95^{13}}$. Check the results with the calculator.

Later, we'll develop further the idea of this section and find a polynomial $P(x)$ of degree $\le n$ which locally approximate the function $f(x)$ in the following way:

$$f(x) = P(x) + o((x-x_0)^n), \qquad x \to x_0.$$

15.2. The tangent line. Given a curve γ in the (x,y)-plane and a point $M_0(x_0,y_0)$ on γ, we want to draw through M_0 a tangent line to γ. For that, we consider another point $M_1(x_1,y_1)$ on γ which is sufficiently close to M_0 and draw the straight line Q through these points. The *tangent line* to γ at M_0 is a limiting position of this straight line when the point M_1 moves to M_0 along γ.

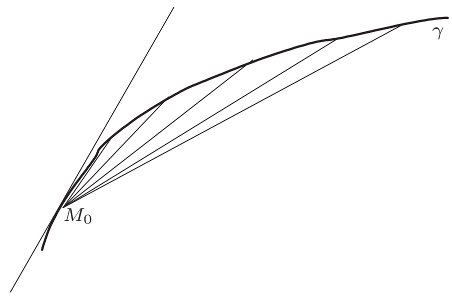

FIGURE 10. `The tangent line to the curve` γ

Now, assume that the line γ is a graph of the function $f(x)$, and let us find equation of the tangent line. The equation of the straight line Q is

$$y = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)\,.$$

We see that if existence of the limiting equation as $x_1 \to x_0$ is equivalent to the differentiability of the function f at x_0. The limiting equation is

$$y = f(x_0) + f'(x_0)(x-x_0)\,.$$

This is the equation of the tangent line we were after. In particular, we see that the *slope* of the tangent line at the point x_0 equals $f'(x_0)$.

Example 15.2.1. Let $f(x) = x^2 \sin\frac{1}{x}$ for $x \ne 0$ and $f(0) = 0$. This function is differentiable at the origin, and $f'(0) = \lim_{\epsilon\to 0} \epsilon \sin\frac{1}{\epsilon} = 0$. We see that the x-axis is the tangent line to the graph of f at the origin. Observe that in this example the graph of f has infinitely many intersections with the tangent line in any neighbourhood of the origin.

Exercise 15.2.2. Find the angles between the graphs of functions $y = 8 - x$ and $y = 4\sqrt{x+4}$ at the point of their intersection.

Exercise 15.2.3. Find the value of parameter a such that the graphs of the functions $y = ax^2$ and $y = \log x$ touch each other (i.e. have a joint tangent line).

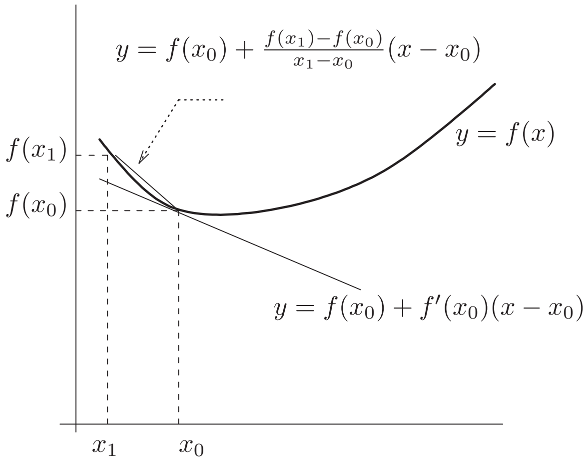

FIGURE 11. The tangent to the graph of the function f

15.3. Lagrange interpolation. From high school, we know how to draw a straight line through two points in the plane. Here, we consider a more general problem: *given a set of $n+1$ points in the plane $M_j(x_j, y_j)$, $0 \leq j \leq n$, find a polynomial $P(x)$ of degree $\leq n$ whose graph passes all these points*; i.e.

$$(a) \qquad P(x_j) = y_j, \qquad 0 \leq j \leq n.$$

A natural restriction is that the points x_j must be disjoint: $x_j \neq x_i$ for $j \neq i$.

To solve the problem we define the polynomial

$$Q(x) = (x - x_0)(x - x_1) \dots (x - x_n)$$

of degree n and observe that

$$(b) \qquad \lim_{x \to x_j} \frac{Q(x)}{(x - x_j)Q'(x_j)} = \lim_{x \to x_j} \frac{Q(x) - Q(x_j)}{(x - x_j)Q'(x_j)} = 1.$$

Now, we can present the solution of the problem:

$$(c) \qquad P(x) = \sum_{k=0}^{n} \frac{y_k Q(x)}{(x - x_k)Q'(x_k)}.$$

First of all, observe that P is indeed a polynomial of degree $\leq n$: since $Q(x)$ vanishes at x_k, the polynomial $Q(x)/(x - x_k)$ is a polynomial of degree n, so that P is a sum of $n+1$ polynomials of degree n, and therefore has degree $\leq n$.

Now, we check that P satisfies conditions (b). When we plug $x = x_j$ in the right hand side of (c), we see that the terms with $k \neq j$ vanish (since the numerator vanishes and the denominator does not). Therefore, the only term with $k = j$ remains on the right hand side. Since this remaining term is a polynomial, it is a continuous function of x, so we can find its value at x_j using (a):

$$P(x_j) = \lim_{x \to x_j} \frac{y_j Q(x)}{(x - x_j)Q'(x_j)} = y_j.$$

Mention, that the solution P we have found is unique: if there are two solutions P_1 and P_2 satisfying (a), then their difference $P_1 - P_2$ vanishes at all $n+1$ points x_j. Being a polynomial of degree $\le n$, it must be the zero function.

It is also worth to mention another form of the formula (c):

$$(d) \qquad \frac{P(x)}{Q(x)} = \sum_{k=0}^{n} \frac{P(x_k)}{(x-x_k)Q'(x_k)}$$

which provides the *partial fraction decomposition* of the rational function P/Q in the case when $\deg P < \deg Q$ (and Q has simple zeroes, i.e. Q' does not vanish at zeroes of Q).

Exercise 15.3.1 (Newton)**.** Show that for $n \ge 1$

$$\sum_{j=0}^{n} \frac{x_j^p}{Q'(x_j)} = \begin{cases} 0, & 0 \le p \le n-1 \\ 1, & p = n. \end{cases}$$

Hint: in the case $p < n$, apply (d) to $P(x) = x^{p+1}$ and set $x = 0$. In the case $p = n$, apply (d) to $P(x) = x^n$, multiply the formula you get by x, and let $x \to \infty$.

15.3.1. *Appendix: the Horner scheme.* In the solution above we used two simple facts which you may not know yet:

15.3.2. *If a polynomial Q of degree $n+1$ vanishes at x_j, then $Q(x) = (x-x_j)Q_1(x)$ where Q_1 is a polynomial of degree n.*

15.3.3. *If a polynomial of degree $\le n$ vanishes at $n+1$ points, then it must be zero everywhere.*

To prove these facts, you should recall the Horner scheme (a fast algorithm of a division of a polynomial by a linear factor) which you've probably known from the high-school. Here it is:

Claim 15.3.4 (Horner's scheme)**.** *Consider the polynomial $p(x) = \sum_{k=0}^{n} p_k x^k$ and the number $c \in \mathbb{R}$. Then there are another polynomial q and a constant $r \in \mathbb{R}$ such that*

$$p(x) = (x-c)q(x) + r\,.$$

Here the degree of q is less than the degree of p by one, and $r = f(c)$.

Proof: We look for q at the form $q(x) = \sum_{k=0}^{n-1} q_k x^k$, we need to find the coefficients q_k. We have

$$p_n x^n + p_{n-1}x^{n-1} + \dots + p_1 x + p_0 = (x-c)(q_{n-1}x^{n-1} + \dots + q_1 x + q_0) + r\,,$$

which is equivalent to the chain of equations:

$$\begin{array}{rcl} p_n & = & q_{n-1} \\ p_{n-1} & = & q_{n-2} - cq_{n-1} \\ p_{n-2} & = & q_{n-3} - cq_{n-2} \\ \dots & & \dots \\ p_1 & = & q_0 - cq_1 \\ p_0 & = & r - cq_0 \,. \end{array}$$

From here, we find one by one the coefficients q_k and the remainder r. $\square$

This yields 15.3.2 and 15.3.3.

Remark 15.3.5. The Horner scheme works without any modifications for polynomials with coefficients in other fields different from $\mathbb{R}$. For instance, the coefficients p_k and the value c can be rational numbers. Then the polynomial q has rational coefficients and the value $r = p(c)$ is rational as well. Similarly, the coefficients of P might be complex numbers.

16. Derivatives of higher orders

16.1. Definition and examples. Let f be a function defined in a neighbourhood of a point x. The derivatives of higher orders of f at x are defined recurrently:

$$f''(x) = (f')'(x) = \frac{d^2 f}{dx^2}$$

(the *second order derivative*),

$$f'''(x) = (f'')'(x) = \frac{d^3 f}{dx^3}$$

(the *third order derivative*) etc, and

$$f^{(n)}(x) = (f^{(n-1)})'(x) = \frac{d^n f}{dx^n}$$

(the *derivative of order* n). Sometimes, it is convenient to agree that the zeroth order derivative is f itself: $f^{(0)} = f$, we'll follow this agreement.

Example 16.1.1. Let

$$P(x) = \sum_{k=0}^{n} c_k x^k$$

be a polynomial of degree n. Then differentiating P, we have:

$$\begin{aligned}
P^{(0)}(x) &= P(x), && P(0) = c_0; \\
P'(x) &= c_1 + 2c_2 x + \, ... \, + n c_n x^{n-1}, && P'(0) = c_1; \\
P''(x) &= 2c_2 + 3 \cdot 2 c_3 x + \, ... \, + n(n-1) c_n x^{n-2}, && P''(0) = 2c_2; \\
P'''(x) &= 3 \cdot 2 c_3 + \, ... \, + n(n-1)(n-2) c_n x^{n-3}, && P'''(0) = 3 \cdot 2 c_3; \\
& ... \\
P^{(n)}(x) &= n! c_n, && P^{(n)}(0) = n! c_n; \\
P^{(k)}(x) &= 0, && \text{for} \quad k > n\,.
\end{aligned}$$

We obtain

$$c_k = \frac{P^{(k)}(0)}{k!}, \qquad k \in \mathbb{Z}_+,$$

and

$$P(x) = P(0) + \frac{P'(0)}{1!} x + \frac{P''(0)}{2!} x^2 + \, ... \, + \frac{P^{(n)}(0)}{n!} x^n\,.$$

From here, we easily get a more general formula

$$P(x) = P(x_0) + \frac{P'(x_0)}{1!}(x - x_0) + \frac{P''(x_0)}{2!}(x - x_0)^2 + \, ... \, + \frac{P^{(n)}(x_0)}{n!}(x - x_0)^n\,.$$

To prove it, we consider the polynomial $Q(x) = P(x + x_0)$, apply the previous boxed formula to the polynomial $Q(y)$, and then replace y be $x - x_0$.

We'll return to these formulas a bit later when we'll begin the study the Taylor expansion.

Exercise 16.1.2. Let $u(x)$ and $v(x)$ be twice differentiable non-vanishing functions of x, and let

$$g(x) = \log \frac{u(x)}{v(x)}\,.$$

Find $g''(x)$.

The next table gives expressions for the higher derivatives of some elementary functions. These expressions are of frequent use. The formulas can be easily checked by induction with respect to the order of derivative.

$f(x)$	$f'(x)$	$f''(x)$	...	$f^{(n)}(x)$
a^x	$a^x \log a$	$a^x \log^2 a$	...	$a^x \log^n a$
e^x	e^x	e^x	...	e^x
$\sin x$	$\cos x$	$-\sin x$	...	$\sin\left(x + \frac{n\pi}{2}\right)$
$\cos x$	$-\sin x$	$-\cos x$	...	$\cos\left(x + \frac{n\pi}{2}\right)$
x^μ	$\mu x^{\mu-1}$	$\mu(\mu-1)x^{\mu-2}$	...	$\mu(\mu-1)...(\mu-n+1)x^{\mu-n}$
$\log\lvert x\rvert$	$\frac{1}{x}$	$-\frac{1}{x^2}$	...	$(-1)^{n-1}(n-1)!x^{-n}$
$\frac{ax+b}{cx+d}$	$\frac{ad-bc}{(cx+d)^2}$	$-\frac{2c(ad-bc)}{(cx+d)^2}$	...	$\frac{(-1)^{n-1}c^{n-1}n!(ad-bc)}{(cx+d)^{n+1}}$
$\frac{1}{\sqrt{ax+b}}$	$-\frac{a}{2(ax+b)^{3/2}}$	$\frac{a^2 1\cdot 3}{2^2(ax+b)^{5/2}}$	...	$\frac{(-1)^n a^n 1\cdot 3\cdot\ldots\cdot(2n-1)}{2^n(ax+b)^{n+\frac{1}{2}}}$

Exercise 16.1.3. Find

$$\left(\frac{\log x}{x}\right)^{(n)}\,.$$

Example 16.1.4. Consider the function

$$f(x) = \frac{1}{x^2 - a^2}\,.$$

First, represent f in the form more convenient for differentiation:

$$f(x) = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

Making use of this form, we easily find that

$$f^{(n)}(x) = \frac{(-1)^n n!}{2a}\left(\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right).$$

Example 16.1.5. Let

$$f(x) = e^{ax}\sin bx\,.$$

Then

$$\begin{aligned} f'(x) &= ae^{ax}\sin bx + be^{ax}\cos bx \\ &= \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\sin bx + \frac{b}{\sqrt{a^2+b^2}}\cos bx\right\}e^{ax} \\ &= \sqrt{a^2+b^2}\sin(bx+\varphi)e^{ax}\,, \end{aligned}$$

where φ is an "auxiliary phase" defined by

$$\sin\varphi = \frac{b}{\sqrt{a^2+b^2}}\,, \qquad \cos\varphi = \frac{a}{\sqrt{a^2+b^2}}\,.$$

Differentiating further, we get

$$f^{(n)}(x) = (a^2+b^2)^{\frac{n}{2}}\sin(bx+n\varphi)e^{ax}\,.$$

Functions which have derivatives of *any order* are called *infinitely differentiable*. The elementary functions are usually infinitely differentiable in the domain of definition. The set of infinitely differentiable functions on an interval I is denoted by $C^\infty(I)$.

Example 16.1.6. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

We show that f is an infinitely differentiable function on $\mathbb{R}$ and that

$$f^{(n)}(x) = \begin{cases} P_n\left(\frac{1}{x}\right)e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0, \end{cases} \tag{1}$$

where $P_n(s)$ is a polynomial of degree $3n$ in s. We shall need a

Claim 16.1.7. *For each p, $p < \infty$,*

$$\lim_{x\to 0} x^{-p}e^{-1/x^2} = 0\,.$$

Proof of the claim: follows by the change of variable: set $t = 1/x^2$, then

$$\lim_{x\to 0} x^{-p}e^{-1/x^2} = \lim_{t\to+\infty} t^{p/2}e^{-t} = 0\,.$$

□

Making use of induction with respect to n, we see that (1) holds for all $n \geq 1$ with $P_0 = 1$ and

$$P_{n+1}(s) = 2s^3 P_n(s) - s^2 P_n'(s)\,, \qquad \mathrm{deg} P_{n+1} = \mathrm{deg} P_n + 3.$$

At the origin, using the claim and again the induction with respect to n, we have

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = 0$$

This completes the argument. $\square$

Exercise 16.1.8. Build the infinitely differentiable function which vanishes outside of the interval $[0,1]$ but does not vanish identically.

Exercise 16.1.9. Suppose

$$f(x) = \begin{cases} x^{2n} \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Show that f is n times differentiable at the origin and $f^{(j)}(0) = 0$, $1 \leq j \leq n$. Show that the $n+1$-st derivative of f at the origin does not exist.

Exercise 16.1.10. Suppose f is an infinitely differentiable function on $\mathbb{R}$ such that, for some $n \in \mathbb{N}$, $f^{(n)}(x) \equiv 0$ on $\mathbb{R}$. Then f is a polynomial.

Problem* 16.1.11.

i. Suppose f is infinitely differentiable function on the real axis such that

$$\forall x \in \mathbb{R} \quad \exists n \in \mathbb{Z}_+ \quad \forall m \geq n \quad f^{(m)}(x) = 0\,.$$

Then f is a polynomial.

ii. Suppose f is infinitely differentiable function on the real axis such that

$$\forall x \in \mathbb{R} \quad \exists n \in \mathbb{Z}_+ \quad f^{(n)}(x) = 0\,.$$

Then f is a polynomial.

16.2. The Leibniz rule. We know that the product of two n times differentiable functions is n times differentiable as well. The Leibnitz formula gives an explicit expression for the n-th derivative of the product:

$$(uv)^{(n)} = \sum_{m=0}^{n} \binom{n}{m} u^{(n-m)} v^{(m)}\,,$$

where, as usual, $\binom{n}{m}$ is the binomial coefficient "n choose m".

Proof: We use induction with respect to n. For $n = 1$ the formula is correct. Suppose it is correct for the n-th derivative, and check its correctness for the

$n+1$-st derivative:

$$\begin{aligned}
(uv)^{(n+1)} &= \left(\sum_{m=0}^{n}\binom{n}{m}u^{(n-m)}v^{(m)}\right)' \\
&= \sum_{m=0}^{n}\binom{n}{m}u^{(n-m+1)}v^{(m)} + \sum_{m=0}^{n}\binom{n}{m}u^{(n-m)}v^{(m+1)} \\
&= u^{(n+1)}v^{(0)} + \sum_{m=1}^{n}\left(\binom{n}{m}+\binom{n}{m-1}\right)u^{(n+1-m)}v^{(m)} + u^{(0)}v^{(n+1)} \\
&= \sum_{m=0}^{n+1}\binom{n+1}{m}u^{(n+1-m)}v^{(m)},
\end{aligned}$$

completing the argument. □

Exercise 16.2.1. Find $(x^2 \cos ax)^{(2008)}$.

Example 16.2.2. Find the n-th order derivative of $g(y) = \arctan y$ at $y = 0$. We'll show that

$$g^{(n)}(0) = \begin{cases} 0 & \text{for } n = 2m \\ \\ (-1)^m(2m)! & \text{for } n = 2m+1. \end{cases}$$

Indeed, since the function $\arctan y$ is odd, its derivatives of even order vanish at the origin (prove it!), so we need to find only derivatives of odd orders. We have

$$g'(y)(1+y^2) = 1.$$

Differentiating this equation $n = 2m$ times and using the Leibnitz rule, we get the recurrence relation

$$(1+y^2)g^{(n+1)} + 2nyg^{(n)} + n(n-1)g^{(n-1)} = 0.$$

Substituting here $y = 0$, we get

$$g^{(2m+1)}(0) + 2m(2m-1)g^{(2m-1)} = 0.$$

Since $g'(0) = 1$, this yields the result. □

Exercise 16.2.3. Show that

$$\left.\frac{d^n \arcsin y}{dy^n}\right|_{y=0} = \begin{cases} 0 & \text{for } n = 2m \\ \\ ((2m-1)!!)^2 & \text{for } n = 2m+1. \end{cases}$$

Here, $(2m-1)!! = 1 \cdot 3 \cdot 5 \cdot \ \ldots\ \cdot (2m-1)$.

Hint: use that $(1-y^2)g''(y) - yg'(y) = 0$ for $g(y) = \arcsin y$.

Exercise 16.2.4. Function $y(x)$ satisfies the differential equation $y'' - xy = 0$ with $y(0) = 0$ and $y'(0) = 1$. Find the derivatives of all orders $y^{(n)}(0)$.

16.3. Derivatives of functions defined in the parametric form. Sometimes, the function $y(x)$ we need to differentiate is given in a parametric form:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad a < t < b.$$

Suppose the function $x(t)$ is invertible, then we denote the inverse function by $t(x)$ and obtain the function $y(x) = y(t(x))$ of variable x. We can differentiate this function using the chain rule and express the derivatives in terms of the parameter t:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} : \frac{dx}{dt} = \frac{y'(t)}{x'(t)},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)\frac{dt}{dx} = \frac{y''(t)x'(t) - y'(t)x''(t)}{x'^3(t)}.$$

If needed, we can continue the process.

Example 16.3.1. Consider the equation of the ellipse:

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \qquad 0 \le t \le 2\pi.$$

To make the function $x(t)$ invertible, we assume that $0 \le t < \pi$, however the formulas we'll obtain below do not depend on the choice of the domain for the parameter t, for example, they also work if $\pi \le t < 2\pi$. We have

$$x'(t) = -a\sin t, \qquad x''(t) = -a\cos t,$$

$$y'(t) = b\cos t, \qquad y''(t) = -b\sin t,$$

and

$$\frac{dy}{dx} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t,$$

$$\frac{d^2y}{dx^2} = \frac{(-b\sin t)(-a\cos t) - (b\cos t)(-a\sin t)}{(-a\sin t)^3} = \frac{b}{a^2\sin^3 t}.$$

17. Basic theorems of the differential calculus: Fermat, Rolle, Lagrange. Applications

17.1. Theorems of Fermat and Rolle. Local extrema. We start with a simple

Claim 17.1.1. *Let the function f has the finite derivative at x_0. If $f'(x_0) > 0$, then there exists a $\delta > 0$ such that*

$$(I)\qquad \begin{cases} f(x) > f(x_0) & \text{for } x_0 < x < x_0 + \delta \\ f(x) < f(x_0) & \text{for } x_0 - \delta < x < x_0. \end{cases}$$

If $f'(x_0) < 0$, then

$$(II)\qquad \begin{cases} f(x) < f(x_0) & \text{for } x_0 < x < x_0 + \delta \\ f(x) > f(x_0) & \text{for } x_0 - \delta < x < x_0. \end{cases}$$

Proof of the claim: If $f'(x_0) > 0$, using the definition of the limit, we choose a $\delta > 0$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} > 0 \qquad \text{for } 0 < |x - x_0| < \delta.$$

This is equivalent to (I). The second case is similar. $\square$

In the case (I) we say that the function f *increases at* x_0, in the case (II) we say that the function f *decreases at* x_0.

Definition 17.1.2. We say that the function f has a *local extremum* at the point x_0, if one of the following holds:

$$f(x) \le f(x_0), \qquad \forall x \in U(x_0),$$

$$f(x) \ge f(x_0), \qquad \forall x \in U(x_0),$$

where $U(x_0)$ is a neighbourhood of x_0. In the first case, we say that f has a local maximum at x_0, and a local minimum in the second case.

Theorem 17.1.3 (Fermat)**.** *Let a function f be defined in a neighbourhood of a point x_0, be differentiable at x_0, and have a local extremum there. Then $f'(x_0) = 0$.*

The proof follows at once from the claim above. $\square$

If $f'(x) = 0$ then the point x is called *a critical point* of the function f. The set of all critical points $\{x\colon f'(x) = 0\}$ is called sometimes *a stationary set* of the function f.

17.1.1. *Classification of local extrema.* Vanishing of the derivative is only a necessary condition for the local extremum, for example, consider the function $f(x) = x^3$ in a neighbourhood of the origin. Its derivative vanishes at the origin, but the function does not have a local extremum there.

Note that if f attains its extremal value on the edge of the interval, then the derivative does not have to vanish. For example, consider the identity function $f(x) = x$ on $[-1, 1]$.

The next figure explains how to recognize what happens at critical points.

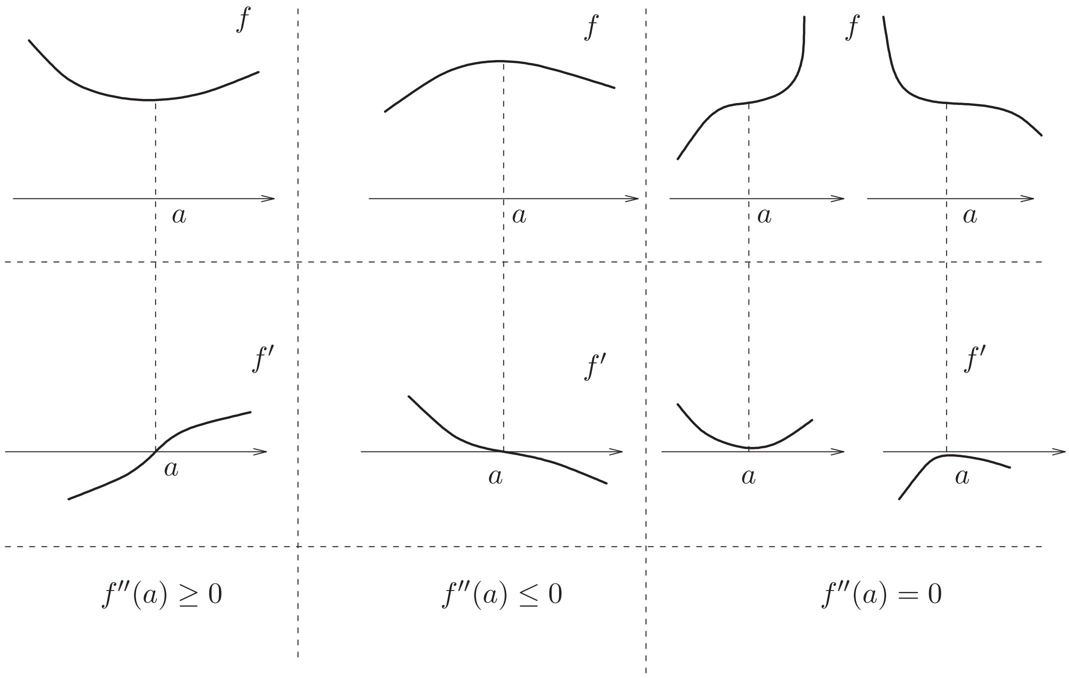

FIGURE 12. Classification of local extrema

Exercise 17.1.4. Find the critical points and their characters for the functions $f(x) = \dfrac{\log^2 x}{x}$, $x > 0$, and $g(x) = x(x-1)^{1/3}$, $x \in \mathbb{R}$. Sketch the graphs of these functions.

Hint: in the second example, the one-to-one change of variables $t = (x-1)^{1/3}$ simplifies the investigation.

17.1.2. *Geometric applications.* Now, we give two geometric applications of Fermat's theorem.

Question 17.1.5. Find x such that the rectangle on the following figure has the maximal area (the radius of the circumference equals one).

To solve this question, denote by $S(x)$ the area which we need to maximize. Then $S(x) = (1+x)\sqrt{1-x^2}$. We need to maximize this function for $-1 \leq$

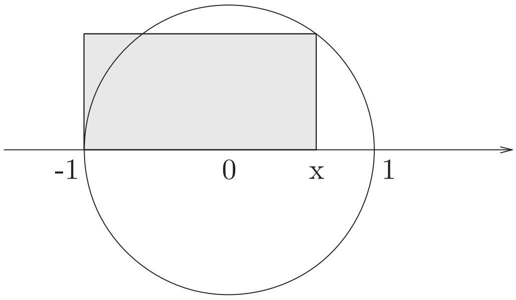

FIGURE 13

$x \le 1$. Since it is non-negative and vanishes at the end points $x = \pm 1$, at achieves its maximum at some inner point $x_0 \in (-1, 1)$. Then $S'(x_0) = 0$; i.e.,

$$\sqrt{1 - x^2} - \frac{x(x+1)}{\sqrt{1 - x^2}} = 0\,,$$

and we get equation

$$2x^2 + x - 1 = 0$$

with solutions $x_1 = \frac{1}{2}$ and $x_2 = -1$. The second root it irrelevant for us, and we see that the function S achieves its maximal value $\frac{3\sqrt{3}}{4}$ at the point $x = \frac{1}{2}$. □

In the second application, we prove the Snellius Law of Refraction. Recall that Fermat's *principle of least action* in optics says that the path of a light ray is determined by the property that the time the light takes to go from point A to point B under the given condition must be the least possible.

Question 17.1.6 (The Law of Refraction)**.** Given two points A and B on the opposite sides of the x-axis. Find the path from A to B that requires the shortest possible time if the velocity on one side of the x-axis is a and on the other side is b.

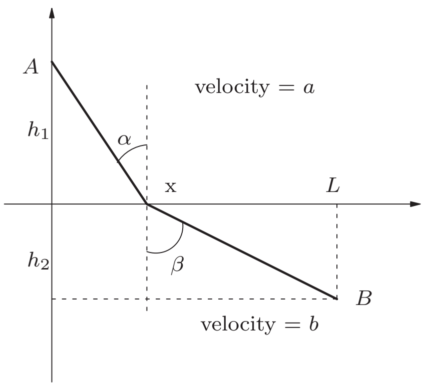

FIGURE 14. Law of refraction

If the light intersects the real axis at x, then the time it takes to go from A to B equals

$$T(x) = \frac{1}{a}\sqrt{h_1^2 + x^2} + \frac{1}{b}\sqrt{h_2^2 + (L-x)^2}\,.$$

We are looking the minimum of this function. We have

$$T'(x) = \frac{1}{a}\frac{x}{\sqrt{h_1^2 + x^2}} - \frac{1}{b}\frac{L-x}{\sqrt{h_2^2 + (L-x)^2}}\,.$$

This function vanishes for

$$\frac{1}{a}\underbrace{\frac{x}{\sqrt{h_1^2 + x^2}}}_{=\sin\alpha} = \frac{1}{b}\underbrace{\frac{L-x}{\sqrt{h_2^2 + (L-x)^2}}}_{=\sin\beta}\,.$$

Hence, the answer:

$$\frac{\sin\alpha}{\sin\beta} = \frac{a}{b}\,.$$

It is easy to see that we've indeed found the minimum of T. For instance, since $T''(x) > 0$ everywhere (`check!`).

Hairer and Wanner write in their book (p. 93) that Fermat himself found the problem too difficult for analytical treatment, and that the computations were performed by Leibniz.

17.1.3. *Rolle's theorem and its applications.*

Theorem 17.1.7 (Rolle)**.** *Let the function f be continuous on the closed interval $[a,b]$, be differentiable on the open interval (a,b), and let $f(a) = f(b)$. Then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof: By the Weierstrass theorem, the continuous function f in the closed interval $[a,b]$ attains its maximal and minimal values:

$$f(x_{\min}) = \min_{x\in[a,b]} f(x), \qquad f(x_{\max}) = \max_{x\in[a,b]} f(x).$$

Consider two cases:

(i) First, assume that $\min_{[a,b]} f = \max_{[a,b]} f$. Then f is the constant function and $f' = 0$ everywhere.

(ii) Now, suppose that $\min_{[a,b]} f \neq \max_{[a,b]} f$. Then at least one of the points $x_{\min}$, $x_{\max}$ must belong to the *open* interval (a,b), and by the Fermat theorem, the derivative of f vanishes at this point. □

Usually, counting zeroes of smooth functions, we are taking into account their *multiplicities*: if

$$f(c) = f'(c) = ... = 0, \qquad \text{but} \quad f^{(n)}(c) \neq 0,$$

then we say that f *has zero of multiplicity* n *at* c. If $n = 1$, we say that c is a *simple* zero of f. For example, the function $x \mapsto x^n$ $(n \in \mathbb{N})$ has zero of multiplicity n at the origin. The function $e^x - 1 - x$ has zero of multiplicity 2 at the origin.

Exercise 17.1.8. Construct a function the has zero of multiplicity m at $x = 0$ and n at $x = 1$. Construct a function the has zeroes of multiplicity 2 at each integer point.

Exercise 17.1.9.
i. Show that if the function f is continuous on the closed interval $[a, b]$, n times differentiable on the open interval (a, b), and has n zeroes in (a, b), then its $n-1$-st derivative has at least one zero in the open interval (a, b).
ii. Show that if a polynomial P of degree n has n real zeroes, then its derivative has $n-1$ real zeroes.
iii. Show that if a polynomial of degree n has at least $n+1$ real zeroes, then it vanishes identically.

Problem 17.1.10. For non-zero c_1, c_2, ..., c_n, and for pairwise distinct α_1, α_2, ..., α_n, prove that the equation

$$c_1x^{\alpha_1} + c_2x^{\alpha_2} + ... + c_nx^{\alpha_n} = 0$$

has at most $n-1$ zeroes in $(0, +\infty)$, and that the equation

$$c_1e^{\alpha_1 s} + c_2e^{\alpha_2 s} + ... + c_ne^{\alpha_n s} = 0$$

has at most $n-1$ real zeroes.
Hint: use induction with respect to n.

This bookkeeping can be made more accurate:

Problem 17.1.11 (Descartes' sign rule)**.** If $\alpha_1 < \alpha_2 < ... < \alpha_n$, then the number of positive zeroes of the function

$$f(x) = \sum_{j=1}^{n} c_jx^{\alpha_j}$$

(with their multiplicities) does not exceed the number of changes of signs in the sequence of coefficients c_1, c_2, ..., c_n.

17.2. Mean-value theorems.

Theorem 17.2.1 (Lagrange's mean value theorem)**.** *Let the function f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a point $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof: Notice, that in the special case $f(b) = f(a)$ the result coincides with the Rolle theorem. Now, using this special case we prove the general one. For this, define a linear function $L(x)$ that interpolates the values of f at the end-points:

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a),$$

and set

$$F(x) = f(x) - L(x).$$

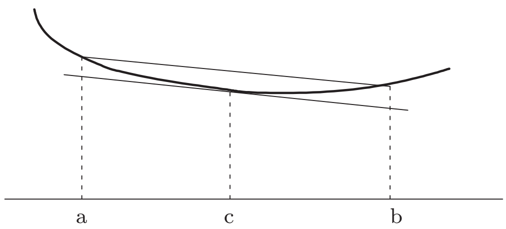

FIGURE 15. Lagrange's MVT

We have $F(a) = F(b) = 0$, so the Rolle theorem can be applied to F. We get an intermediate point $c \in (a, b)$ such that $F'(c) = 0$, or

$$f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a},$$

completing the proof. □

Corollary 17.2.2.
If the function f is differentiable on an open interval (a, b) and has a positive derivative there, then f is strictly increasing. If f' is negative, then f is strictly decreasing. If f' is non-negative, then f does not decrease, and if f' is not positive, then f does not increase.
If $f' \equiv 0$ on (a, b), then f is a constant function.
If f is n times differentiable and $f^{(n)} \equiv 0$, then f is a polynomial of degree $n - 1$ or less.

Corollary 17.2.3. *If f is a differentiable function, and $f' = f$. Then $f(x) = Ce^x$ (C is a constant).*

Proof: Consider the function $F(x) = f(x)e^{-x}$. Then $F'(x) = f'(x)e^{-x} - f(x)e^{-x} = 0$, therefore, F is a constant function. □

We've just learnt how to solve the simplest differential equations. The next problem looks more complicated (but in a year, after the course of ordinary differential equations you will recall it with a smile).

Problem 17.2.4. Let f be a twice differentiable function such that $f'' + f = 0$. Show that $f(z) = C_1 \sin x + C_2 \cos x$ where C_1 and C_2 are constants.
Hint: multiply the equation by $2f'$, deduce that $(f'^2 + f^2)' = 0$, hence $f'^2 + f^2$ is the constant function.

Exercise 17.2.5. Let $f\colon (0, +\infty) \to \mathbb{R}$ be a twice differentiable function, such that $f''(x) > 0$ everywhere. Prove that for each $x > 0$,

$$f(2x) - f(x) < f(3x) - f(2x)\,.$$

Exercise 17.2.6. Let the function f be defined on the interval I, and for some $\alpha > 1$ and $K < \infty$ satisfy

$$|f(x) - f(y)| \le K|x - y|^\alpha, \qquad \forall x, y \in I.$$

Then f is a constant function.

Problem 17.2.7 (Darboux)**.** Let the function f be differentiable everywhere in the segment $[a, b]$. Then f' attains every intermediate value between $f'(a)$ and $f'(b)$.

Notice that we do not require here that the derivative f' is continuous.

Hint: consider first a special case when $f'(a) < 0$ and $f'(b) > 0$, and prove that there exists $c \in (a, b)$ such that $f'(c) = 0$.

Warning: the obvious idea is that c must be an extremal point of f. In general, the idea is correct, but before applying the Fermat theorem, do not forget to check that c is not the end-point of the interval $[a, b]$.

Problem 17.2.8. Prove that if f is an unbounded differentiable function on an interval (a, b), then its derivative f' is also unbounded.

Whether the converse is true?

Problem 17.2.9. Prove that if f is a differentiable function on an interval (a, b) (finite or infinite) with the bounded derivative, then f is uniformly continuous on this interval.

Whether the converse is true; i.e. whether the uniformly continuous differentiable function must have a bounded derivative?

The next theorem slightly generalizes Lagrange's theorem:

Theorem 17.2.10 (Cauchy's extended mean value theorem)**.** *Let f and g be continuous functions on $[a, b]$ differentiable in the open interval (a, b). Then there exists a point $c \in (a, b)$ such that*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

If $g' \neq 0$ on (a, b), then $g(b) \neq g(a)$, and

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof: Notice, that if $g(x) = x$ then we get the previous result. The strategy of the proof is similar: define an auxiliary function

$$F(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

This function vanishes at the end-points: $F(b) = F(a) = f(a)g(b) - f(b)g(a)$, and applying the Rolle theorem, we get the result. □

17.3. L'Hospital's rule. Here we prove a theorem which in many cases simplifies calculation of limits.

Theorem 17.3.1. *Let f and g be differentiable functions defined on an interval (a, b) with*

$$f(a + 0) = g(a + 0) = 0.$$

If $g'(x) \neq 0$ for $x \in (a, b)$, and the limit

$$\lim_{x \downarrow a} \frac{f'(x)}{g'(x)}$$

exists, then the limit

$$\lim_{x\downarrow a}\frac{f(x)}{g(x)}$$

also exists and has the same value.

Proof: Set $f(a)=g(a)=0$, then the functions f and g are continuous on $[a,b)$. By Cauchy's extended mean value theorem, for $x\in(a,b)$ there is an intermediate value c between a and x such that

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(c)}{g'(c)}.$$

As x decreases to a, $c=c(x)$ also tends to a. By the assumption, the limit of the right hand side exists, so $f(x)/g(x)$ has the same limit. □

There are many other versions of L'Hospital rule. The limit point a can be replaced by $-\infty$ or $+\infty$. The limit values of f and g can be $+\infty$ or $-\infty$ instead of 0. The limit of $f'(x)/g'(x)$ also can be equal $+\infty$ or $-\infty$. In all these cases, the l'Hospital rule persists.

Here, we explain how to modify the proof if

$$f(a+0)=g(a+0)=+\infty.$$

Let $a<x<y<b$. Then

$$\frac{f(y)-f(x)}{g(y)-g(x)}=\frac{f'(c)}{g'(c)}$$

for some $c\in(x,y)$. From here, we find that

$$\frac{f(x)}{g(x)}:\frac{f'(c)}{g'(c)}=\frac{1-g(y)/g(x)}{1-f(y)/f(x)}.$$

Set

$$A=\lim_{t\downarrow a}\frac{f'(t)}{g'(t)},$$

and fix an arbitrary small positive ϵ. First, we choose y so close to a that

$$1-\epsilon<\frac{f'(c)}{g'(c)}:A<1+\epsilon.$$

Then we choose x such that

$$1-\epsilon<\left|\frac{1-g(y)/g(x)}{1-f(y)/f(x)}\right|<1+\epsilon\,.$$

(`why this is possible?`) We obtain

$$(1-\epsilon)^2<\frac{f(x)}{g(x)}:A<(1+\epsilon)^2.$$

Letting $\epsilon\to 0$, we complete the proof of this case. □

The other cases are left as an exercise.

Examples:

i.
$$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x} = \lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-1}{1-\cos x} = \lim_{x\to 0}\frac{1}{\cos^2 x}\frac{1-\cos^2 x}{1-\cos x} = 2.$$

ii.
$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right) &= \lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}\\ &= \lim_{x\to 0}\frac{\sin x + x\cos x}{\sin x}\cdot\lim_{x\to 0}\frac{\sin x - x\cos x}{x^2\sin x}\\ &= 2\cdot\lim_{x\to 0}\frac{x\sin x}{2x\sin x + x^2\cos x} = \frac{2}{3}.\end{aligned}$$

Exercise 17.3.2. Find the limits
$$\lim_{x\to 0}\frac{a^x + a^{-x} - 2}{x^2}\quad (a>0),\qquad \lim_{x\to 0}\frac{a^x-b^x}{c^x-d^x}\quad (c\neq d)\,.$$

Problem 17.3.3. Prove that if f is differentiable on $(a,+\infty)$ and
$$\lim_{x\to+\infty} f'(x) = 0,$$
then $f(x) = o(x)$ when $x\to+\infty$.

Problem 17.3.4. Prove that if the function f has the second derivative at x, then
$$f''(x) = \lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}.$$

Whether existence of the limit on the right hand side yields existence of the second derivative of f at x?

17.4. Appendix: Algebraic numbers. Lagrange's MVT has a nice application in the algebraic number theory.

Definition 17.4.1. The number $t\in\mathbb{R}$ is *algebraic* if there exist $a_0, a_1, ..., a_n\in\mathbb{Z}$, $a_n\neq 0$, with
$$\sum_{j=0}^{n} a_j t^j = 0\,.$$
The *degree* of the algebraic number t is the least possible n with this property.

The number $t\in\mathbb{R}$ is *transcendental* if it is not algebraic.

For instance, the rational numbers are algebraic numbers of degree 1, $\sqrt{2}$ is an algebraic number of degree 2. The number $10^{3/17}$ is also algebraic.

Note that if a rational number satisfies some algebraic equation with rational coefficients, then it satisfies another equation of the same degree with integer coefficients and hence is algebraic.

The first question is natural: *do the transcendental numbers exist?*

Exercise 17.4.2 (Cantor)**.** The set of algebraic numbers is countable. Hence, the transcendental numbers exist.

Unfortunately, this neat argument does not give us explicit examples of transcendental numbers.

Theorem 17.4.3 (Liouville)**.** *Suppose t is an algebraic number of degree $n \geq 2$. Then there exist a positive constant c (depending on t) such that*

$$\big|t - \frac{p}{q}\big| \geq \frac{c}{q^n}$$

for any $p, q \in \mathbb{Z}$.

The theorem says that algebraic numbers are badly approximated by the rational ones.

Proof: We assume that $\big|t - \dfrac{p}{q}\big| < 1$ (otherwise, any $c \leq 1$ works).

Suppose that $P(x) = \sum_{j=0}^{n} a_j x^j$ is a polynomial of degree n with integer coefficients such that $P(t) = 0$.

Claim 17.4.4. *The polynomial P cannot have rational roots.*

Proof of Claim: Indeed, suppose that $P(\frac{p}{q}) = 0$. Then

$$P(x) = P(x) - P(\frac{p}{q}) = (x - \frac{p}{q})Q(x)$$

where Q is a polynomial with *rational* coefficients of degree $n-1$. Since

$$Q(t) = \frac{P(t)}{t - p/q} = 0$$

we arrive at the contradiction (t cannot satisfy an algebraic equation of degree less than n). This proves the claim. □

The claim yields that, for any integers p and q, the number $P(p/q)$ is a non-zero rational number of the form r/q^n with integer $r \neq 0$. Hence

$$\big|P(\frac{p}{q})\big| \geq \frac{1}{q^n}\,.$$

Now, we have

$$\frac{1}{q^n} \leq \big|P(\frac{p}{q})\big| = \big|P(\frac{p}{q}) - P(t)\big| \overset{\text{MVT}}{=} \big|\frac{p}{q} - t\big|\big|P'(\xi)\big|\,.$$

The point ξ lies in the interval with the end-points at t and p/q, hence, it belongs to the larger interval $(t-1, t+1)$. Denoting by M the maximum of $|P'|$ over the closed interval $[t-1, t+1]$, we get

$$\frac{1}{Mq^n} \leq \big|\frac{p}{q} - t\big|\,.$$

Hence, the result. □

The numbers $t \in \mathbb{R}$ such that

$$\forall n \geq 2 \quad \exists \frac{p}{q} \in \mathbb{Q} \quad \big|t - \frac{p}{q}\big| \leq \frac{1}{q^n}$$

are called the *Liouville numbers.* The Liouville theorem says that they are transcendental.

Example 17.4.5. The number

$$t = \sum_{k=1}^{\infty} \frac{1}{10^{k!}}$$

is the Liouville number.

Indeed, let

$$\frac{p}{q} = \sum_{k=1}^{n} \frac{1}{10^{k!}}\,.$$

Then $q = 10^{n!}$, and

$$0 < t - \frac{p}{q} = \sum_{k=n+1}^{\infty} \frac{1}{10^{k!}} < \frac{2}{10^{(n+1)!}}\,,$$

while

$$\frac{1}{q^n} = \frac{1}{10^{n \cdot n!}}\,.$$

Since $10^{n!} > 2$ (`sic!`), we have

$$10^{(n+1)!} = \left(10^{n!}\right)^{n+1} > 2 \cdot 10^{n \cdot n!}\,,$$

i.e.,

$$0 < t - \frac{p}{q} < \frac{1}{q^n}\,.$$

Done! ◻

It is worth mentioning that the numbers e and π are transcendental but the proofs are not so simple (they are due to Hermite and Lindemann) and they were found *after* Liouville proved his theorem.

18. Inequalities

Here, we show how the differential calculus helps to prove useful inequalities.

18.1. $\frac{2}{\pi}x \le \sin x \le x, \qquad 0 \le x \le \frac{\pi}{2}$**.** The right inequality we already know. In order to prove the left inequality, consider the function

$$\varphi(x) = \frac{\sin x}{x}, \qquad 0 \le x \le \frac{\pi}{2}.$$

We have

$$\varphi'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x^2}(x - \tan x).$$

Since $x \le \tan x$ on the interval $[0, \frac{\pi}{2})$, $\varphi'(x) \le 0$. Therefore, the function φ does not increase, and

$$\varphi(x) \ge \varphi\left(\frac{\pi}{2}\right) = \frac{2}{\pi},$$

proving the inequality. □

Exercise 18.1.1. Show that the equality signs attains only at the end-points $x = 0$ and $x = \frac{\pi}{2}$.

18.2. $\frac{x}{1+x} < \log(1+x) < x$**,** $x > -1$**,** $x \neq 0$**.** In order to prove the right inequality, consider the function $\psi(x) = \log(1+x) - x$. Its derivative equals

$$\psi'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}.$$

Therefore, the function ψ increases on $(-1, 0)$, has a local maximum at $x = 0$ and decreases for $x > 0$. At the end-points it equals $-\infty$:

$$\lim_{x\downarrow -1} \psi(x) = \lim_{x\uparrow +\infty} \psi(x) = -\infty.$$

So that, the function ψ attains its global maximum at the origin, and hence $\log(1+x) < x$ for $x > -1$, $x \neq 0$.

To prove the left inequality, we set

$$\psi(x) = \log(1+x) - \frac{x}{1+x}.$$

In this case,

$$\psi'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}.$$

Now, ψ' is positive for $x > 0$, vanishes at the origin and is negative for $-1 < x < 0$. Therefore, ψ decreases for $-1 < x < 0$ and increases for $x > 0$. The limiting values of ψ equals $+\infty$:

$$\lim_{x\downarrow -1} \psi(x) = \lim_{x\uparrow +\infty} \psi(x) = +\infty.$$

So that, ψ attains its global minimum at the origin, and

$$\log(1+x) > \frac{x}{1+x}, \qquad x > -1, \quad x \neq 0,$$

completing the argument. □

Exercise 18.2.1. Show that

$$\frac{a-b}{a} < \log\frac{a}{b} < \frac{a-b}{b}$$

for positive a and b.

The inequality we proved has an interesting application:

Corollary 18.2.2. *There exists the limit*

$$\gamma = \lim_{n\to\infty}\Big(\sum_{j=1}^{n}\frac{1}{j} - \log n\Big).$$

The constant γ is called the Euler constant. Its approximate value is $\gamma \approx 0.5772$.

Proof of Corollary: Consider the series

$$(S) \qquad \sum_{j=1}^{\infty}\left(\frac{1}{j} - \log\frac{j+1}{j}\right).$$

We'll show that the terms of this series are positive and that the series is convergent.

Indeed,

$$\frac{1}{j+1} = \frac{1/j}{1+1/j} < \log\left(1+\frac{1}{j}\right) < \frac{1}{j},$$

so that

$$0 < \frac{1}{j} - \log\left(1+\frac{1}{j}\right) < \frac{1}{j} - \frac{1}{j+1} < \frac{1}{j^2},$$

and the series (S) converges since the series $\sum_{j\ge1}\frac{1}{j^2}$ is convergent.

Denote by γ the sum of the series S. Then

$$\begin{aligned}\sum_{j=1}^{n}\frac{1}{j} &= \sum_{j=1}^{n}\left(\frac{1}{j} - \log\frac{j+1}{j}\right) + \log(n+1)\\ &= \gamma + o(1) + \log n + o(1) = \gamma + \log n + o(1), \qquad n\to\infty,\end{aligned}$$

proving the corollary. □

18.3. Bernoulli's inequalities. We prove that for $x > 0$

$$\begin{aligned} x^\alpha - \alpha x &\le 1-\alpha, \qquad 0<\alpha<1,\\ x^\alpha - \alpha x &\ge 1-\alpha, \qquad \alpha<0, \quad \text{or } \alpha>1,\end{aligned}$$

with strong inequalities for $x \ne 1$.

Consider the function

$$f(x) = x^\alpha - \alpha x + \alpha - 1, \qquad x>0.$$

Then $f'(x) = \alpha(x^{\alpha-1} - 1)$. If $0<\alpha<1$, then f' is positive on $(0,1)$, vanishes at $x=1$ and is negative for $x>1$, and the limiting values of f are negative:

$$f(+0) = \alpha - 1 < 0,$$

$$\lim_{x\to+\infty} f(x) = -\infty.$$

So that

$$f(x) < f(1) = 0, \qquad \text{for } x > 0, \quad x \neq 1.$$

Similarly, if $\alpha < 0$ or $\alpha > 1$, f decreases on $(0,1)$ and increases on $(1,+\infty)$, and the limiting values of f are positive. So that, in this case

$$f(x) > f(1) = 0, \qquad \text{for } x > 0, \quad x \neq 1,$$

completing the proof. □

Exercise 18.3.1. Prove inequalities:

$$x^m(1-x)^n \leq \frac{m^m n^n}{(m+n)^{m+n}}, \qquad m,n > 0, \quad 0 \leq x \leq 1\,,$$

$$(x+1)2^{-\frac{n-1}{n}} \leq (x^n+1)^{\frac{1}{n}} \leq x+1\,, \qquad n \geq 1, \quad x > 0\,.$$

Exercise 18.3.2. Prove that equation $\log x = cx$
(i) has no solutions if $c > \frac{1}{e}$;
(ii) has a unique solution if $c = \frac{1}{e}$ or if $c \leq 0$;
(iii) has two solutions if $0 < c < \frac{1}{e}$.

Exercise 18.3.3. Prove that equation $\log(1+x^2) = \arctan x$ has two real solutions.

18.4. Young's inequality. Here, we prove that

$$(Y) \qquad ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

for $a,b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p,q > 1$, and the equality sign attains for $a^p = b^q$ only.

Introduce the function

$$h(a) = ab - \frac{a^p}{p}.$$

Then

$$h'(a) = b - a^{p-1}.$$

We see that

$$h'(a) \begin{cases} < 0, & \text{for } a < b^{1/(p-1)} \\ = 0, & \text{for } a = b^{1/(p-1)} \\ > 0, & \text{for } a > b^{1/(p-1)}. \end{cases}$$

Therefore,

$$h(a) \leq h\left(b^{1/(p-1)}\right) = b^{1+\frac{1}{p-1}} - \frac{b^{\frac{p}{p-1}}}{p} = \frac{b^q}{q},$$

and the equality sign attains only when $a = b^{1/(p-1)}$. This proves the statement. □

If $p > 1$, the value $q = \frac{p}{p-1}$ is called sometimes the *dual* to p. I.e., if p and q are dual to each other, then $\frac{1}{p} + \frac{1}{q} = 1$.

Exercise 18.4.1. Prove the inequality

$$ab \le e^a + b \log \frac{b}{e}, \qquad a, b > 0.$$

18.5. Hölder's inequality. The Hölder inequality says that

$$(H) \qquad \sum_{j=1}^{n} x_j y_j \le \left(\sum_{j=1}^{n} x_j^p \right)^{1/p} \left(\sum_{j=1}^{n} y_j^q \right)^{1/q}$$

provided that $x_j, y_j \ge 0$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, with the equality sign only in the case when

$$\frac{x_j^p}{y_j^q} = \text{const}, \qquad 1 \le j \le n.$$

When $p = q = 2$, with get the Cauchy-Schwarz inequality

$$\sum_{j=1}^{n} x_j y_j \le \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} \left(\sum_{j=1}^{n} y_j^2 \right)^{1/2}.$$

Proof of (H): Set

$$X = \left(\sum_{j=1}^{n} x_j^p \right)^{1/p}, \qquad Y = \left(\sum_{j=1}^{n} y_j^q \right)^{1/q},$$

and

$$a = \frac{x_j}{X}, \qquad b = \frac{y_j}{Y}.$$

Applying the Young inequality (Y), we get

$$\frac{x_j}{X} \cdot \frac{y_j}{Y} \le \frac{1}{p} \frac{x_j^p}{X^p} + \frac{1}{q} \frac{y_j^q}{Y^q}, \qquad 1 \le j \le n.$$

Adding these inequalities, we obtain

$$\frac{1}{X \cdot Y} \sum_{j=1}^{n} x_j y_j \le \frac{1}{p} \cdot 1 + \frac{1}{q} \cdot 1 = 1,$$

which yields (H).

There is the equality sign in (H) if and only if for each j we applied (Y) with the equality sign, that is

$$\left(\frac{x_j}{X} \right)^p = \left(\frac{y_j}{Y} \right)^q,$$

or setting $\lambda = X^p / Y^q$, we obtain

$$x_j^p = \lambda y_j^q, \qquad 1 \le j \le n,$$

completing the argument. $\square$

18.6. Minkowski's inequality. Minkowski's inequality says

$$(M)\qquad \left(\sum_{j=1}^{n}(x_j+y_j)^p\right)^{1/p} \le \left(\sum_{j=1}^{n}x_j^p\right)^{1/p} + \left(\sum_{j=1}^{n}y_j^p\right)^{1/p}$$

provided that $x_j, y_j > 0$ and $p \ge 1$.

Proof of (M): Let the index q be dual to p. Then

$$\begin{aligned}
\sum_{j=1}^{n}(x_j+y_j)^p &= \sum_{j=1}^{n}x_j(x_j+y_j)^{p-1} + \sum_{j=1}^{n}y_j(x_j+y_j)^{p-1} \\
&\le \left(\sum_{j=1}^{n}x_j^p\right)^{1/p}\left(\sum_{j=1}^{n}(x_j+y_j)^{(p-1)q}\right)^{1/q} \\
&\qquad + \left(\sum_{j=1}^{n}y_j^p\right)^{1/p}\left(\sum_{j=1}^{n}(x_j+y_j)^{(p-1)q}\right)^{1/q} \\
&= \left(\sum_{j=1}^{n}x_j^p\right)^{1/p}\left(\sum_{j=1}^{n}(x_j+y_j)^{p}\right)^{1/q} \\
&\qquad + \left(\sum_{j=1}^{n}y_j^p\right)^{1/p}\left(\sum_{j=1}^{n}(x_j+y_j)^{p}\right)^{1/q},
\end{aligned}$$

whence (M) follows at once. □

We finish this lecture mentioning two beautiful and deep inequalities proven by Swedish mathematicians:

Problem* 18.6.1 (Carleman)**.** Let $\sum_{j\ge1} a_j$ be a convergent series with positive terms. Then the series

$$\sum_{j\ge1}\{a_1 ... a_j\}^{1/j}$$

also converges and its sum is

$$< e\sum_{j\ge1}a_j.$$

The constant e in this inequality cannot be replaced by a smaller one.

Problem* 18.6.2 (Carlson)**.**

$$\left(\sum_{j\ge1}a_j\right)^4 \le \pi^2\left(\sum_{j\ge1}a_j^2\right)\left(\sum_{j\ge1}j^2a_j^2\right).$$

The constant π on the right hand side is optimal.

Try to solve these with *some* constants on the right hand side. This is also not easy. If you want to learn more about the inequalities, you should look at the classical book:

HARDY, LITTLEWOOD, POLYA "Inequalities"

or at the recent book

J.M.STEELE " Cachy-Schwarz master class".

19. Convex functions. Jensen's inequality

19.1. Definition. Let I be an interval, open or closed, finite or infinite. The function $f\colon I \to \mathbb{R}$ is called *convex* if its graphs lies below the chord between any two points on the graph.

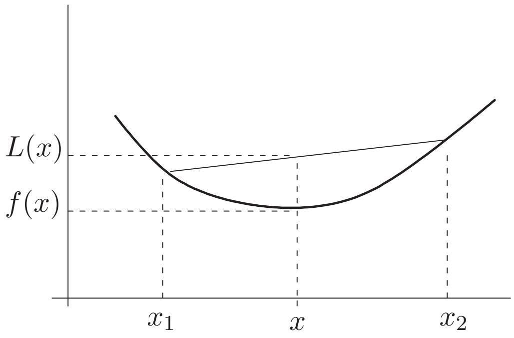

Figure 16. Convexity

Now, we'll find an analytic form of this condition. We fix two points $x_1, x_2 \in I$, $x_1 < x_2$, and let x be an intermediate point between x_1 and x_2; i.e. $x_1 \le x \le x_2$. Let $y = L(x)$ be an equation of the chord which joins the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Then the definition says

$$f(x) \le L(x) \qquad \forall x \in [x_1, x_2].$$

The affine function L is given by the equation

$$L(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1),$$

so that we get the inequality

$$(a) \qquad (x_2 - x_1) f(x) \le (x_2 - x) f(x_1) + (x - x_1) f(x_2),$$

which holds for any triple of points $x_1 \le x \le x_2$ from I. We set

$$x = \lambda x_1 + (1 - \lambda) x_2, \quad \lambda = \frac{x - x_1}{x_2 - x_1},$$

and get

$$(a') \qquad f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1 - \lambda) f(x_2)$$

for each $\lambda \in [0, 1]$ and each $x_1 < x_2$ in I. Obviously, (a) and (a') are equivalent. Taking $\lambda = \frac{1}{2}$, we get

$$f\big(\frac{x+y}{2}\big) \le \frac{f(x) + f(y)}{2}$$

for each $x, y \in I$. This property is "almost equivalent" to convexity of f:

Exercise 19.1.1. If the function f is continuous on an interval I and if for any pair of points $x, y \in I$, $x < y$:

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2},$$

then f is convex on I.

It is convenient way to rewrite condition (a) as a double inequality between the slopes of three chords which join the points $(x_1, f(x_1))$, $(x, f(x))$ and $(x_2, f(x_2))$ on the graph of f:

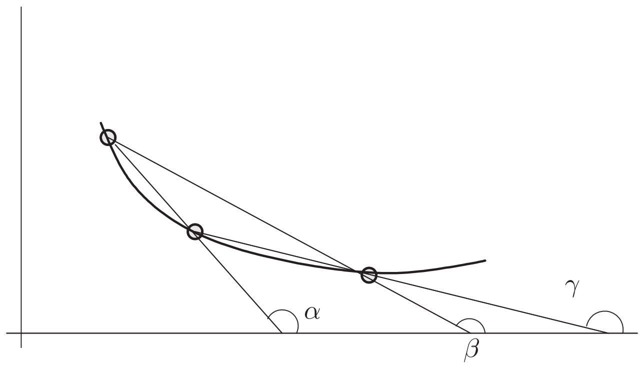

FIGURE 17. $\alpha < \beta < \gamma$

$$(b) \qquad \frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_2) - f(x)}{x_2 - x}.$$

Each of these two inequalities after a simple transformation reduces to (a).

Exercise 19.1.2. If f and g are two convex functions defined on the same interval I, then the functions $cf(x)$, where c is a positive constant, $f(x)+g(x)$ and $\max\{f(x), g(x)\}$ are convex as well.

From this exercise we see that the function $|x|$ is convex on $\mathbb{R}$, and more generally, if $L_1(x)$, ..., $L_n(x)$ are affine functions, then the function $\max_{1\le j\le n} L_j(x)$ is also convex.

The other examples will be given a bit later after we'll find a simple way to verify that a twice-differentiable function is convex.

Problem 19.1.3 (Geometric meaning of convexity)**.** The set $F \subset \mathbb{R}^2$ is called *convex* if, for any two points $A, B \in F$, the whole segment $[A, B]$ that connects these two points also belongs to F. For instance, the disk, the triangle and the rectangle are convex sets, while the annulus is not convex.

Suppose $f\colon I \to \mathbb{R}$, I is an open interval. Consider the set $\Gamma_+(f) = \{(x, y)\colon x \in I, y \ge f(x)\}$. This is a set of points $P(x, y)$ that lie *above* the graph of f.

Prove that the function f is convex iff the set $\Gamma_+(f)$ is convex.

19.2. Fundamental properties of convex functions.

Claim 19.2.1. *Any convex function on an open interval is continuous.*

Proof: Fix two points $t, x \in I$, $t > x$ which are not the end-points of I. Choose a subinterval $[a, b] \subset I$ such that $[x, t] \subset (a, b)$. Then applying condition (b) to the triple $x < t < b$, we get

$$\frac{f(t) - f(x)}{t - x} \le \frac{f(b) - f(x)}{b - x}$$

and

$$\frac{f(x) - f(a)}{x - a} \le \frac{f(t) - f(x)}{t - x}.$$

Thus

$$(t - x)\frac{f(x) - f(a)}{x - a} \le f(t) - f(x) \le (t - x)\frac{f(b) - f(x)}{b - x},$$

which yields continuity of f. □

Question 19.2.2. Suppose the function f is convex on a closed interval $[a, b]$. Whether it has to be continuous at the end-points a and b?

Exercise 19.2.3. If f is convex on the closed interval $[a, b]$, then f attains its maximal value at one of the end-points:

$$\max_{x \in [a,b]} f(x) = \max\{f(a), f(b)\}.$$

Claim 19.2.4. *Set*

$$m_f(x, y) = \frac{f(y) - f(x)}{y - x}.$$

If f is convex, then the functions $x \mapsto m_f(x, y)$ and $y \mapsto m_f(x, y)$ are increasing.

Proof: is a reformulation of (b). □

In the next claim, we'll use one-sided derivatives of the function f defined by

$$f'_+(x) = \lim_{t \downarrow x} \frac{f(t) - f(x)}{t - x}$$

(the right derivative) and

$$f'_-(x) = \lim_{t \uparrow x} \frac{f(t) - f(x)}{t - x}$$

(the left derivative). The (usual) derivative $f'(x)$ exists if and only if the right and left derivatives exist and equal to each other.

Claim 19.2.5. *If f is convex on I, then f has the right and left derivatives, and*

$$f'_-(x) \le f'_+(x) \le f'_-(y),$$

for any $x < y$, $x, y \in I$.

Proof: follows from the previous claim. □

Remark 19.2.6. The same argument shows that if f is convex on the closed interval $[a, b]$, then the one-sided derivatives $f'_+(a)$ and $f'_-(b)$ exist, and

$$f'_+(a) \le f'_-(x), \qquad \forall x \in (a, b],$$
$$f'_-(b) \ge f'_+(x), \qquad \forall x \in [a, b).$$

Exercise 19.2.7. Prove that the set of points x where the derivative of a convex function does not exist is at most countable.

Claim 19.2.8. *If f is differentiable on I, then f is convex if and only if f' does not decrease.*

Proof: In one direction, this follows from the inequalities between the one-sided derivatives. Now, assume that f' does not decrease. Then using the Lagrange mean value theorem we get for any triple $x_1 < x < x_2$ there are points $\xi_1 \in [x_1, x]$, and $\xi_2 \in [x, x_2]$ such that

$$\frac{f(x) - f(x_1)}{x - x_1} = f'(\xi_1) \qquad \text{and} \qquad f'(\xi_2) = \frac{f(x_2) - f(x)}{x_2 - x}.$$

Since $f(\xi_1) \le f(\xi_2)$, this yields inequality (a). □

Claim 19.2.9. *If f is twice differentiable on I, then it is convex if and only if $f'' \ge 0$.*

Proof: follows from the previous claim. □

Problem 19.2.10. Let $f \in C^2(\mathbb{R})$ and

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$

Prove that there exist at least two points c_1 and c_2 such that

$$f''(c_1) = f''(c_2) = 0\,.$$

19.3. A function f is called *concave* if the function $-f$ is convex. The affine function is the only one which is convex and concave at the same time.

- The function $f(x) = x^a$ is convex on $[0, +\infty)$ for $a \ge 1$, is convex on $(0, +\infty)$ for $a \le 0$, and is concave on $[0, +\infty)$ for $0 \le a \le 1$.
- The exponent $f(x) = a^x$ is a convex function on $\mathbb{R}$.
- The logarithmic function $f(x) = \log x$ is a concave function on $(0, +\infty)$.
- The function $f(x) = \sin x$ is concave on $[0, \pi]$ and convex on $[\pi, 2\pi]$.

Exercise 19.3.1. Suppose that $t \ge 1$. Show that

$$2t^p \le (t-1)^p + (t+1)^p$$

for $p \ge 1$, and

$$2t^p \ge (t-1)^p + (t+1)^p$$

for $0 \le p \le 1$.

Exercise 19.3.2. If g is the inverse function to a convex one, then g is concave.

19.4. Jensen's inequality.

Theorem 19.4.1. *Let f be a convex function in the interval I, and let x_1, x_2, ..., $x_n \in I$. Then*

$$(J) \qquad f\left(\sum_{j=1}^{n} \alpha_j x_j\right) \le \sum_{j=1}^{n} \alpha_j f(x_j)$$

provided that $\alpha_1, ..., \alpha_n \ge 0$ and $\sum_{j=1}^{n} \alpha_j = 1$.

Proof: We shall use induction with respect to n. The case $n = 2$ corresponds to inequality (a') proved above.

Now, assuming that (J) is proven for $n = m - 1$, we prove it for $n = m$. We assume that $\alpha_m > 0$ (if $\alpha_m = 0$, then we have already the result), and take $\beta = \alpha_2 + ... + \alpha_m > 0$. Notice that $\alpha_1 + \beta = 1$ and that

$$\frac{\alpha_2}{\beta} + ... + \frac{\alpha_m}{\beta} = 1.$$

Then applying (J) first with $n = 2$ and then with $n = m - 1$ we get

$$\begin{aligned} f(\alpha_1 x_1 + ... + \alpha_n x_n) &= f\left(\alpha_1 x_1 + \beta\left(\frac{\alpha_2}{\beta}x_2 + ... + \frac{\alpha_m}{\beta}x_m\right)\right) \\ &\le \alpha_1 f(x_1) + \beta f\left(\frac{\alpha_2}{\beta}x_2 + ... + \frac{\alpha_m}{\beta}x_m\right) \\ &\le \alpha_1 f(x_1) + ... + \alpha_m f(x_m), \end{aligned}$$

completing the proof. □

Problem 19.4.2. Prove that if $\alpha_j > 0$ for every j, then there is equality in (J) if and only if f is the affine function in the interval $[\min x_j, \max x_j]$.

Examples:

i. Take $f(x) = \log x$. This function is concave, so (J) works with the opposite inequality:

$$\alpha_1 \log x_1 + ... + \alpha_n \log x_n \le \log\left(\alpha_1 x_1 + ... + \alpha_n x_n\right).$$

Taking the exponent of the both sides, we get

$$x_1^{\alpha_1} \cdot \ ... \ \cdot x_n^{\alpha_n} \le \alpha_1 x_1 + \ ... \ + \alpha_n x_n,$$

provided that $\alpha_1, ..., \alpha_n \ge 0$ and $\sum_{j=1}^{n} \alpha_j = 1$.

Consider a special case with

$$\alpha_1 = \alpha_2 = \ ... \ = \alpha_n = \frac{1}{n}.$$

We get celebrated Cauchy's inequality between the geometric and arithmetic means:

$$\sqrt[n]{x_1 \cdot ... \cdot x_n} \le \frac{x_1 + ... + x_n}{n}.$$

ii. Now, we apply the Jensen inequality to the function $f(x) = x^p$, $p > 1$, again with $\alpha_1 = ... = \alpha_n = \frac{1}{n}$. Recall, that f is convex for such p's. We obtain that for any $x_1, ..., x_n > 0$

$$\frac{1}{n}\sum_{j=1}^{n} x_j \leq \left(\frac{1}{n}\sum_{j=1}^{n} x_j^p\right)^{1/p}, \qquad p > 1\,.$$

Note that this inequality also follows from Hólder's inequality.

Problem 19.4.3. For $x_1, ..., x_n > 0$ and $p \in \mathbb{R} \setminus \{0\}$, set

$$\mathcal{M}_p(x_1, ..., x_n) = \left\{\frac{1}{n}\sum_{j=1}^{n} x_j^p\right\}^{1/p}.$$

This quantity is called the p-th mean of the values x_1, x_2, ..., x_p.

i. Find the limits

$$\lim_{p\to 0} \mathcal{M}_p(x_1, ..., x_n), \qquad \lim_{p\to +\infty} \mathcal{M}_p(x_1, ..., x_n), \qquad \text{and} \qquad \lim_{p\to -\infty} \mathcal{M}_p(x_1, ..., x_n).$$

ii. Show that the function $p \mapsto \mathcal{M}_p(x_1, ..., x_n)$ is strictly increasing unless all x_j are equal, in that case $\mathcal{M}_p(x_1, ..., x_n)$ is their common value for all p.

20. The Taylor expansion

In this lecture we develop the polynomial approximation to smooth functions which works both locally and globally.

20.1. Local polynomial approximation. Peano's theorem.

The starting point of this lecture is the following

Problem. Let the function f has n derivatives[6] at x_0. Find the polynomial $P_n(x)$ of degree $\le n$ such that

$$f(x) = P_n(x) + o((x-x_0)^n), \qquad x \to x_0.$$

In the case $n = 1$, we know that the solution is given by the linear function

$$P_1(x) = f(x_0) + (x-x_0)f'(x_0).$$

Juxtaposing this with another formula

$$P(x) = \sum_{j=0}^{n} \frac{P^{(j)}(x_0)}{j!}(x-x_0)^j$$

which we proved in Section 16 for an arbitrary polynomial P of degree n, we can guess that the answer to our problem is given by the polynomial

$$P_n(x) = P_n(x; x_0, f) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

called *the Taylor polynomial of degree* n of the function f at x_0. The difference

$$R_n(x) = R_n(x; x_0, f) = f(x) - P_n(x)$$

called *the remainder.*

The Taylor polynomial of degree n *interpolates* at the point x_0 the value of f and of its first n derivatives:

$$P_n^{(j)}(x_0) = f^{(j)}(x_0), \qquad 0 \le j \le n.$$

Therefore, the remainder vanishes at x_0 with its first n derivatives:

$$R_n^{(j)}(x_0) = 0, \qquad 0 \le j \le n.$$

The following claim finishes the job:

Claim 20.1.1. *Suppose the function g has n derivatives at x_0, and*

$$g(x_0) = g'(x_0) = \ldots = g^{(n)}(x_0) = 0.$$

Then

$$g(x) = o((x-x_0)^n), \qquad x \to x_0.$$

[6] This means that f is differentiable $n-1$ times in a neighbourhood of x_0 and the n-th derivatives exists at x_0.

Proof: We shall use induction in n. For $n = 1$, we have

$$\lim_{x\to x_0} \frac{g(x)}{x-x_0} = \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = g'(x_0) = 0.$$

Now, having the claim for n, we'll prove it for $n+1$, using the Lagrange mean value theorem:

$$g(x) = g(x) - g(x_0) = g'(c)(x-x_0),$$

where c is an intermediate point between x_0 and x. By the inductive assumption,

$$g'(x) = o((x-x_0)^{n-1}), \qquad x \to x_0,$$

hence

$$g'(c) = o((c-x_0)^{n-1}) = o((x-x_0)^{n-1}), \qquad x \to x_0.$$

This proves the claim. $\square$.

Theorem 20.1.2 (Peano). *Let the function f have n derivatives at x_0. Then*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + o((x-x_0)^n), \qquad x \to x_0.$$

20.2. The Taylor remainder. Theorems of Lagrange and Cauchy. The Peano theorem shows that the Taylor polynomial $P_n(x)$ well approximates the function f *locally* in a small neighbourhood of x_0 (which generally speaking may shrink as $n \to \infty$). It appears, that in many cases $P_n(x)$ is close to f *globally*, that is in a fixed interval containing x_0 whose size does not depend on n. In order to prove this, we need to find a convenient expression good for the remainder $R_n(x)$.

First, we introduce some notations: let I be an interval (it can be open or close, finite or infinite). By $C^n(I)$ we denote the class of all n-times differentiable functions on I such that the n-th derivative is continuous on I. By $C^\infty(I)$ we denote the class of all infinitely differentiable functions on I.

Theorem 20.2.1. *Let $f \in C^n[x_0, x]$, and let $f^{(n+1)}$ exist on (x_0, x). Let the function φ be continuous on $[x_0, x]$, be differentiable on (x_0, x), and the derivative φ' do not vanish on (x_0, x). Then there exists an intermediate point c between x_0 and x such that*

$$(R) \qquad R_n(x) = \frac{\varphi(x) - \varphi(x_0)}{\varphi'(c)n!} f^{(n+1)}(c)(x-c)^n.$$

Proof: Fix x and consider the function

$$F(t) \stackrel{\text{def}}{=} f(x) - \left\{ f(t) + \frac{f'(t)}{1!}(x-t) + \dots + \frac{f^{(n)}(t)}{n!}(x-t)^n \right\}.$$

Then $F(x) = 0$, $F(x_0) = R_n(x; x_0)$, and

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

So that

$$\frac{R_n(x;x_0)}{\varphi(x)-\varphi(x_0)} = -\frac{F(x)-F(x_0)}{\varphi(x)-\varphi(x_0)}$$

$$\overset{\text{Cauchy'sMVT}}{=} -\frac{F'(c)}{\varphi'(c)} = \frac{f^{(n+1)}(c)}{n!\varphi'(c)}(x-c)^n$$

completing the proof. □

In what follows, we use two special cases of (R). Taking

$$(L) \qquad \varphi(t) = (x-t)^{n+1},$$

we arrive at the *Lagrange formula* for the remainder:

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

This immediately yields a good estimate of the remainder:

Corollary 20.2.2. *Suppose the function f is the same as in Theorem 2. Then*

$$|R_n(x)| \le \frac{|x-x_0|^{n+1}}{(n+1)!} \sup_{c\in I} |f^{(n+1)}(c)|.$$

Taking in (R) $\varphi(t) = x - t$, we arrive at another representation for the remainder $R_n(x)$ called the *Cauchy formula*:

$$(C) \qquad R_n(x) = \frac{(x-c)^n(x-x_0)}{n!} f^{(n+1)}(c),$$

which sometimes gives a better result than the Lagrange formula. The both forms will be extensively used in the next lecture.

Exercise 20.2.3. Find the approximation error:

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}, \qquad 0 \le x \le 1.$$

Problem* 20.2.4. Suppose that the function f is twice differentiable on $[0,1]$, $f(0) = f(1) = 0$, and $\sup |f''| \le 1$. Show that $|f''| \le \frac{1}{2}$ everywhere on $[0,1]$.

Problem* 20.2.5 (Hadamard's inequality)**.** Suppose that the function f is twice differentiable on $\mathbb{R}$, and set $M_k = \sup_{\mathbb{R}} |f^{(k)}|$, $k = 0, 1, 2$. Show that $M_1^2 \le 2M_0M_2$.

C^∞-functions whose derivatives do not grow too fast with n:

$$\sup_I |f^{(n)}| \le C^n n!, \qquad n \in \mathbb{Z}_+,$$

are called *real analytic*.

Problem 20.2.6. Let f be a real analytic function on the interval I.

(i) Show that the Taylor series of f at x_0 converges to f on the set $\{x \in I : |x-x_0| < C^{-1}\}$ (C is the same constant as in the real analyticity condition).

(ii) Show that if f vanishes with all its derivatives at some point x_0 of I:

$$f^{(n)}(x_0) = 0, \qquad j \in \mathbb{Z}_+,$$

then f is the zero function.

In Lecture 15 we defined the Lagrange interpolation polynomial of degree n with the interpolation nodes at the pairwise distinct points $\{x_j\}_{0\le j\le n}$:

$$L_n(x) = L_n(x; x_0, f) = \sum_{j=0}^{n} \frac{f(x_j)Q(x)}{Q'(x_j)(x - x_j)},$$

where

$$Q(x) = (x - x_0)(x - x_1)...(x - x_n).$$

Problem* 20.2.7. Show that if $f \in C^n[a, b]$ and $f^{(n+1)}$ exists on (a, b), then for any choice of nodes $\{x_j\} \subset [a, b]$ there exists a point $c \in (a, b)$ such that

$$f(x) - L_n(x) = \frac{Q(x)}{(n+1)!} f^{(n+1)}(c).$$

In particular,

$$\max_I |f - L_n| \le \frac{\max_I |Q|}{(n+1)!} \sup_I |f^{(n+1)}|.$$

Hint: Take $r = f - L_n$, and consider the function

$$t \mapsto r(x)Q(t) - r(t)Q(x).$$

This function has $n + 2$ zeroes on $[a, b]$, so that its $n + 1$-st derivative vanishes at an intermediate point c.

21. Taylor expansions of elementary functions

Let f be a C^∞-function on I. In many cases, using one of the formulas for the remainder, we can conclude that

$$\lim_{n\to\infty} R_n(x;x_0) = 0$$

for any point x from the interval $I \ni x_0$. This means that

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j, \qquad x \in I. \tag{T}$$

The series on the right hand side is called the *Taylor series* of f at x_0. The formula (T) says the Taylor series converges to f everywhere on I.

We should warn that even if the Taylor series converges, it does not have to represent the function f. For example, the Taylor series at the origin of the C^∞-function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

has only zero coefficients (since $f^{(j)}(0) = 0$, $j \geq 0$), and does not represent the function f anywhere outside the origin.

In the rest of this lecture we consider examples of the Taylor series for elementary functions. In all examples below, we choose $x_0 = 0$ and set $R_n(x) = R_n(x;0,f)$.

21.1. The exponential function. We start with the exponential function $f(x) = e^x$. Then by Lagrange's estimate for the remainder, for any $M < +\infty$,

$$\max_{[-M,M]} |R_n(x)| \leq \frac{M^{n+1}e^M}{(n+1)!}.$$

The right hand side converges to zero as $n \to \infty$, hence

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}, \qquad x \in \mathbb{R}.$$

In particular, we obtain that

$$e = \sum_{j=0}^{\infty} \frac{1}{j!},$$

with a good estimate for the remainder:

$$0 < e - \sum_{j=0}^{n} \frac{1}{j!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}.$$

Exercise 21.1.1. Which n one should take to compute e with error at most 10^{-10}?

Claim 21.1.2. *The number e is irrational.*

Proof: Let $e = \frac{m}{n}$ and $s_n = \sum_{k=1}^{n}(k!)^{-1}$. Then

$$n!(e - s_n) = (n-1)!m - \sum_{k=1}^{n} \frac{n!}{k!}$$

is a natural number and hence is ≥ 1. On the other hand,

$$\begin{aligned} n!(e - s_n) &= \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + ... \\ &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + ... \\ &< \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = 1\,. \end{aligned}$$

Contradiction! □

Exercise 21.1.3. Prove that $n! > \left(\frac{n}{e}\right)^n$.

21.2. The sine and cosine functions. In this case, the Lagrange estimate for the remainder gives us

$$\max_{[-M,M]} |R_n(x)| \leq \frac{M^{n+1}}{(n+1)!}$$

which yields the formulas:

$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}, \qquad x \in \mathbb{R}$$

and

$$\cos x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!}, \qquad x \in \mathbb{R}\,.$$

Similar formulas hold for the hyperbolic sine and cosine:

$$\sinh x \stackrel{\text{def}}{=} \frac{e^x - e^{-x}}{2} = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!}, \qquad x \in \mathbb{R},$$

and

$$\cosh x \stackrel{\text{def}}{=} \frac{e^x + e^{-x}}{2} = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!}, \qquad x \in \mathbb{R}.$$

Exercise 21.2.1. Prove these two formulas and bound the reminder using the Lagrange estimate.

Exercise 21.2.2. Check that $\cosh^2 x - \sinh^2 = 1$, and that the both functions satisfy the differential equation $f'' = f$.

21.3. The logarithmic function. Consider the function $f(x) = \log(1+x)$ defined for $x > -1$. We have

$$f^{(j)}(x) = (-1)^{j-1}\frac{(j-1)!}{(1+x)^j},$$

so that $f^{(j)}(0) = (-1)^{j-1}(j-1)!$. Lagrange's estimate for the remainder yields the convergence of the Taylor expansion for $0 \le x \le 1$:

$$\max_{0\le x\le 1} |R_n(x)| \le \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Therefore, for $0 \le x \le 1$,

$$\log(1+x) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{x^j}{j}. \tag{21.3.1}$$

In particular, we find the formula which was promised in Lecture 8:

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots .$$

For $x > 1$ the Taylor series diverges (its terms tend to infinity with n). For the negative x's, we have to use Cauchy's formula for the remainder. If $|x| < 1$, then for some intermediate c between 0 and x:

$$|R_n(x)| = \left|\frac{(x-c)^n x}{(1+c)^n}\right| = |x|\left|\frac{x-c}{1+c}\right|^n .$$

Claim 21.3.2.

$$\left|\frac{x-c}{1+c}\right| < |x|.$$

Proof of Claim: since c is an intermediate point between 0 and x, $|x-c| = |x| - |c|$. Then

$$\left|\frac{x-c}{1+c}\right| = \frac{|x|-|c|}{|1+c|} \le \frac{|x|-|c|}{1-|c|} < \frac{|x|-|c||x|}{1-|c|} = |x|.$$

proving the claim. □

Making use of the claim, we continue the estimate for the remainder $R_n(x)$ and get

$$|R_n(x)| < |x|^{n+1}.$$

Since $|x| < 1$, we see that the remainder goes to zero with n. Therefore, the Taylor expansion converges to $\log(1+x)$ for $-1 < x \le 1$. □

It is curious, that the remainder in Cauchy's form gives us the result for $|x| < 1$ but to get the expansion at the end-point $x = 1$ we have to use Lagrange's estimate of the remainder. There is another way to find the Taylor expansion for $\log(1+x)$. The derivative of this function equals

$$\frac{1}{1+x} = \sum_{j=0}^{\infty}(-1)^j x^j .$$

Recalling that $\log(1+x) = 0$ at $x = 0$ and that $\left(x^{j+1}\right)' = (j+1)x^j$, we immediately arrive at the expansion (21.3.1). This idea will be justified in the second semester.

Exercise 21.3.3. Find the Taylor expansion of the function $\log \frac{1+x}{1-x}$ and investigate its convergence.

21.4. The binomial series. In this section, we consider the function $f(x) = (1+x)^a$ defined for $x > -1$. Now,

$$f^{(j)}(x) = a(a-1)...(a-j+1)(1+x)^{a-j},$$

and we get (at least, formally) the Newton formula

$$(1+x)^a = \sum_{j=0}^{\infty} \frac{a(a-1)...(a-j+1)}{j!} x^j \, .$$

Of course, if $a \in \mathbb{N}$, then there are only finitely many non-zero terms in the series on the right hand side, and we arrive at the familiar binomial formula.

We shall prove convergence of this formula for $|x| < 1$. The formula is also valid at $x = 1$ and (for $a \geq 0$) at $x = -1$. This will follow from the Abel convergence theorem that you'll learn in the second semester course.

So we fix $s < 1$, assume that $|x| < s$, and estimate the remainder using the Cauchy formula:

$$\begin{aligned} |R_n(x)| &= \left| \frac{a(a-1)...(a-n)}{n!} (1+c)^{a-n-1} (x-c)^n x \right| \\ &= \left| a\left(1-\frac{a}{1}\right) ... \left(1-\frac{a}{n}\right) \right| (1+c)^{a-1} \left| \frac{x-c}{1+c} \right|^n |x| \\ &\leq (1+c)^{a-1} \cdot \left| a\left(1-\frac{a}{1}\right) ... \left(1-\frac{a}{n}\right) \right| |x|^{n+1} = (1+c)^{a-1} \cdot q_n \end{aligned}$$

(in the passage from the second to the third line we used the claim from the previous section). If n is big enough, we have

$$\frac{q_{n+1}}{q_n} = \left| \left(1 - \frac{a}{n+1}\right) x \right| \leq s < 1,$$

so that q_n and hence $R_n(x)$ tend to zero for $|x| < 1$.

21.5. The Taylor series for $\arctan x$. Let $f(x) = \arctan x$, $|x| \leq 1$. To arrive at the Taylor expansion, recall that

$$f'(x) = \frac{1}{1+x^2} = \sum_{j=0}^{\infty} (-1)^j x^{2j} \, .$$

Hence, the guess:

$$\arctan x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{2j+1} \, .$$

To justify our guess, we need to bound the remainder. For this, we need a formula for the j-th derivative $f^{(j)}(x)$.

Claim 21.5.1. *For each $j \geq 1$,*

$$f^{(j)} = (j-1)!\cos^j f \sin j\left(f + \frac{\pi}{2}\right). \qquad (C)$$

Proof of the claim: We'll use the induction with respect to j. For $j = 1$ we have

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\tan^2 f} = \cos^2 f = \cos f \sin\left(f + \frac{\pi}{2}\right).$$

Suppose the claim is verified for $j = n$, then

$$\begin{aligned} f^{(n+1)} &= (n-1)!\cos^{n-1} f \cdot nf' \left\{ -\sin f \sin n\left(f + \frac{\pi}{2}\right) + n\cos f \cos n\left(f + \frac{\pi}{2}\right)\right\} \\ &= n!\cos^{n+1} f \cos\left((n+1)f + n\frac{\pi}{2}\right) \\ &= n!\cos^{n+1} f \sin\left((n+1)\left(f + \frac{\pi}{2}\right)\right), \end{aligned}$$

proving the claim. □

Corollary 21.5.2. *For each $n \geq 1$,*

$$\sup_{[-1,1]} |f^{(n)}| \leq n!.$$

Then, by the Lagrange estimate for the remainder,

$$\sup_{x\in[-1,1]} |R_n(x)| \leq \frac{1}{(n+1)!} \sup_{[-1,1]} |f^{(n+1)}| \leq \frac{1}{n}.$$

That is, the Taylor expansion converges to $\arctan x$ everywhere on $[-1, 1]$.

Plugging the value $x = 0$ into (C), we get

$$f^{(j)}(0) = (j-1)!\sin\frac{j\pi}{2} = \begin{cases} (-1)^m(2m)!, & j = 2m+1 \\ 0, & j = 2m \end{cases}$$

(we got this expression in Lecture 16 by a different calculation). So that we obtain the Taylor expansion for $\arctan x$

$$\arctan x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{2j+1}$$

valid on $[-1, 1]$.

Taking $x = 1$, we arrive at a remarkable formula of Leibnitz:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots.$$

Problem 21.5.3. Prove that

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!(2n+1)} x^{2n+1}, \qquad -1 \le x \le 1.$$

Plugging $x = \frac{1}{2}$ into the expansion of $\arcsin x$, we get

$$\frac{\pi}{6} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!(2n+1)2^{2n+1}}.$$

This expansion of $\frac{\pi}{6}$ is essentially better than the previous one of $\frac{\pi}{4}$. Why?

21.6. Some computations. There are many elementary functions for which it is not easy to find a good expression for coefficients in the Taylor expansion. In most of applications, one usually needs only a few first terms in the Taylor expansion which can be found directly (sometimes, this requires a patience). Consider several examples:

21.6.1. $f(x) = \tan x$. This is an odd function, so in its Taylor expansion all even coefficients vanish. We'll find first three non-vanishing odd coefficients. We have

$$f'(x) = \cos^{-2} x, \qquad f'(0) = 1,$$

then

$$f''(x) = 2\sin x \cos^{-3} x,$$

$$f'''(x) = 2\cos^{-2} x + 6\sin^2 x \cos^{-4} x = -4\cos^{-2} x + 6\cos^{-4} x, \qquad f'''(0) = 2,$$

$$f^{(iv)}(x) = -8\sin x \cos^{-3} x + 24 \sin x \cos^{-5} x,$$

and at last

$$\begin{aligned} f^{(v)}(x) &= -8\cos^{-2} x + 24\sin^2 x \cos^{-4} x + 24\cos^{-4} x + 120\sin^2 \cos^{-6} x \\ &= 16\cos^{-2} x - 120\cos^{-4} x + 120\cos^{-6} x, \end{aligned}$$

so that $f^{(v)}(0) = 16$. We find that

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6), \qquad x \to 0.$$

Exercise 21.6.1. Find the approximation error

$$\tan x \approx x + \frac{x^3}{3}, \qquad |x| \le \frac{1}{10}.$$

21.6.2. $f(x) = \log\cos x$. Sometimes, when f is a superposition of functions with known Taylor expansions, instead of the direct differentiation it is easier to use formal algebraic manipulations.

The function f is an even function, we find the first three non-vanishing terms of its Taylor expansion. We know that

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + o(x^7), \qquad x \to 0.$$

Therefore,

$$\begin{aligned}
\log\cos x &= \log\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+o(x^7)\right)\\
&= \log(1+u) \qquad\qquad u=-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+o(x^7)\\
&= u-\frac{u^2}{2}+\frac{u^3}{3}+o(x^7)\\
&= \left(-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\right)-\frac{1}{2}\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)^2+\frac{1}{3}\left(-\frac{x^2}{2}\right)^3+o(x^7)\\
&= \left(-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\right)-\frac{1}{2}\left(\frac{x^2}{4}-\frac{x^6}{24}\right)+\frac{1}{3}\left(-\frac{x^6}{8}\right)+o(x^7)\\
&= -\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}+o(x^7).
\end{aligned}$$

Exercise 21.6.2. Find the Taylor polynomials of degree n at the point x_0 to the following functions

$$\begin{array}{ll}
\frac{1+x+x^2}{1-x+x^2} \quad (n=4, x_0=0) & \sqrt[m]{a^m+x} \quad (a>0) \quad (n=4, x_0=0)\\
\sqrt{2x-x^2} \quad (n=3, x_0=1) & e^{2x-x^2} \quad (n=4, x_0=0)\\
\sin(\sin x) \quad (n=3, x_0=0) & x^x-1 \quad (n=3, x_0=1)\,.
\end{array}$$

21.7. Application to the limits. In many cases, knowledge of the Taylor expansion simplifies computation of limits. For example, making use of the expansions of $\tan x$ and $\log\cos x$ we easily find

$$\lim_{x\to 0}\frac{\sin x - x}{\tan x - x}=\lim_{x\to 0}\frac{-x^3/6+o(x^3)}{-x^3/6+o(x^3)}=1,$$

and

$$\lim_{x\to 0}\frac{\log\cos x}{x^2}=-\frac{1}{2}.$$

Exercise 21.7.1. Find the limits

$$\lim_{x\to 0}\frac{\sin x-\arcsin x}{\tan x-\arctan x} \qquad \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} \qquad \lim_{x\to 0}\frac{\cos x-e^{-\frac{1}{2}x^2}}{x^4}$$

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) \qquad \lim_{x\to 1}\frac{1-x+\log x}{1-\sqrt{2x-x^2}} \qquad \lim_{x\to+\infty}\left(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5}\right).$$

22. The complex numbers

In this lecture we introduce the complex numbers and recall they basic properties.

22.1. Basic definitions and arithmetics. As you probably remember from the high-school, the complex numbers are the expressions $z = x + iy$ with $i^2 = -1$. We can add and multiply the complex numbers as follows

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2)\,,\\ (x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\,.\end{aligned}$$

If $z = x+iy$, then the value $\overline{z} = x-iy$ is called *the conjugate* to z, x is *the real part* of z, $x = \operatorname{Re} z = \frac{z+\overline{z}}{2}$, and y is *the imaginary part* of z, $y = \operatorname{Im} z = \frac{z-\overline{z}}{2i}$. Note that $z\overline{z} = x^2 + y^2$ is always non-negative, and vanishes iff $z = 0$. The non-negative number $\sqrt{z\overline{z}}$ is called *the absolute value* of z, denoted $r = |z| = \sqrt{x^2 + y^2}$. If $z \neq 0$, then there is the inverse to z:

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\,.$$

Then, for $z_2 \neq 0$, we can define

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2}\,.$$

I.e., the complex number form a field denoted by $\mathbb{C}$. Any real number x can be regarded as a complex number $x + i0$ with zero imaginary part. I.e., $\mathbb{R} \subset \mathbb{C}$.

Exercise 22.1.1. Check:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\,, \qquad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}\,.$$

Claim 22.1.2 (Triangle inequality)**.**

$$|z + w| \leq |z| + |w|\,.$$

Proof: We have

$$\begin{aligned}|z + w|^2 = (z + w)\overline{(z + w)} &= (z + w)(\overline{z} + \overline{w})\\ &= z\overline{z} + w\overline{w} + z\overline{w} + w\overline{z} = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})\,.\end{aligned}$$

Note that $-|a| \leq \operatorname{Re} a \leq |a|$, whence

$$|z + w|^2 \leq |z|^2 + |w|^2 + 2|z|\,|w| = (|z| + |w|)^2\,.$$

Done! ◻

Exercise 22.1.3.

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)\,.$$

Exercise 22.1.4 (Cauchy-Schwarz inequality)**.**

$$\left|\sum z_j w_j\right|^2 \leq \left(\sum |z_j|^2\right)\left(\sum |w_j|^2\right)\,.$$

22.2. Geometric representation of complex numbers. The argument. We can represented complex numbers by two-dimensional vectors:

$$z = x + iy \mapsto \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then, the addition law for the complex numbers corresponds to the addition

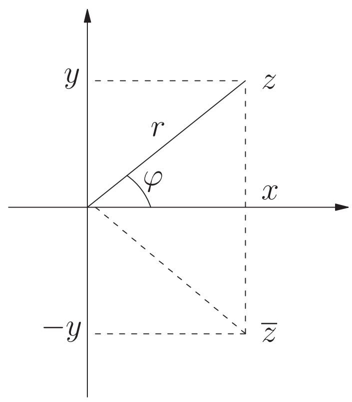

FIGURE 18. `Complex plane`

law for the vectors, and the absolute value of the complex number is the same as the length of the corresponding vector. However, the vector representation is not very convenient when we need to multiply the complex number. In this case, it is more convenient to use the polar coordinates.

Definition 22.2.1 (argument)**.** For $z \neq 0$, *the argument* of z is the angle $\varphi = \arg z$ the point z is seen from the origin. The angle is measured counter-clockwise, started with the positive ray.

We have

$$\tan \varphi = \frac{y}{x},$$

$$x = r \cos \varphi, \qquad y = r \sin \varphi$$

(as above, $r = |z|$), and

$$z = r(\cos \varphi + \sin \varphi).$$

This representation is consistent with multiplication: if $z_j = r_j(\cos \varphi_j + \sin \varphi_j)$, $j = 1, 2$, are non-zero complex numbers, then

$$z_1 \cdot z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)).$$

I.e., multiplying the complex numbers, we multiply their absolute values and add their arguments.

Corollary 22.2.2 (Moivre)**.** *If* $z = r(\cos \varphi + i \sin \varphi)$, *then*

$$z^n = r^n (\cos n\varphi + i \sin n\varphi), \qquad n \in \mathbb{N}.$$

Warning: the angles are measured up to $2\pi k$, $k \in \mathbb{Z}$. Hence, the argument is not the number but rather a set of real numbers, such that the difference between any two numbers from this set equals $2\pi k$ with some integer k. The most popular choice for the representative from this set is $\varphi \in [0, 2\pi)$.

Example 22.2.3. Let us solve the equation $z^n = a$. Here, $n \in \mathbb{N}$. We suppose that $a \neq 0$, otherwise, the equation has only the zero solution. Denote $a = \rho(\cos\theta + i\sin\theta)$. Then

$$r^n(\cos n\varphi + i\sin n\varphi) = \rho(\cos\theta + i\sin\theta)\,,$$

i.e., $r^n = \rho$ and $n\varphi = \theta + 2k\pi$ with some $k \in \mathbb{Z}$. Hence, $r = \sqrt[n]{\rho}$. The obvious solution for the second equation is $\varphi = \theta/n$. However, after a minute reflection we realize that it has n distinct solutions:

$$\varphi_k = \frac{\theta}{n} + \frac{2k\pi}{n}\,, \qquad k = 0, 1, ..., n-1\,.$$

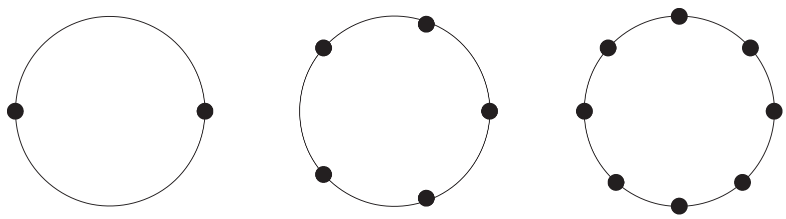

FIGURE 19. The roots of unity, $n = 2$, $n = 5$, and $n = 8$

Consider the special case $a = 1$. In this case, $\rho = 1$ and $\theta = 0$. We get n points

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\,, \qquad k = 0, 1, ..., n-1$$

called *the roots of unity.*

Exercise 22.2.4. Solve the equations $z^4 = i$, $z^2 = i$, $z^2 = 1 + i$. Find the absolute value and the argument of the solutions, as well as their real and imaginary parts. Mark the solutions on the complex plane.

Exercise 22.2.5. Let

$$\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\,.$$

Compute the sums

$$1 + \omega + \omega^2 + \;...\; + \omega^{n-1} =?\,,$$

$$1 + 2\omega + 3\omega^2 + \;...\; + n\omega^{n-1} =?\,,$$

and

$$1 + \omega^h + \omega^{2h} + \;...\; + \omega^{(n-1)h} =?$$

(h is a positive integer).

22.3. Convergence in $\mathbb{C}$. The *distance* between the complex numbers z_1 and z_2 is $|z_1 - z_2|$.

Definition 22.3.1. The sequence z_n converges to z (denoted by $z_n \to z$ or $z = \lim\limits_{n\to\infty} z_n$), if $\lim\limits_{n\to\infty} |z - z_n| = 0$.

Since

$$\max\{|x - x_n|, |y - y_n|\} \le \underbrace{\sqrt{(x - x_n)^2 + (y - y_n)^2}}_{=|z-z_n|} \le |x - x_n| + |y - y_n|\,,$$

the sequence z_n converges to z iff the corresponding real and imaginary parts converge:

$$x_n \to x\,, \qquad y_n \to y\,.$$

Exercise 22.3.2. Check that the Cauchy criterion of convergence works for the complex sequences.

Definition 22.3.3 (continuity)**.** The complex valued function f is continuous at z, if for each sequence $z_n \to z$, $f(z_n) \to f(z)$.

Exercise 22.3.4. Check that the sum and the product of continuous functions is continuous. Check that the quotient of continuous functions is continuous in the points where the denominator does not vanish.
Hint: the proofs are the same as in the real case.

We see that the polynomials are continuous functions in the whole complex plane. That's all we need to prove in the next lecture the fundamental theorem of algebra.

Exercise 22.3.5. If $f = u + iv$, then f is continuous iff its real and imaginary parts u and v are continuous. If f is continuous, then $|f|$ is also continuous.

23. The fundamental theorem of algebra and its corollaries

23.1. The theorem and its proof.

Theorem 23.1.1. *Any polynomial $P(z) = c_0 + c_1 z + \dots + c_n z^n$ of positive degree has at least one zero in $\mathbb{C}$.*

Proof: WLOG, we assume that $c_n = 1$. Denote $m = \inf\limits_{z\in\mathbb{C}} |P(z)|$.

Claim 23.1.2. *There is a sufficiently big R such that $|P(z)| > m + 1$ for $|z| > R$.*

Indeed, we have
$$P(z) = z^n \left(1 + \frac{c_{n-1}}{z} + \dots + \frac{c_0}{z^n}\right),$$
whence
$$\begin{aligned} |P(z)| &\ge |z|^n \left(1 - \left|\frac{c_{n-1}}{z} + \dots + \frac{c_0}{z^n}\right|\right) \\ &\ge |z|^n \Bigg(1 - \underbrace{\left(\frac{|c_{n-1}|}{|z|} + \dots + \frac{|c_0|}{|z|^n}\right)}_{\le 1/2}\Bigg) \ge \frac{1}{2}|z|^n \overset{|z|\ge R}{\ge} \frac{1}{2}R^n \ge m+1 \end{aligned}$$
provided that R is sufficiently big. $\square$

Therefore, $m = \inf\limits_{|z|\le R} |P(z)|$. Next, using the Bolzano-Weierstrass lemma, we will check that the infimum is actually attained:

Claim 23.1.3. *There exists z_0 with $|z_0| \le R$ such that $|P(z_0)| = m$.*

Indeed, choose a sequence of points z_k, $|z_k| \le R$, such that
$$|P(z_k)| \le m + \frac{1}{k}.$$
The sequences $x_k = \operatorname{Re} z_k$ and $y_k = \operatorname{Im} z_k$ are bounded $\max\{|x_k|, |y_k|\} \le R$. Hence, they have convergent subsequences. Hence, the sequence $\{z_k\}$ has a convergent subsequence $z_{k_j} \to z_0$. Then by continuity of the polynomial P, we have
$$P(z_0) = \lim_{j\to\infty} P(z_{k_j}),$$
whence $|P(z_0)| = m$. $\square$

Suppose that P does not have zeroes in $\mathbb{C}$, i.e., $m > 0$, and consider the polynomial
$$Q(z) \overset{\text{def}}{=} \frac{P(z+z_0)}{P(z_0)}.$$
Then $1 = Q(0) \le |Q(z)|$, $z \in \mathbb{C}$.

To complete the proof, we show that there are points z where $|Q(z)| < Q(0)$. This will lead to the contradiction. We have
$$Q(z) = 1 + q_k z^k + q_{k+1} z^{k+1} + \dots + q_n z^n \qquad \text{with} \quad |q_k| \ne 0.$$

Set $\psi = \arg q_k$ and consider the points z with $\arg z = \dfrac{\pi - \psi}{k}$. Then

$$\arg(q_k z^k) = \psi + (\pi - \psi) = \pi,$$

so that $q_k z^k = -r^k |q_k|$. Let's estimate $|Q(z)|$ assuming on each step that r is chosen sufficiently small:

$$\begin{aligned}|Q(z)| &\le \left|1 + q_k z^k\right| + |q_{k+1}| r^{k+1} + \ldots + |q_n| r^n \\ &= 1 - r^k |q_k| + r^{k+1} |q_{k+1}| + \ldots + r^n |q_n| \\ &= 1 - r^k \left(|q_k| - r|q_{k+1}| - \ldots - r^{n-k}|q_n|\right) < 1\,,\end{aligned}$$

and we are done! □

23.2. Factoring the polynomials. In Lecture 15, we discussed the Horner scheme of the polynomial division. This scheme also works for the polynomials with complex coefficients. It yields, that if P is a polynomial of degree $n \ge 1$, then

$$P(z) = (z - a)P_1(z) + P(a)$$

where P_1 is a polynomial of degree $n - 1$. In particular, if P vanishes at a, then

$$P(z) = (z - a)P_1(z)\,.$$

Using induction with respect to the degree of P, we arrive at

Corollary 23.2.1 (factorization of polynomials)**.** *Every polynomial of degree $n \ge 1$ can be factored:*

$$P(z) = c(z - z_1)\ldots(z - z_n)\,.$$

Note that some of the zeroes z_1,...,z_n of P may coincide. We say that a is a zero of P of multiplicity k if

$$P(z) = (z - a)^k P_1(z)$$

where the polynomial P_1 does not vanish at a. Usually, we count zeroes of the polynomials with their multiplicities[7]. Then we can write down the factorization in the following form

$$P(z) = c(z - z_1)^{k_1} \ldots (z - z_m)^{k_m}$$

where the zeroes z_1, ..., z_m are pairwise different, and $\sum k_j = n$.

Exercise 23.2.2. If a polynomial of degree P has more than n zeroes in $\mathbb{C}$ (counting with the multiplicities), then it vanishes identically.

[7]For instance, the polynomial $P(z) = z(z-1)^2(z-2)^{10}$ has 1 zero at the origin, 2 zeroes at $z = 1$, and 10 zeroes at $z = 2$.

23.3. Rational functions. Partial fraction decomposition. *Rational functions* are functions represented as the quotients of the polynomials:

$$R(z) = \frac{P(z)}{Q(z)}\,.$$

Usually, writing this representation we assume that the polynomials P and Q have no common zeroes. Then $\deg R \overset{\text{def}}{=} \max\{\deg P, \deg Q\}$. The rational functions form a *field* with usual addition and multiplication.

The rational function R is defined everywhere except of the zeroes of Q. The zeroes of the polynomial Q are called *the poles* of R. Note that if a is a pole of R, then

$$\lim_{z\to a} |R(z)| = +\infty\,.$$

If a is a zero of Q of multiplicity k, then we say that the pole of R at a also has multiplicity k. The polynomials are the rational functions without poles.

Claim 23.3.1. *If a is a pole of R of multiplicity, then there are the unique coefficients A_1, ..., A_k such that*

$$R(z) - \left(\frac{A_1}{z-a} + ... + \frac{A_k}{(z-a)^k}\right)$$

has no pole at a.

The sum on the RHS is called *the singular part* of R at a. We denote it by $S_a(z)$.

Proof:

i (existence): Consider the rational function $U(z) = (z-a)^k R(z)$, it has no pole at a. We set $A_k = U(a)$. Then

$$(z-a)^k R(z) - A_k = U(z) - A_k = (z-a)V(z)$$

where V is a rational function without pole at a, or

$$R(z) - \frac{A_k}{(z-a)^k} = \frac{V(z)}{(z-a)^{k-1}}$$

and the RHS has a pole at a of multiplicity $k-1$ or less. Then we apply the same procedure to the function V.

ii (uniqueness): Suppose that the expression

$$R(z) - \left(\frac{B_1}{z-a} + ... + \frac{B_k}{(z-a)^k}\right)$$

also has no pole at a. Then the difference of the two expressions

$$F(z) = \frac{B_1 - A_1}{z-a} + ... + \frac{B_k - A_k}{(z-a)^k}$$

also has no pole at a. Suppose that some $A_l \neq B_l$ and set $j = \max\{l\colon A_l \neq B_l\}$. Then

$$F(z) = \frac{1}{(z-a)^j} \underbrace{\left\{(B_j - A_j) + (B_{j-1} - A_{j-1}(z-a) + \ \dots\ + (B_1 - A_1)(z-a)^{j-1}\right\}}_{=T(z)}$$
$$= \frac{T(z)}{(z-a)^j}$$

where T is a polynomials, and $T(a) = B_j - A_j \neq 0$ by our assumption. Hence, F has a pole at a, arriving at the contradiction. Hence, the claim. □

Applying the claim, one by one, to all poles of R, we get

Theorem 23.3.2 (partial fraction decomposition)**.** *Every rational function R can be uniquely represented in the following form:*

$$R(z) = \sum_a S_a(z) + W(z)$$

where the sum is taken over the set of all poles a of R, S_{a_j} are the corresponding singular parts, and W is a polynomial.

Exercise 23.3.3. If $R = \frac{P}{Q}$ where the polynomials P and Q has no common zeroes, then $\deg W = \deg P - \deg Q$, if the latter is non-negative; otherwise $W = 0$.

Example 23.3.4. Let

$$R(z) = \frac{z^4+1}{z(z+1)(z+2)}\,.$$

This function has simple poles at the points $z = 0,\ -1,\ -2$. Hence,

$$R(z) = \frac{A_0}{z} + \frac{A_{-1}}{z+1} + \frac{A_{-2}}{z+2} + W(z)$$

where W is a (linear) polynomial. We have

$$A_0 = \lim_{z\to 0} R(z)z = \lim_{z\to 0} \frac{z^4+1}{(z+1)(z+2)} = \frac{1}{2}\,,$$

$$A_{-1} = \lim_{z\to -1} R(z)(z+1) = \lim_{z\to -1} \frac{z^4+1}{z(z+2)} = -2\,,$$

$$A_{-2} = \lim_{z\to -2} R(z)z = \lim_{z\to -2} \frac{z^4+1}{z(z+1)} = \frac{17}{2}\,,$$

and

$$W(z) = \frac{z^4+1}{z(z+1)(z+2)} - \left(\frac{1}{2z} - \frac{2}{z+1} + \frac{17}{2(z+2)}\right) = \ \dots\ = z - 3\,,$$

and finally

$$\frac{z^4+1}{z(z+1)(z+2)} = \frac{1}{2z} - \frac{2}{z+1} + \frac{17}{2(z+2)} + z - 3\,.$$

There a more simple way to compute the linear polynomial $W(z) = az + b$:

$$a = \lim_{z\to\infty} \frac{R(z)}{z} = 1\,,$$

and

$$b = \lim_{z\to\infty} (R(z) - z) = \lim_{z\to\infty} \frac{z^4 + 1 - z^2(z+1)(z+2)}{z(z+1)(z+2)} = -3\,.$$

23.3.1. *Simple poles and Lagrange interpolation.* If the poles of R are *simple* (i.e., have multiplicity 1), then we get a representation of R as a sum of simple fractions and a polynomial:

$$R(z) = \sum_j \frac{A_j}{z - a_j} + W(z)\,. \tag{23.3.5}$$

In this case[8],

$$A_j = \lim_{z\to a_j} R(z)(z - a_j) = \lim_{z\to a_j} \frac{P(z)(z - a_j)}{Q(z)} = \frac{P(a_j)}{Q'(a_j)}\,,$$

and we get

$$\frac{P(z)}{Q(z)} = \sum_j \frac{P(a_j)}{(z - a_j)Q'(a_j)} + W(z)$$

where the sum is taken over the zeroes of the polynomial Q. If $\deg P < \deg Q$, then W is zero, and we arrive at the Lagrange interpolation formula with nodes at the zeroes of Q proven in Lecture 15.

$$P(z) = \sum_j \frac{P(a_j)Q(z)}{(z - a_j)Q'(a_j)}\,.$$

That is, Lagrange interpolation formula is a special case of the partial fraction decomposition of rational functions!

23.4. Appendix: real polynomials and real rational functions. The polynomial P is *real* if $P(z) = c_n z^n + \dots + c_1 z + c_0$ with the real coefficients $c_0, \dots, c_n$. Then

$$\overline{P(z)} = P(\overline{z})\,. \tag{23.4.1}$$

[8]Here we use the derivative of the polynomial Q at $a \in \mathbb{C}$. It is defined as usual:

$$Q'(a) = \lim_{z\to a} \frac{Q(z) - Q(a)}{z - a}\,.$$

It is easy to see that this limit always exists. If

$$Q(z) = \sum_{0\le j\le n} q_j z^j\,,$$

then

$$Q'(a) = \sum_{0\le j\le n-1} (j+1) q_{j+1} z^j\,.$$

It's easy to see that (23.4.1) is also a necessary condition for the polynomial P to have the real coefficients. Indeed, (23.4.1) yields that P maps $\mathbb{R}$ to $\mathbb{R}$, hence, its coefficients must be real (recall that $c_k = \dfrac{P^k(0)}{k!}$).

By condition (23.4.1), if a real polynomial vanishes at some point a with $\operatorname{Re} a \neq 0$, then it also vanishes at the conjugate point $\overline{a}$:

$$P(a) = 0 \Longrightarrow P(\overline{a}) = 0\,.$$

In this case, the product

$$(z-a)(z-\overline{a}) = z^2 - (a+\overline{a})z + |a|^2$$

appears in the factorization of P. We arrive at

Corollary 23.4.2 (factorization of real polynomials)**.** *Every real polynomial of degree $n \geq 1$ can be factorized as*

$$P(x) = c(x-x_1)\dots(x-x_s)(x^2+p_1x+q_1)\dots(x^2+p_lx+q_l)$$

with $s + 2l = n$.

Example 23.4.3. Consider the real polynomial $x^{2n}+1$. It has zeroes at

$$z_k = \sqrt[2n]{-1} = \cos\frac{(2k-1)\pi}{2n} + i\sin\frac{(2k-1)\pi}{2n}\,,$$

for $k = 1, 2, \dots, 2n$. All these zeroes are not real. The zeroes z_1, ..., z_n have the argument less than π and are located in the upper half-plane, while the zeroes z_{n+1}, ..., z_{2n} are located in the lower half-plane, $\overline{z_k} = z_{2n-k}$. Then

$$(x-z_k)(x-\overline{z_k}) = x^2 - 2x\operatorname{Re} z_k + |z_k|^2 = x^2 - 2x\cos\frac{(2k-1)\pi}{2n} + 1\,,$$

and

$$x^{2n} - 1 = \prod_{k=1}^{n}\left(x^2 - 2x\cos\frac{(2k-1)\pi}{2n} + 1\right).$$

We say that R is a *real rational function* if it is represented as a quotient $R = P/Q$ of two real polynomials. This is equivalent to $\overline{R(z)} = R(\overline{z})$, $z \in \mathbb{C}$. Suppose that the real rational function R has a real pole at a. By the proof of the existence in Claim 23.3.1, the coefficients A_j, $1 \leq j \leq k$, in the corresponding singular part S_a are also real. (Indeed, A_k is real since it equals $\lim\limits_{x\to a}(x-a)^kR(x)$ etc). Now, we look at the complex conjugated poles of R.

Claim 23.4.4. *Suppose R is a real rational function with a pole at $w \neq \overline{w}$ of multiplicity k. Then the sum of the singular parts of R at w and $\overline{w}$ equals*

$$S_w(z) + S_{\overline{w}}(z) = \frac{A_1 + B_1 z}{z^2+pz+q} + \dots + \frac{A_z + B_k z}{(z^2+pz+q)^k}$$

with $z^2 + pz + q = (z-w)(z-\overline{w})$ and with real coefficients A_j and B_j, $1 \leq j \leq k$.

Proof: As above, we are looking for the coefficients A_k and B_k such that the rational function

$$R(z) - \frac{A + Bz}{(z-w)^k(z-\overline{w})^k} \tag{23.4.5}$$

has poles at w and $\overline{w}$ of multiplicity at most $k-1$. This means that after multiplication by $(z-w)^k(z-\overline{w})^k$ expression (23.4.5) vanishes as $z \to w$, i.e.

$$\lim_{z\to w}(z-w)^k(z-\overline{w})^k R(z) - (A+Bw) = 0\,.$$

Denote the limit on the RHS by α. Then we have $A+Bw=\alpha$. Since we are looking for real A and B, we conclude that $\operatorname{Im}\alpha = B\operatorname{Im} w$, whence

$$B = \frac{\operatorname{Im}\alpha}{\operatorname{Im} w}, \qquad \text{and } A = \alpha - \frac{\operatorname{Im}\alpha}{\operatorname{Im} w}w\,.$$

It remains to check that with this choice of A and B, expression (23.4.5) has a pole of multiplicity at most $k-1$ at $\overline{w}$. We leave this as an exercise. □

Exercise 23.4.6. Find (real) decompositions of the rational functions

$$\frac{1}{x^2(x-1)}, \qquad \frac{x^3}{x^2+1}, \qquad \frac{1}{x^4+1}\,.$$

24. Complex exponential function

24.1. Absolutely convergent series. Here we deal with absolutely convergent series $\sum a_k$ with complex terms a_k.

24.1.1. *Rearrangement of the series.* A series $\sum a'_k$ is a *rearrangement* of the series $\sum a_k$ if every term in the first series appears exactly once in the second and conversely. In other words, there is a bijection $p\colon \mathbb{N} \to \mathbb{N}$ such that $a'_k = a_{p(k)}$.

Theorem 24.1.1 (Dirichlet)**.** *If the series $\sum a_k$ is absolutely convergent, then all its rearrangements converge to the same sum.*

Proof: First, we prove the result in the case when assume the terms a_k are non-negative. Set

$$S = \sum_{k=1}^{\infty} a_k\,, \qquad S_n = \sum_{k=1}^{n} a_k\,.$$

Let $\{a'_k\}$ be an arbitrary rearrangement of the sequence $\{a_k\}$. Set

$$S'_n = \sum_{k=1}^{n} a'_k\,.$$

Then, for each $n \in \mathbb{N}$, $S'_n \le S$. Hence, the series $\sum a'_k$ converges to the sum S', and $S' \le S$.

In turn, the series $\sum a_k$ is a rearrangement of the series $\sum a'_k$, whence $S \le S'$. Hence, $S = S'$.

Now, consider the general case when the terms a_k are complex. Observe that

$$a_k = \alpha_k + i\beta_k = \alpha_k^+ - \alpha_k^- + i\beta_k^+ - i\beta_k^-\,.$$

Here we've used notations $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. In these notations, $x = x^+ - x^-$, and $|x| = x^+ + x^-$. Hence, we can represent the series $\sum a_k$ by a linear combination of four convergent series with non-negative terms:

$$\sum a_k = \sum \alpha_k^+ - \sum \alpha_k^- + i \sum \beta_k^+ - i \sum \beta_k^-\,.$$

Applying the special case proven above, we get the result. $\Box$.

24.1.2. *Multiplication of series.* Having two absolutely convergent series

$$(A) \qquad \sum_k a_k$$

and

$$(B) \qquad \sum_l b_l\,,$$

we want to learn how to multiply them. Intuitively, the product (AB) should be a double sum

$$(AB) \qquad \sum_{k,l} a_k b_l\,.$$

The first question is *how to understand this expression?* The second question is *does it converges to the product $A \cdot B$?*

Consider the two-dimensional array of all possible products $a_k b_l$:

$$\begin{array}{cccccc}
a_1b_1 & a_1b_2 & a_1b_3 & \dots & a_1b_n & \dots \\
a_2b_1 & a_2b_2 & a_2b_3 & \dots & a_2b_n & \dots \\
a_3b_1 & a_3b_2 & a_3b_3 & \dots & a_3b_n & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
a_mb_1 & a_mb_2 & a_mb_3 & \dots & a_mb_n & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

Recall that we know how enumerate the elements of this array by the naturals $\mathbb{N}$ and each enumeration leads to a different series. Luckily, the previous theorem tells us, that if the series we get in this way are absolutely convergent, then different enumerations will lead to the same answer, so we'll be able to choose the most convenient one.

Absolute convergence: observe that we can bound the finite sums

$$|a_{k_1}b_{l_1}| + \dots + |a_{k_s}b_{l_s}| \le \big(|a_1| + \dots + |a_n|\big)\big(|b_1| + \dots + |b_n|\big)$$

with $n = \max\{k_1, ..., k_s, l_1, ..., l_s\}$. Hence, an arbitrary finite sum $|a_{k_1}b_{l_1}| + \dots + |a_{k_s}b_{l_s}|$ is bounded by $\big(\sum |a_k|\big)\big(\sum |b_l|\big)$. Therefore, for any rearrangement of the terms, the series (AB) is absolutely convergent, and its sum does not depend on the rearrangement.

Cauchy's product: the most popular rearrangement is the one called Cauchy's product:

$$a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \dots ,$$

or

$$\sum_{k,l=1}^{\infty} a_k b_l = \sum_{n=1}^{\infty} \sum_{k+l=n} a_k b_l = \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_k b_{n-k} .$$

Here is our chief example:

Example 24.1.2. Suppose we have two absolutely convergent Taylor series

$$\sum_{k=0}^{\infty} a_k z^k , \qquad \sum_{l=0}^{\infty} b_l z^l .$$

Then their product is represented by another absolutely convergent Taylor series

$$\sum_{k=0}^{\infty} a_k z^k \cdot \sum_{l=0}^{\infty} b_l z^l = \sum_{n=0}^{\infty} c_n z^n$$

with

$$c_n = \sum_{k+l=n} a_k b_l .$$

24.2. The complex exponent. Define the functions

$$e^z \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{z^n}{n!}\,, \qquad \sin z \overset{\text{def}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\,, \qquad \cos z \overset{\text{def}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\,.$$

First, note that the series on the RHS absolutely converge at any point $z \in \mathbb{C}$, and that for real z's the new definitions coincide with the ones we know. Now, the miracle comes:

Claim 24.2.1 (Euler)**.**

$$e^{iz} = \cos z + i \sin z\,, \qquad z \in \mathbb{C}\,.$$

Proof: by inspection. We have

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{m=0}^{\infty} \frac{\overbrace{(iz)^{2m}}^{i^{2m}=(-1)^m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{\overbrace{(iz)^{2m+1}}^{i^{2m+1}=i(-1)^n}}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m+1}}{(2m+1)!} = \cos z + i \sin z\,.$$

Done! $\square$

Note that the cosine function is even, while the sine function is odd. Hence,

Corollary 24.2.2. $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$.

Corollary 24.2.3. *Any non-zero complex number z can be represented in the form $z = re^{i\varphi}$ where $r = |z|$, and $\varphi = \arg z$.*

Corollary 24.2.4. $e^{2\pi i} = 1$.

Corollary 24.2.5 (Euler's formula)**.** $e^{i\pi} = -1$.

This miraculous identity connects the numbers $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$, π defined as the quotient of the length of the circumference to its diameter, and $i = \sqrt{-1}$.

Exercise 24.2.6. Define

$$\sinh z \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\,, \qquad \cosh z \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\,.$$

Check the following relations:

i. $\cosh z = \dfrac{e^z + e^{-z}}{2}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$.

ii. $\sin(iz) = i \sinh z$, $\cos(iz) = \cosh z$.

iii. $\sin^2 z + \cos^2 z = 1$, $\cosh^2 z - \sinh^2 = 1$.

iv. $\sin\left(\dfrac{\pi}{2} - z\right) = \cos z$.

The fundamental properties of the exponential function e^x on the real axis are the functional equation $e^{x+y} = e^x \cdot e^y$ and the differential equation $(e^x)' = e^x$. As we know, each of these properties characterizes the exponential function. Now, we'll check that this two properties persist for the function e^z on $\mathbb{C}$.

Claim 24.2.7. $e^{z+w} = e^z \cdot e^w$.

Proof: by inspection.

$$e^z \cdot e^w = \sum_{n=0}^{\infty} \sum_{k+l=n} \frac{z^k}{k!} \cdot \frac{w^l}{l!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{(z+w)!}{n!} = e^{z+w}$$

and we are done. $\square$

Corollary 24.2.8. $e^{z+2\pi i} = e^z$; i.e., e^z is a periodic function with the period $2\pi i$.

The function $f\colon \mathbb{C} \to \mathbb{C}$ is said to be (complex) differentiable at the point z if there exists the limit

$$f'(z) = \lim_{\mathbb{C} \ni \epsilon \to 0} \frac{f(z+\epsilon) - f(z)}{\epsilon}\,.$$

It is important that the limit does not depend on the direction at which ϵ approaches 0.

Claim 24.2.9. *The function e^z is differentiable in $\mathbb{C}$ and $(e^z)' = e^z$.*

Proof: We have

$$\frac{e^{z+\epsilon} - e^z}{\epsilon} = e^z \, \frac{e^\epsilon - 1}{\epsilon}\,.$$

Note that

$$\left| \frac{e^\epsilon - 1}{\epsilon} - 1 \right| \le \sum_{n=1}^{\infty} \frac{|\epsilon|^n}{(n+1)!} = o(1)$$

as $\epsilon \to 0$. Done! $\square$