

# 1 Euclidean space $\mathbb{R}^n$

We start the course by recalling prerequisites from the courses Hedva 1 and 2 and Linear Algebra 1 and 2.

## 1.1 Scalar product and Euclidean norm

During the whole course, the  $n$ -dimensional linear space over the reals will be our home. It is denoted by  $\mathbb{R}^n$ . We say that  $\mathbb{R}^n$  is an Euclidean space if it is equipped with a *scalar product*, that is a function  $(x, y): \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying conditions

- (i)  $(x, x) \geq 0$ ;
- (ii)  $(x, x) = 0$  if and only if  $x = 0$ ;
- (iii)  $(x, y) = (y, x)$ ;
- (iv) for any  $t \in \mathbb{R}$ ,  $(tx, y) = t(x, y)$ ;
- (v)  $(x + y, z) = (x, z) + (y, z)$ .

If  $\{e_1, \dots, e_n\}$  is a basis in  $\mathbb{R}^n$ , and  $x = \sum x_i e_i$ ,  $y = \sum y_j e_j$ , then  $(x, y) = \sum_{i,j} x_i y_j (e_i, e_j)$ . In particular, if  $\{e_1, \dots, e_n\}$  is an orthonormal basis in  $\mathbb{R}^n$ , then  $(x, y) = \sum x_i y_i$ .

Having a chosen scalar product<sup>1</sup>, we can define the *length* (or the *Euclidean norm*) of the vector  $\|x\| = \sqrt{(x, x)}$ , and the *angle* between two vectors:

$$\angle(x, y) = \arccos \frac{(x, y)}{\|x\| \cdot \|y\|}.$$

Of course, the angle  $\angle(x, y)$  is defined only for  $x, y \neq 0$ .

The norm enjoys the following properties:

- (a)  $\|x\| > 0$  for  $x \neq 0$ ;
- (b)  $\|tx\| = |t| \|x\|$  for  $t \in \mathbb{R}$ ;
- (c)  $\|x + y\| \leq \|x\| + \|y\|$ .

**Exercise 1.1.** Prove the property (c). Describe when the equality sign attains in (c).

There are many other functions  $\|\cdot\|: \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying conditions (a), (b) and (c). They are also called the *norms*. For example, the  $l^p$ -norm

$$\|x\|_p \stackrel{def}{=} \left\{ \sum_{i=1}^n |x_i|^p \right\}^{1/p}, \quad 1 \leq p < \infty,$$

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<sup>1</sup>Of course, there are many different scalar products in  $\mathbb{R}^n$ . If we fix one of them, scalar then any other has the form  $(x, y)_* = (Ax, y)$ , where  $A$  is a positive linear operator in  $\mathbb{R}^n$ .

and

$$\|x\|_\infty \stackrel{def}{=} \max_{1 \leq i \leq n} |x_i|,$$

here  $\{x_i\}$  are the coordinates of  $x$  in a fixed basis in  $\mathbb{R}^n$ .

It is easy to see that the  $l^p$ -norm meets conditions (a) and (b). In the extreme cases  $p = 1$  and  $p = \infty$ , condition (c) is also obvious. Now, we sketch the proof of (c) for  $1 < p < \infty$ , splitting it into three simple exercises.

**Exercise 1.2.** Let  $q$  be a dual exponent to  $p$ :  $\frac{1}{p} + \frac{1}{q} = 1$ ,

1. Prove Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

for  $a, b > 0$ . The equality sign attains for  $a^p = b^q$  only. Hint: maximize the function  $h(a) = ab - \frac{a^p}{p}$ .

2. Prove Hölder's inequality

$$\sum_i a_i b_i \leq \left( \sum_i a_i^p \right)^{1/p} \cdot \left( \sum_i b_i^q \right)^{1/q},$$

for  $a_i, b_i > 0$ . The equality sign attains only for  $a_i/b_i = \text{const}$ ,  $1 \leq i \leq n$ . Hint: assume, WLOG, that

$$\sum_i a_i^p = \sum_i b_i^q = 1,$$

and apply Young's inequality to  $a_i \cdot b_i$ .

3. Check that

$$\left( \sum_i (a_i + b_i)^p \right)^{1/p} \leq \left( \sum_i a_i^p \right)^{1/p} + \left( \sum_i b_i^p \right)^{1/p},$$

Hint: write

$$\sum_i (a_i + b_i)^p = \sum_i a_i (a_i + b_i)^{p-1} + \sum_i b_i (a_i + b_i)^{p-1},$$

and apply Hölder's inequality.

Having the norm  $\|\cdot\|$ , we define the distance  $\rho(x, y) = \|x - y\|$ . In this course, we will mainly use the Euclidean norm <sup>2</sup>, and the Euclidean distance. Normally, we denote the Euclidean norm by  $|x|$ .

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<sup>2</sup>Though, as we will see in the next lecture, all the norms in  $\mathbb{R}^n$  are *equivalent* and the choice of the norm does not affect the problems we are studying.

## 1.2 Open and closed subsets of $\mathbb{R}^n$

Recall a familiar terminology:

*open ball*  $B(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$ ;

*closed ball*  $\overline{B(a, r)} = \{x \in \mathbb{R}^n : |x - a| \leq r\}$ ;

*sphere*  $S(a, r) = \{x \in \mathbb{R}^n : |x - a| = r\} = \partial B(a, r)$ ;

*brick*  $\{x \in \mathbb{R}^n : \alpha_i < x_i < \beta_i\}$ . This brick is open. Sometimes, we'll need closed and semi-open bricks.

*complement* (to the set  $E$ )  $E^c = \mathbb{R}^n \setminus E$ .

The next definition is fundamental:

**Definition 1.3.** A set  $A \subset \mathbb{R}^n$  is open, if for each  $a \in X$  there is an open ball  $B$  centered at  $a$  such that  $B \subset X$ . A set  $X \subset \mathbb{R}^n$  is closed if its complement is open.

The empty set  $\emptyset$  and the whole  $\mathbb{R}^n$  are open and closed at the same time. (These two subsets of  $\mathbb{R}^n$  are often called *trivial* subsets).

**Exercise 1.4.** Union of any family of open sets is open, intersection of any family of closed sets is closed. Finite intersection of open sets is open, finite union of closed subsets is closed.

**Exercise 1.5.** Give an example of infinite family of open sets with non-trivial and closed intersection, and an example of infinite family of closed sets with non-trivial and open union.

Let us proceed further.

Any open set  $O$  containing the point  $a$  is called a *vicinity/neighbourhood* of  $a$ , the set  $O \setminus \{a\}$  is called a *punctured neighbourhood* of  $a$ . The (open) ball  $B(a, \delta)$  is called a  $\delta$ -*neighbourhood* of  $a$ .

*Interior*  $\text{int}E$  is a set of points  $a$  which belong to  $E$  with some neighbourhood. *Exterior*  $\text{ext}E$  is a set of points  $a$  having a neighbourhood which does not belong to  $E$  (equivalently,  $a$  belongs to the interior of the complement  $E^c$ ). *Boundary*  $\partial E$  is a set of points which are neither interior nor exterior points of  $E$ .

In other words, we always have a decomposition  $\mathbb{R}^n = \text{int}E \cup \partial E \cup \text{ext}E$ .

*Closure:*  $\overline{E} = E \cup \partial E$ .

Equivalently,  $\overline{E}$  is a union of  $E$  with the set of all accumulation points of  $E$ . Recall that  $x$  is an accumulation point of the set  $E$  if in any punctured neighbourhood of  $x$  there is at least one (and therefore, infinitely many) points of  $E$ .

- Exercise 1.6.** 1. The set  $E$  is closed if and only if it contains all its accumulation points; i.e.  $\overline{E} = E$ .
2. The closure  $\overline{E}$  is always closed, and the second closure coincides with the first one  $\overline{\overline{E}} = \overline{E}$ .
3. Only trivial subsets are simultaneously closed and open.

*Convergence:* a sequence  $\{x^k\} \subset \mathbb{R}^n$  converges to  $x$  if  $\lim_{k \rightarrow \infty} |x^k - x| = 0$ . Equivalently<sup>3</sup>, all coordinates must converge:  $x_i^k \rightarrow x_i$  for  $1 \leq i \leq n$ .

The Cauchy criterion and the Bolzano-Weierstrass lemma are the same as in the 1D-case. Cauchy's criterion says that *the sequence  $\{x^k\}$  converges to  $x$  iff<sup>4</sup> for an arbitrary small  $\epsilon > 0$  there exists a sufficiently large  $N$  such that for all  $k, m \geq N$ ,  $|x^k - x^m| < \epsilon$ .*

The BW-lemma says that *any bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.*

**Exercise 1.7.** Prove Cauchy's criterion and the Bolzano-Weierstrass lemma.

### 1.3 Compact sets

**Definition 1.8.** A set  $K \subset \mathbb{R}^n$  is compact, if any sequence  $\{x^m\} \subset K$  has a subsequence  $\{x^{m_j}\}$  convergent to a point from  $K$ .

**Claim 1.9.** A set  $K$  is compact if and only if it is closed and bounded.

*Proof:*

(a) Assume that  $K$  is compact. Then  $K$  must contain all its accumulation points and therefore  $K$  is closed.

Assume that  $K$  is unbounded. Then there is a sequence  $\{x^m\} \subset K$  such that  $\|x^m\| \geq m$ . If a subsequence  $\{x^{m_j}\}$  converges to a point  $x$ , then by the triangle inequality

$$|x| \geq |x^{m_j}| - |x^{m_j} - x| \geq m_j - \epsilon \uparrow +\infty.$$

Contradiction.

(b) Assume that the set  $K$  is closed and bounded. By the BW-lemma, each sequence  $\{x^m\} \subset K$  has a convergent subsequence. Let  $x$  be its limit. Then  $x$  is an accumulation point of  $K$ , and since  $K$  is closed,  $x \in K$ . Thus,  $K$  is compact.  $\square$

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<sup>3</sup>by the elementary inequality

$$\max_{1 \leq i \leq n} |x_i - y_i| \leq |x - y| \leq \sqrt{n} \max_{1 \leq i \leq n} |x_i - y_i|.$$

<sup>4</sup>iff = if and only if

**Exercise 1.10.** Any nested sequence of compact sets  $K_1 \supset K_2 \supset \dots \supset K_j \supset \dots$  has a non-empty intersection.

Hint: consider a sequence  $\{x_j\}$  such that  $x_j \in K_j$ .

If the *diameters* of  $K_j$  converge to zero, then the intersection is a singleton. Here,

$$\text{diameter}(K) = \max_{x,y \in K} |x - y|.$$

Continue with definitions.

*Open covering* of  $X$ :

$$X \subset \bigcup_{j \in J} U_j,$$

where the sets  $U_j$  are open. If the set of indices  $J$  is finite, then the covering is called *finite*.

The next lemma is fundamental:

**Lemma 1.11** (Heine - Borel). *The set  $K$  is compact if and only if*

$$\boxed{\forall \text{ open covering of } K \quad \exists \text{ a finite subcovering}}$$

In topology, the boxed formula is taken as the definition of compact sets.

I'll prove this result only in one direction, assuming that  $K$  is a compact set.

*Proof:* First, we enclose the compact  $K$  by a brick  $I$ , and then will follow the standard 'dissection procedure', as in the one-dimensional case.  $\square$

The other direction probably will not be used later, so I leave it as an exercise.

**Exercise 1.12.** If for any open covering of the set  $K \subset \mathbb{R}^n$  there exists a finite subcovering, then  $K$  is a compact set.

**Exercise 1.13.** Show that the result fails for bounded (but non-closed) sets, and for closed (but unbounded) sets. It also fails for coverings of compact set by non-open sets.

## 2 Continuous mappings. Curves in $\mathbb{R}^n$

We continue with prerequisites.

### 2.1 Continuous mappings

Let  $X \subset \mathbb{R}^n$ , and  $x_0$  be an accumulation point of  $X$ . Let  $f: \overline{X} \mapsto \mathbb{R}^m$ . If there is  $a \in \mathbb{R}^m$  such that  $|f(x) - a| \rightarrow 0$  when  $x \rightarrow x_0$ ,  $x \in X$ , then we say that  $f$  has a limit  $a$  when  $x \rightarrow x_0$  along  $X$ , and write

$$\lim_{x \rightarrow x_0, x \in X} f(x) = a.$$

Usually, we assume that  $x \in \text{int}X$ , then there is no need to indicate that  $x \rightarrow x_0$  along  $X$ .

It is important to keep in mind that existence of such a limit, generally speaking, does not imply existence of iterated limits and vice versa (see the exercise in the very end of this subsection).

The mapping  $f: \overline{X} \rightarrow \mathbb{R}^m$  is *continuous* at  $x_0$ , if  $f(x) \rightarrow f(x_0)$  for  $x \rightarrow x_0$  along  $X$ . It is always possible to check continuity using the coordinate functions. Fix a basis  $\{e_1, \dots, e_m\}$  in  $\mathbb{R}^m$ , and consider the coordinate functions  $f_j(x)$ ,  $1 \leq j \leq m$  (that is,  $f(x) = \sum_j f_j(x)e_j$ ). It is easy to see that the function  $f$  is continuous at  $x_0$  iff all the functions  $f_j$  are continuous at  $x_0$ .

**Exercise 2.1.** Write down a formal proof.

We say that  $f$  is continuous on  $X$  if it is continuous at every point of  $X$ . By  $C(X)$  we denote the class of all continuous functions on  $X$ .

**Exercise 2.2.** Prove or disprove: let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^1$  be a mapping with the following properties: for each  $y \in \mathbb{R}$ , the function  $x \mapsto f(x, y)$  is continuous on  $\mathbb{R}$ , and for each  $x \in \mathbb{R}$ , the function  $y \mapsto f(x, y)$  is continuous on  $\mathbb{R}$ . Then  $f$  is continuous on  $\mathbb{R}^2$ .

If  $X$  is a compact set, then continuous mappings defined on  $X$  enjoy many properties of continuous functions defined on closed segments.

**Exercise 2.3.** Let  $f: K \rightarrow \mathbb{R}^m$  be a continuous mapping on a compact set  $K \subset \mathbb{R}^n$ . Prove:

(i)  $f$  is uniformly continuous, that is

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall x, y \in K, \|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon.$$

- (ii)  $f$  is bounded on  $K$ , i.e.  $\exists M$  such that  $\|f(x)\| \leq M$  for all  $x \in K$ ;
- (iii) if  $m = 1$  (that is,  $f$  is a scalar function),  $f$  attains its maximal and minimal values on  $K$ .

**Exercise 2.4.** A subset  $K \in \mathbb{R}^n$  is compact if and only if *any* continuous function map  $f: K \rightarrow \mathbb{R}$  is bounded on  $K$ .

**Exercise 2.5.** If  $f$  is a continuous mapping, and  $K$  is a compact set, then its image  $fK$  is compact as well. If the set  $V$  is open, then its preimage  $f^{-1}V$  is also open.

**Exercise 2.6.** Check:

- (i) for the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$$

the iterated limits exist and equal to each other

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = 0$$

but the limit

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

does not exist;

- (ii) for the function

$$f(x, y) = \begin{cases} x + y \sin \frac{1}{x}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$$

the limits

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \quad \text{and} \quad \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y)$$

exist and equal zero, but the second iterated limit

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y)$$

does not exist;

- (iii) the function

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$$

has the following property: for each  $a$  and  $b$  in  $\mathbb{R}^2$

$$\lim_{t \rightarrow 0} f(ta, tb) = 0,$$

but the limit

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

does not exist.

**Exercise 2.7.** Let  $E \subset \mathbb{R}^n$  be a closed set,  $f: E \rightarrow \mathbb{R}^m$  a continuous function. Show that its *graph*

$$\Gamma_f \stackrel{\text{def}}{=} \{(x, f(x)): x \in E\}$$

is a closed subset of  $\mathbb{R}^{n+m}$ .

**Exercise 2.8.** Show that there is a mapping  $f$  from the unit ball  $\mathbb{B} \in \mathbb{R}^n$  onto the whole  $\mathbb{R}^n$  such that  $f$  and  $f^{-1}$  are continuous and  $f$  is one-to-one. (Such maps are called *homeomorphisms*).

### 2.1.1 Linear mappings

We denote by  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$  the space of all linear mappings (= transformations = operators) from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . Since we can add mappings:  $(A + B)x \stackrel{\text{def}}{=} Ax + Bx$ , this is a linear space. This space can be identified with  $\mathbb{R}^{mn} = \underbrace{\mathbb{R}^m \times \dots \times \mathbb{R}^m}_{n \text{ times}}$ . For identification, we use the *matrix representation* which we recall.

Let  $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ . Fix bases:  $\{e_j\} \subset \mathbb{R}^n$ , and  $\{e_k^*\} \subset \mathbb{R}^m$ . Then

$$Ae_j = \sum_{k=1}^m a_{kj} e_k^*.$$

The matrix of  $A$  consists of  $n$  columns of height  $m$ , the  $j$ -th column consists of the coordinates of the vector  $Ae_j$  in the basis  $\{e_k^*\}$ :

$$\text{Mat}(A) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

**Exercise 2.9.** What happens with the matrix of  $A$  under the change of the bases in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ ?



By  $\text{Mat}_{\mathbb{R}}(m \times n)$  we denote the linear space of  $m \times n$  matrices with real entries. Thus we get three isomorphic linear spaces:

$$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \simeq \text{Mat}_{\mathbb{R}}(m \times n) \simeq \mathbb{R}^{mn}.$$

We can also *multiply* elements from  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \simeq \text{Mat}_{\mathbb{R}}(m \times n)$  by elements from  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^p) \simeq \text{Mat}_{\mathbb{R}}(p \times m)$  taking the composition of linear mappings, or, what is the same, the product of the correspondent matrices.

If  $m = 1$ , then the linear mapping  $L: \mathbb{R}^n \rightarrow \mathbb{R}^1$  is called a *linear functional* on  $\mathbb{R}^n$ . The space of linear functionals is an  $n$ -dimensional space called the *dual* space. Usually, it is denoted by  $(\mathbb{R}^n)^*$ . The important representation theorem from the Linear Algebra says that if  $\mathbb{R}^n$  has an Euclidean structure, then for any linear functions  $L \in (\mathbb{R}^n)^*$  there exists a vector  $l \in \mathbb{R}^n$  such that  $Lx = (l, x)$  for any  $x \in \mathbb{R}^n$ .

**Exercise 2.10.** Every linear map  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous.

Hint: first, prove this for linear functionals (i.e., scalar linear functions) on  $\mathbb{R}^n$ : if  $x = \sum x_j e_j$ , then  $L(x) = \sum L(e_j)x_j$ , and  $L(x) - L(y) = \sum L(e_j)(x_j - y_j)$ . The rest is clear.

Since the unit ball is a compact subset of  $\mathbb{R}^n$ , as a corollary, we obtain that, for any linear mapping  $L$ , the quantity

$$N(L) \stackrel{\text{def}}{=} \sup_{|x| \leq 1} |Lx|$$

is finite, and actually the maximum on the RHS is attained somewhere on the unit sphere. Actually,  $N(L)$  can be defined as the best possible constant in the estimate  $|Lx| \leq N(L)|x|$ ; i.e.

$$N(L) = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|Lx|}{|x|}.$$

This quantity is called the *operator norm* of the mapping  $L$ , and is denoted by  $\|L\|$ .

**Exercise 2.11.** Check that this is a norm, i.e.,  $\|L\| = 0$  iff  $L = 0$ , and  $\|L_1 + L_2\| \leq \|L_1\| + \|L_2\|$ . Check that  $\|L \circ M\| \leq \|L\| \cdot \|M\|$ .

It is easy to show that

$$(2.12) \quad \|L\| \leq \sqrt{\sum_{j,k} l_{j,k}^2},$$

where  $(l_{j,k})$  are matrix elements of  $L$ .

**Exercise 2.13.** Prove (2.12).

Hint: set  $y = Lx$ , using the Cauchy-Schwarz inequality, estimate first  $y_k^2$  and then  $\sum_{k=1}^m y_k^2$ . Here  $y_k = \sum_j l_{kj}x_j$ .

If  $m > 1$ , estimate (2.12) is not sharp. Later, using Lagrange multipliers, we'll give a sharp expression for  $\|L\|$ . (Check, maybe, you already know it from the Linear Algebra course?)

If  $m = 1$ , that is  $L \in (\mathbb{R}^n)^*$  is a linear function, there is a vector  $b$  such that  $Lx = \sum b_j x_j$  for all  $x$  (simply  $b_j = L(e_j)$ ). In this case,  $\|L\| = |b|$ . (Check this!)

The expression  $\sqrt{\sum_{j,k} l_{j,k}^2}$  we met above is called the *Hilbert-Schmidt norm* of the operator  $L$  and denoted by  $\|L\|_{\text{HS}}$ . There is a more natural definition of the Hilbert-Schmidt norm:

**Exercise 2.14.** Show that  $\|L\|_{\text{HS}} = \text{trace}(L^*L)$ , and that  $\|\cdot\|_{\text{HS}}$  does not depend on the choice of the bases in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ .

**Exercise 2.15.** Show that  $\|L\|_{\text{HS}} \leq \sqrt{n}\|L\|$ .

Hint:  $|Le_j| \leq \|L\|$  for each  $j$ ,  $1 \leq j \leq n$ .

## 2.1.2 Continuity of norms

Another important class of continuous functions is given by norms, i.e. functions  $\|\cdot\|: \mathbb{R}^n \rightarrow \mathbb{R}_+$  satisfying conditions (a) – (c) from Lecture 1. It will be convenient for us to prove simultaneously the next two claims.

**Claim 2.16.** Any norm  $\|\cdot\|_*$  in the Euclidean space  $\mathbb{R}^n$  is equivalent to the Euclidean one: there are positive constants  $c$  and  $C$  depending on the norm such that for any  $x \in \mathbb{R}^n$

$$c|x| \leq \|x\|_* \leq C|x|.$$

**Claim 2.17.** Any norm is a continuous function on  $\mathbb{R}^n$ .

*Proof:* First, we check the second inequality in Claim 2.16. Let  $\{e_i\}$  be the standard orthonormal basis in  $\mathbb{R}^n$ . Then writing  $x = \sum x_i e_i$ , we get

$$\|x\|_* \leq \sum_{i=1}^n |x_i| \cdot \|e_i\|_* \leq \underbrace{\sqrt{\sum_i x_i^2}}_{=|x|} \cdot \underbrace{\sqrt{\sum_i \|e_i\|_*^2}}_{=C} = C|x|.$$

Now, we check continuity of the norm  $\|\cdot\|_*$ :

$$\|x\|_* - \|y\|_* = \|y + (x - y)\|_* - \|y\|_* \leq \|x - y\|_* \leq C|x - y|,$$

and by symmetry

$$\|y\|_* - \|x\|_* \leq C|x - y|.$$

To get the first inequality in Claim 2.16, observe that the function  $x \mapsto \|x\|_*$  attains its minimal value on the unit (Euclidean) sphere, and this value must be positive (why?). Let us denote it by  $c$ . Then, writing  $x = |x|\hat{x}$ ,  $|\hat{x}| = 1$ , we get

$$\|x\|_* = |x| \cdot \|\hat{x}\|_* \geq c|x|,$$

completing the proofs.  $\square$

### 2.1.3 Norms and symmetric compact convex bodies

There is an intimate relation between the norms in  $\mathbb{R}^n$  and a special class of compact convex bodies in  $\mathbb{R}^n$ . The closed ‘unit ball’ with respect to any norm in  $\mathbb{R}^n$  is defined as

$$(2.18) \quad K = \{x : \|x\| \leq 1\}$$

A straightforward inspection shows that  $K$  always has the following four properties:

- (i)  $K$  is compact;
- (ii)  $K$  is convex, that is, if  $a, b \in K$ , then the whole segment with the end-points at  $a$  and  $b$ ,  $\{ta + (1 - t)b : 0 \leq t \leq 1\}$ , belongs to  $K$ ;
- (iii)  $K$  is symmetric about the origin, i.e. if  $x \in K$ , then  $-x \in K$  as well;
- (iv)  $K$  contains a Euclidean neighbourhood of the origin.

**Exercise 2.19.** Check the properties (i)–(iv). Draw the unit ball for the  $l^p$  norms in  $\mathbb{R}^2$ , for  $1 \leq p \leq \infty$ . If the norm is generated by a scalar product, how the body  $K$  looks like?

Assume that we know the compact convex body  $K$ . How to recover the norm  $\|\cdot\|$ ? Given  $x \neq 0$ , observe that  $tx \in K$  for  $t \leq \frac{1}{\|x\|}$ , and  $tx \notin K$  for  $t > \frac{1}{\|x\|}$ . Thus

$$\|x\|^{-1} = \max\{t : tx \in K\}.$$

**Problem 2.20.** Let  $K$  be a set with properties (i)–(iv). For every  $x \neq 0$ , let

$$\|x\| \stackrel{\text{def}}{=} \frac{1}{\max\{t : tx \in K\}}.$$

Then  $\|\cdot\|$  is a norm, and (2.18) holds.

## 2.2 Continuous curves in $\mathbb{R}^n$

**Definition 2.21.** (*Continuous*) curve in  $\mathbb{R}^n$  is a continuous mapping  $\gamma: I \rightarrow \mathbb{R}^n$ , where  $I \subset \mathbb{R}^1$  is an interval. If the interval  $I$  is closed,  $I = [a, b]$ , then  $\gamma(a)$  and  $\gamma(b)$  are end-points of the curve  $\gamma$ . The curve  $\gamma$  is *closed* if  $\gamma(a) = \gamma(b)$ . The curve  $\gamma$  is *simple* if the function  $\gamma|_{(a,b)}$  is one-to-one.

Intervals have a natural orientation which induces orientation on curves. Each curve  $\gamma$  is oriented. We can always change the orientation defining the ‘inverse’ curve  $-\gamma$ . For example, if  $\gamma$  is defined on  $[0, 1]$ , then  $(-\gamma)(t) \stackrel{def}{=} \gamma(1 - t)$ .

The curves  $\gamma_1: I_1 \rightarrow \mathbb{R}^n$  and  $\gamma_2: I_2 \rightarrow \mathbb{R}^n$  are called *equivalent*, if there exists a homeomorphism (i.e., a continuous bijection)  $\varphi: I_2 \rightarrow I_1$  which preserves the orientation of the intervals, and such that  $\gamma_2(s) = \gamma_1(\varphi(s))$ . Normally, we identify equivalent curves.

### Examples:

1. The segment with end-points at  $x$  and  $y$ :  $\gamma(t) = tx + (1 - t)y$ ,  $0 \leq t \leq 1$ . The segment with the inverse orientation is  $(-\gamma)(t) = ty + (1 - t)x$ .
2. The circle with the natural (counter clock-wise) orientation  $\gamma(t) = (\cos t, \sin t)$ ,  $0 \leq t \leq 2\pi$ . The circle with the opposite orientation  $(-\gamma)(t) = (\cos t, -\sin t)$ . This is also the circle  $\gamma(t) = (\cos 10t, \sin 10t)$ , but run 10 times.
3. Archimedes spiral  $\gamma(t) = (t \cos t, t \sin t)$ ,  $0 \leq t \leq 2\pi$ .

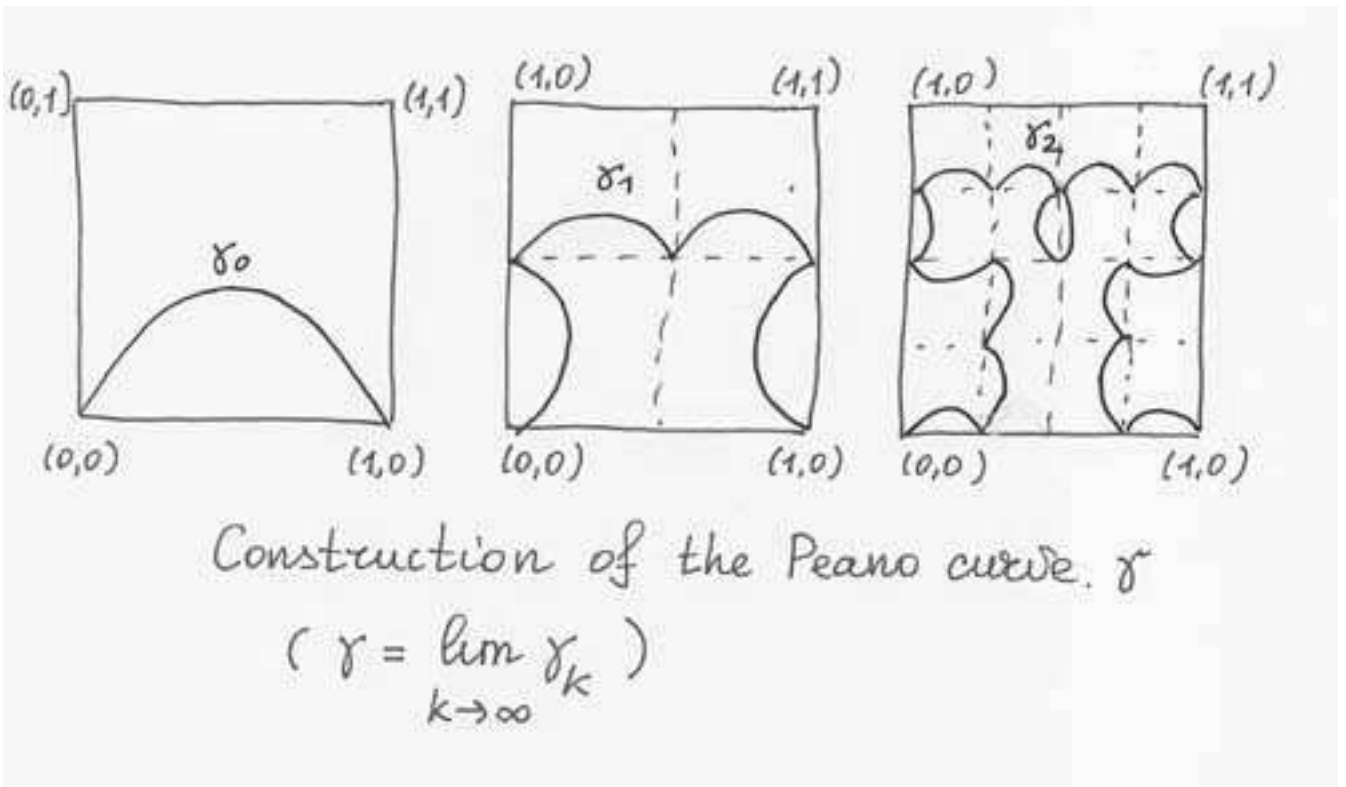
**Exercise 2.22.** Draw the images (with orientation) of the following curves given in the polar coordinates:  $r = 1 - \cos 2t$  ( $0 \leq t \leq 2\pi$ ),  $r^2 = 4 \cos t$  ( $|t| \leq \pi/2$ ),  $r = 2 \sin 3t$  ( $0 \leq t \leq \pi$ ). Draw the image (with orientation) of the curve in  $\mathbb{R}^3$  defined as  $\gamma(t) = (\cos t, \sin t, t)$ ,  $-\infty < t < \infty$ .

### 2.2.1 Peano curve

In 1890, Peano discovered a remarkable example of a (continuous) curve filling the whole unit square in  $\mathbb{R}^2$ . The following construction is taken from the book by Hairer and Wanner.

Start with an arbitrary curve  $\gamma(t) = (x(t), y(t))$ ,  $0 \leq t \leq 1$  which lies in the unit square and has the end-points  $\gamma(0) = (0, 0)$ ,  $\gamma(1) = (1, 0)$ . Now, applying rotation and rescaling, we define a new curve

$$(\Phi\gamma)(t) = \begin{cases} \frac{1}{2}(y(4t), x(4t)), & \text{if } 0 \leq t \leq \frac{1}{4}; \\ \frac{1}{2}(x(4t - 1), 1 + y(4t - 1)), & \text{if } \frac{1}{4} \leq t \leq \frac{2}{4}; \\ \frac{1}{2}(1 + x(4t - 2), 1 + y(4t - 2)), & \text{if } \frac{2}{4} \leq t \leq \frac{3}{4}; \\ \frac{1}{2}(2 - y(4t - 3), 1 - x(4t - 3)), & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$



This curve has the same end-points as  $\gamma$ .

Now, we iterate the procedure defining the curves  $\gamma_1 = \Phi\gamma$ ,  $\gamma_2 = \Phi\gamma_1$ , and so on. We need to show that the iterations converge. For a mapping  $\lambda: [0, 1] \rightarrow \mathbb{R}^n$ , we set

$$\|\lambda\|_\infty = \max_{t \in [0,1]} |\lambda(t)|.$$

This is again the norm, but this time defined on continuous mappings from  $[0, 1]$  to  $\mathbb{R}^2$ . Usually, it is called the *uniform* norm.

Observe, that if we start iterate another curve  $\mu$ , with  $\|\gamma - \mu\|_\infty = M$ , then  $\|\Phi\gamma - \Phi\mu\|_\infty \leq M/2$ , and hence

$$(2.23) \quad \|\gamma_k - \mu_k\|_\infty \leq M \cdot 2^{-k}.$$

Putting here  $\mu = \gamma_m$ , we get

$$\|\gamma_k - \gamma_{k+m}\|_\infty \leq M \cdot 2^{-k}.$$

Now, applying Cauchy's criterion, we see that the sequence of curves  $\gamma_k$  converges uniformly, and therefore has a continuous limit  $\gamma_\infty$ .

The limiting curve  $\gamma_\infty$  is independent of the initial curve  $\gamma$  (look at (2.23)), and fills the whole unit square. Indeed, the set  $\gamma([0, 1])$  is a compact (why?) and dense (why?) subset of the unit square, hence, it coincides with the unit square.  $\square$

**Problem 2.24.** Show that the coordinates of the limiting curve  $x_\infty(t)$  and  $y_\infty(t)$  are continuous nowhere differentiable functions on  $[0, 1]$ .

First examples of such functions were constructed by Weierstrass.

**Problem 2.25.** Show that there is no one-to-one continuous mapping from the interval  $[0, 1]$  onto the unit square.

## 2.3 Arc-wise connected sets in $\mathbb{R}^n$

**Definition 2.26.** The set  $X \subset \mathbb{R}^n$  is called (*arc-wise*) *connected* if for each pair of points  $x, y \in X$  there is a curve  $\gamma: [a, b] \rightarrow X$  such that  $\gamma(a) = x$  and  $\gamma(b) = y$ .

**Examples:**  $\mathbb{R}^1 \setminus \{0\}$ ,  $\mathbb{R}^2 \setminus \{x_1 = 0\}$ ,  $\mathbb{S}^n$ ,  $\mathbb{R}^2 \setminus \{0\}$ ,  $\mathbb{R}^3 \setminus \{x_1 = 0\}$ . The first two sets are disconnected, the others are arc-wise connected.

The next four claims are very useful and have straightforward proofs:

**Claim 2.27** (continuous images of connected sets are connected). *Let  $A \subset \mathbb{R}^n$  be a connected set, and  $f: A \rightarrow \mathbb{R}^m$  be a continuous function. Then the image  $B = f(A)$  is also connected.*

**Claim 2.28** (mean-value property). *Suppose  $A$  is an arc-wise connected set, and  $f: A \rightarrow \mathbb{R}$  is a continuous function. If  $\inf_A f < 0$  and  $\sup_A f > 0$ , then there exists a point  $x \in A$  such that  $f(x) = 0$ .*

**Exercise 2.29.** Prove these claims.

An open connected subset of  $\mathbb{R}^n$  is called *domain* (= *region*).

**Exercise 2.30.** Any open set  $U \subset \mathbb{R}^n$  can be decomposed into at most countable union of disjoint domains.

**Claim 2.31** (polygonal connectivity). *Each domain  $\Omega$  in  $\mathbb{R}^n$  is polygonal-connected.*

That is, for any two points  $a, b \in \Omega$  there exists a polygonal line (a curve which consists of finitely many segments) which starts at  $a$  and terminates at  $b$ . An equivalent statement is that for any two points  $a, b \in \Omega$  can be connected within  $\Omega$  by a finite chain of open balls  $B_0, B_1, \dots, B_N$ :  $B_0$  is centered at  $a$ ,  $B_N$  is centered at  $b$ ,  $B_i \cap B_{i+1} \neq \emptyset$ ,  $0 \leq i \leq N-1$ , and  $\overline{B_i} \subset \Omega$ ,  $0 \leq i \leq N$ . The proof follows from the Heine-Borel lemma.

A function  $f$  on an open set  $X$  is called *locally constant* if for any  $x \in X$  there is a neighbourhood  $U$  of  $x$  such that  $f$  is constant on  $U$ .

**Claim 2.32.** *An open set  $X \subset \mathbb{R}^n$  is connected iff any locally constant function is a constant.*

**Exercise 2.33.** Check that any curve  $\gamma$  starting at  $x$ ,  $|x| < 1$ , and terminating at  $y$ ,  $|y| > 1$ , intersects the unit sphere.

**Exercise 2.34.**  $f$  is a continuous function on the unit sphere  $\mathbb{S}^2 \subset \mathbb{R}^3$ . Prove that there is a point  $x \in \mathbb{S}^2$  such that  $f(x) = f(-x)$ .

## 3 Differentiation

### 3.1 Derivative

In this lecture,  $U$  is always an open subset of  $\mathbb{R}^n$ , and  $a \in U$ .

**Definition 3.1.** *The mapping  $f: U \rightarrow \mathbb{R}^m$  is differentiable at the point  $a$  if in a neighbourhood of  $a$  it can be well approximated by a linear mapping  $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ :*

$$(3.2) \quad f(x) = f(a) + L(x - a) + o(|x - a|), \quad x \rightarrow a.$$

It is important that the mapping  $L$  does not depend on the direction in which  $x$  approach  $a$  in (3.2). If such a map  $L$  exists, then it is unique. Indeed, set  $x = a + th$ , where  $|h| = 1$  and  $t > 0$ . Then

$$f(a + th) = f(a) + tLh + o(t), \quad t \downarrow 0,$$

and we can recover  $L$ :

$$Lh = \lim_{t \downarrow 0} \frac{f(a + th) - f(a)}{t}, \quad |h| = 1.$$

The linear map  $L$  is called the *derivative* (or, sometimes, the *differential*) of  $f$  at  $a$ . There are several customary notations for the derivative:  $f'(a)$ ,  $df(a)$ ,  $D_f(a)$ . Usually, we shall try to stick with the latter one.

At this point we need to slightly revise the one-dimensional definition. In Hedva 1, we learn that if  $f$  is a real-valued function of one real variable, then its derivative  $f'(a)$  is a real number. From now on, we should think about it as of a linear mapping from  $\mathbb{R}^1$  to  $\mathbb{R}^1$  defined as multiplication by  $f'(a)$ .

If  $f$  is differentiable everywhere in  $U$ , the derivative is a map

$$D_f: U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m).$$

By  $C^1(U)$  (or sometimes, by  $C^1(U, \mathbb{R}^m)$ ) we denote the class of mappings  $f$  such that the map  $D_f$  is a continuous one.

Now, several simple properties of the derivative.

1. If  $f$  is a constant map, then  $D_f = 0$  everywhere in  $U$ . If  $U$  is a domain and  $D_f = 0$  everywhere in  $U$ , then  $f$  is a constant mapping.
2. If  $f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ , then  $f$  is differentiable everywhere and  $D_f = f$ . In the opposite direction, if  $U$  is a domain, and  $f: U \rightarrow \mathbb{R}^m$  has a constant derivative in  $U$ , then there exist  $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$  and  $b \in \mathbb{R}^m$  such that  $f(x) = Lx + b$ .
3.  $D_{f+g} = D_f + D_g$ .



**Exercise 3.3.** If the mapping  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

**Exercise 3.4.** If  $f \in C^1(U)$ , and  $K \subset U$  is a compact subset, then  $f|_K$  is a Lipschitz function; i.e. there exists a constant  $M$  such that for every  $x, y \in K$

$$|f(x) - f(y)| \leq M|x - y|.$$

**Theorem 3.5** (The Chain Rule). *Let  $f: U \rightarrow \mathbb{R}^m$ ,  $f(U) \subset V \subset \mathbb{R}^m$ , let  $g: V \rightarrow \mathbb{R}^k$ , and let  $h = g \circ f$ . If  $f$  is differentiable at  $a$ , and  $g$  is differentiable at  $b = f(a)$ , then  $h$  is differentiable at  $a$ , and*

$$D_h(a) = D_g(b) \cdot D_f(a).$$

*Proof:* Set  $A = D_f(a)$ ,  $B = D_g(b)$ . We need to check that

$$r(x) \stackrel{\text{def}}{=} h(x) - h(a) - B \cdot A(x - a) \stackrel{??}{=} o(|x - a|), \quad x \rightarrow a.$$

We have

$$u(x) \stackrel{\text{def}}{=} f(x) - f(a) - A(x - a) = o(|x - a|), \quad x \rightarrow a,$$

$$v(y) \stackrel{\text{def}}{=} g(y) - g(b) - B(y - b) = o(|y - b|), \quad y \rightarrow b.$$

Therefore,  $r(x)$  can be written as

$$r(x) = \underbrace{g(f(x)) - g(b) - B(f(x) - b)}_{v(f(x))} + \underbrace{B(f(x) - f(a) - A(x - a))}_{Bu(x)}.$$

We estimate the terms on the RHS.

For an arbitrary small positive  $\epsilon$ , we choose  $\eta > 0$  such that

$$|v(y)| \leq \epsilon|y - b|, \quad |y - b| \leq \eta.$$

Then we choose  $\delta > 0$  so small that, for  $|x - a| \leq \delta$ ,

$$|f(x) - f(a)| \leq \eta, \quad \text{and} \quad |u(x)| \leq \epsilon|x - a|.$$

With this choice, we have

$$|v(f(x))| \leq \epsilon|f(x) - b| = \epsilon|A(x - a) + u(x)| \leq \epsilon \cdot \|A\| \cdot |x - a| + \epsilon^2|x - a|,$$

and

$$|Bu(x)| \leq \|B\| \cdot |u(x)| \leq \epsilon \cdot \|B\| \cdot |x - a|.$$

Putting together, this gives us

$$|r(x)| \leq (\epsilon \cdot \|A\| + \epsilon^2 + \epsilon \cdot \|B\|) |x - a| = o(|x - a|).$$

Done! □

As a special case, we get that if  $f: U \rightarrow \mathbb{R}^m$  is differentiable,  $T \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$ , and  $Q \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^k)$ , then the mappings  $f \circ T$  and  $Q \circ f$  are differentiable, and  $D_{f \circ T} = D_f \cdot T$ ,  $D_{Q \circ f} = Q \cdot D_f$ . In particular, we see that if  $f_j = (f, e_j)$ ,  $1 \leq j \leq m$  are coordinate functions of the mapping  $f$ , then  $f$  is differentiable iff all the functions  $f_j$  are differentiable. (Prove!)

**Exercise 3.6.** Let  $f_j: \mathbb{R} \rightarrow \mathbb{R}$  be differentiable functions,  $1 \leq j \leq n$ , and  $f(x) = \sum_j f_j(x_j)$  ( $x_j$  are the coordinates of  $x$ ). Then  $f$  is differentiable and  $Df(x)h = \sum f'_i(x_i)h_i$ .

**Exercise 3.7.** Show that the function  $f(x) = |x|$ ,  $x \in \mathbb{R}^n$ , is differentiable on  $\mathbb{R}^n \setminus \{0\}$ , and find its derivative.

Differentiate the function  $x \mapsto |x - y|^2$ ,  $x \in \mathbb{R}^n$ .

**Exercise 3.8.** Suppose  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a non-constant homogeneous  $C^1$ -function; i.e.  $f(tx) = t^k f(x)$  for all  $x \in \mathbb{R}^n$ ,  $t > 0$ . Prove:

(i)  $k \geq 1$ ,

(ii)

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = kf$$

(Euler's identity).

**Exercise 3.9.** If  $f$  and  $g$  are differentiable mappings, and  $\varphi = \langle f, g \rangle$ , then

$$D\varphi h = \langle D_f h, g \rangle + \langle f, D_g h \rangle.$$

Now, we shall turn to the derivatives of scalar functions of several variables.

## 3.2 The gradient

Now,  $f: U \rightarrow \mathbb{R}$ , and the derivative  $D_f(a)$  is a linear functional on  $\mathbb{R}^n$ . Therefore, there exists the vector, denoted  $\nabla f(a)$ , such that

$$(3.10) \quad D_f(a)h = (\nabla f(a), h), \quad h \in \mathbb{R}^n.$$

This vector called the *gradient* of the function  $f$  at the point  $a$ . For  $|h| = 1$ , the expression (3.10) is called the *derivative of  $f$  in direction  $h$* . We have

$$f(a + th) = f(a) + t(\nabla f(a), h) + \dots$$

This yields the following *geometric properties* of the gradient:

- $f$  increases in the directions  $h$  where  $(\nabla f(a), h) > 0$ , and decreases in the directions  $h$  where  $(\nabla f(a), h) < 0$ . Direction of the fastest increase is  $h = \frac{\nabla f(a)}{|\nabla f(a)|}$ , direction of the steepest descent is the opposite one:  $h = -\frac{\nabla f(a)}{|\nabla f(a)|}$ . The rate of increase (decay) in  $f$  is measured by the length  $|\nabla f(a)|$ .
- If  $f$  has a local extremum at  $a$ , then  $\nabla f(a) = 0$ . The points where the gradient vanish are called the *critical points* of the function  $f$ .
- $\nabla f(a)$  is orthogonal to the level set  $S = \{x: f(x) = f(a)\}$ .

Let us comment the last claim. Consider a differentiable curve  $\gamma: I \rightarrow S$ ,  $I = (-c, c)$ ,  $\gamma(0) = a$ . Then  $f(\gamma(t)) = f(a)$ , and

$$0 = \left. \frac{d}{dt} f(\gamma(t)) \right|_{t=0} = (\nabla f(a), \gamma'(0)) .$$

That is, the vectors  $\nabla f(a)$  and  $\gamma'(0)$  are orthogonal to each other<sup>5</sup>.

**Question 3.11.** Whether the gradient depends on the choice of the inner product in  $\mathbb{R}^n$ ? Justify your answer.

**Exercise 3.12** (Rolle theorem in  $\mathbb{R}^n$ ). Let  $U \subset \mathbb{R}^n$  be a bounded domain,  $f: \bar{U} \rightarrow \mathbb{R}^1$  be a continuous function differentiable on  $U$ , and vanishing on the boundary  $\partial U$ . Show that there exists  $x \in U$  such that  $D_f(x) = 0$ .

### 3.3 The partial derivatives

Choose the orthogonal coordinates in  $\mathbb{R}^n$ , i.e. fix an orthonormal basis  $\{e_1, \dots, e_n\}$  in  $\mathbb{R}^n$ . Then

$$D_f(a)h = D_f(a) \sum_i h_i e_i = \sum_i h_i \cdot D_f(a) e_i .$$

---

<sup>5</sup>The vector  $\gamma'(0)$  is called the *tangent vector* to  $S$  at the point  $a$ .

The real numbers  $D_f(a)e_i = (\nabla f(a), e_i)$  are called the *partial derivatives* of  $f$  at  $a$ , and are denoted by  $\frac{\partial f}{\partial x_i}(a)$ ,  $f_{x_i}(a)$ , or by  $\partial_i f(a)$ . That is,

$$\nabla f(a) = \begin{pmatrix} \partial_1 f(a) \\ \dots \\ \partial_n f(a) \end{pmatrix}.$$

Equivalently, the partial derivatives can be defined as

$$\partial_i f(a) = \lim_{t \rightarrow 0} \frac{f(a + te_i) - f(a)}{t}.$$

Probably, you've started with this definition in the course Hedva 2.

Existence of partial derivatives *does not* imply differentiability of  $f$ . Look at the function

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

Its partial derivatives exist everywhere in the plane and vanish at the origin ( $f(x, 0) = f(0, y) = 0 \implies \partial_x f(0, 0) = \partial_y f(0, 0) = 0$ ). Thus, if  $f$  would be differentiable, its derivative at the origin must be the zero linear functional.

Then

$$\frac{\left| f(x, y) - f(0, 0) - \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right|}{\left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right|} = \frac{x^2 |y|}{(x^2 + y^2)^{3/2}}.$$

Substituting  $x = y = t$ , we get

$$\frac{t^2 |t|}{(2t^2)^{3/2}} = \frac{1}{2^{3/2}} \neq 0.$$

Thus,  $f$  is not differentiable at the origin.

Existence of partial derivatives does not yield even continuity, as the function

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0) \end{cases}$$

shows. This function has partial derivatives everywhere in the plane but is discontinuous at the origin.

**Exercise 3.13.** Fill the details.

If the partial derivatives are continuous, then the function must be continuously differentiable:

**Theorem 3.14.** *TFAE:*

(i)  $f \in C^1(U)$ ;

(ii) Everywhere in  $U$ , there exist continuous partial derivatives  $\partial_i f$ .

*Proof:*

(i)  $\implies$  (ii):

$$|\partial_i f(a) - \partial_i f(b)| = |(D_f(a) - D_f(b)) e_i| \leq \|D_f(a) - D_f(b)\|, \quad 1 \leq i \leq n.$$

(ii)  $\implies$  (i). First, we prove that  $f$  is differentiable at  $U$ . To simplify notations, we assume that we deal with the function of two variables. Then

$$\begin{aligned} f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) &= f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2 + h_2) + f(a_1, a_2 + h_2) - f(a_1, a_2) \\ &= f_{x_1}(a_1 + \theta_1 h_1, a_2 + h_2) h_1 + f_{x_2}(a_1, a_2 + \theta_2 h_2) h_2. \end{aligned}$$

In the last line, we used the mean value property for differentiable functions of one variable  $t \mapsto f(t, a_2 + h_2)$  and  $t \mapsto f(a_1, t)$ . Using the continuity of the partial derivatives, we get

$$\begin{aligned} f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) &= (f_{x_1}(a_1, a_2) + o(1)) h_1 + (f_{x_2}(a_1, a_2) + o(1)) h_2 \\ &= f_{x_1}(a_1, a_2) h_1 + f_{x_2}(a_1, a_2) h_2 + o(\sqrt{h_1^2 + h_2^2}). \end{aligned}$$

That is, the function  $f$  is differentiable at the point  $a = (a_1, a_2)$ .

It remains to check the continuity of the derivative  $D_f$ . It is obvious, since

$$\|D_f(a) - D_f(b)\| = \sqrt{\sum_{i=1}^n (\partial_i f(a) - \partial_i f(b))^2}.$$

Done! □

**Corollary 3.15.**  $f \in C^1(U, \mathbb{R}^m)$  iff all partial derivatives are continuous:  $\frac{\partial f_j}{\partial x_i} \in C^1(U)$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ .

The matrix consisting of partial derivatives

$$\begin{pmatrix} \partial_1 f_1 & \cdots & \partial_n f_1 \\ \vdots & \ddots & \vdots \\ \partial_1 f_m & \cdots & \partial_n f_m \end{pmatrix}$$

is called the *Jacobi matrix* of the mapping  $f$ . In the case  $m = n$ , the determinant of this matrix is called the *Jacobian* of the mapping  $f$ . Both will play a very important role in this course.

**Exercise 3.16.** Let  $f(x, y)$  be a differentiable function, and  $g(r, \theta) = f(r \cos \theta, r \sin \theta)$ . Show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial g}{\partial \theta}\right)^2.$$

**Exercise 3.17.** Let  $\text{Mat}_n(\mathbb{R})$  be the linear space of  $n \times n$  matrices with real entries, and  $\det: \text{Mat}_n(\mathbb{R}) \rightarrow \mathbb{R}$  be the determinant.

- (i) The function  $\det$  is continuously differentiable on  $\text{Mat}(\mathbb{R})$ . For each  $A$  the derivative at  $A$ , i.e.  $(D\det)(A)$ , is a linear functional on  $n \times n$  matrices.
- (ii) if  $I$  is the unit matrix, then  $(D\det)(I)H = \text{tr}(H)$ ;
- (iii) if the matrix  $A$  is invertible, then  $(D\det)(A)H = (\det A)^{-1} \text{tr}(A^{-1}H)$ ;
- (iv) in the general case,  $(D\det)(A)H = \text{tr}(A^\sharp H)$ , where  $A^\sharp$  is the *complementary matrix* to  $A$ , that is,  $A^\sharp_{i,j}$  equals  $(-1)^{i-j}$  times the determinant of the  $(n-1) \times (n-1)$  matrix obtained from  $A$  by deleting the  $i$ -th row and the  $j$ -th column.

### 3.4 The mean-value theorem

**Theorem 3.18.** Let  $f: U \rightarrow \mathbb{R}$  be a differentiable function, and let  $[a, b] \subset U$  be a closed segment. Then there exists  $\xi \in (a, b)$  such that

$$f(b) - f(a) = (\nabla f(\xi), b - a) = D_f(\xi)(b - a).$$

*Proof:* Define the function

$$\varphi(t) \stackrel{\text{def}}{=} f(a + t(b - a)), \quad 0 \leq t \leq 1,$$

and apply the one-dimensional mean-value theorem. □

**Corollary 3.19.** In the assumptions of the previous theorem,

$$|f(b) - f(a)| \leq \sup_{\xi \in (a, b)} \|D_f(\xi)\| \cdot |b - a|.$$

**Exercise 3.20.** 1. Suppose  $U \subset \mathbb{R}^n$  is an open convex set,  $f: U \rightarrow \mathbb{R}^1$  is a differentiable function, such that  $\|D_f\| \leq M$  everywhere in  $U$ . Then, for any  $a, b \in U$ ,  $|f(b) - f(a)| \leq M|b - a|$ .

2. Construct a non-convex domain  $U \subset \mathbb{R}^2$  and a function  $f \in C^1(U)$  such that  $\|D_f\| \leq 1$  everywhere in  $U$ , but  $|f(b) - f(a)| > |b - a|$  for some  $a, b \in U$ .

**Exercise 3.21.** 1. Let  $U \subset \mathbb{R}^2$  be a convex domain. Let  $f$  be a  $C^1$ -function in  $U$ . If  $\partial_1 f \equiv 0$  everywhere in  $U$ , then  $f$  does not depend on  $x_1$ .

2. Whether the result from item 1. persists if  $U$  is an arbitrary domain in  $\mathbb{R}^n$ ? (Prove or disprove by a counterexample).

### 3.5 Derivatives of high orders

In this course, we shall use the high order derivatives only occasionally. So here, we restrict ourselves by few comments.

Partial derivatives of higher orders are defined recursively; e.g.

$$\partial_{ij}^2 = \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \right)$$

etc. A very important fact, which you certainly know from Hedva 2 says that under certain assumptions it does not matter in which order to take the partial derivatives.

**Theorem 3.22.** *If the mixed derivatives  $\partial_{ij}^2 f$ ,  $\partial_{ji}^2 f$  of a scalar function  $f$  exist and continuous at  $a$ , then*

$$\partial_{ij}^2 f(a) = \partial_{ji}^2 f(a).$$

If you are not sure that you remember this result, I strongly suggest to look at any analysis textbook.

The next six exercises also pertain to Hedva-2, rather than to Hedva-3.

**Exercise 3.23.** Suppose  $\Delta = \sum_{j=1}^n \partial_j^2$  (this is the second order differential operator in  $\mathbb{R}^n$  called *Laplacian*).

(i) Check that  $\Delta(f \cdot g) = f \Delta g + 2(\nabla f, \nabla g) + g \Delta f$ .

(ii) Compute Laplacian of the functions  $f(x, y) = \log(x^2 + y^2)$ ,  $(x, y) \in \mathbb{R}^2 \setminus \{0\}$ , and  $f(x) = |x|^{-n+2}$ ,  $x \in \mathbb{R}^n \setminus \{0\}$ .

(iii) Let  $\mathcal{R}_\theta$  be the (counterclockwise) rotation of the plane by angle  $\theta$ , and  $\mathcal{F}_\theta: f \mapsto f \circ \mathcal{R}_\theta$  be the composition operator. Check that the operators  $\Delta$  and  $\mathcal{F}_\theta$  commute, i.e.,  $\Delta \circ \mathcal{F}_\theta = \mathcal{F}_\theta \circ \Delta$ .

**Exercise 3.24.** The function

$$f(x, t) = \begin{cases} \frac{1}{\sqrt{t}} e^{-x^2/4t} & t > 0 \\ 0 & t \leq 0, \quad (x, t) \neq (0, 0) \end{cases}$$

is infinitely differentiable in  $\mathbb{R}^2 \setminus \{(0, 0)\}$  and satisfies therein the *heat equation*  $f_{xx} = f_t$ .

**Exercise 3.25** (combinatorics). How many  $m$ -th order partial derivatives has an infinitely differentiable function of 3 variables? of  $n$  variables?

**Exercise 3.26.** Let  $f$  be an infinitely differentiable function on  $\mathbb{R}^3$ , and  $\varphi(t) = f(t, t^2, t^3)$ . Find the first 4 terms of the Taylor expansion of  $\varphi$  at  $t = 0$ .

**Exercise 3.27.** Show that the point  $(0, 0)$  is a local extremum of the function

$$\frac{1}{\sqrt{(1-x)^2 + (1-y)^2}} + \frac{1}{\sqrt{(1+x)^2 + (1-y)^2}} \\ + \frac{1}{\sqrt{(1-x)^2 + (1+y)^2}} + \frac{1}{\sqrt{(1+x)^2 + (1+y)^2}}.$$

Find out whether this is a local minimum or local maximum.

**Exercise 3.28.** Suppose  $f$  is a  $C^2$ -function such that  $\Delta f \geq 0$  ( $\Delta$  is the Laplacian). Prove that  $f$  does not have strong local maxima

Hint: start with a stronger assumption  $\Delta f > 0$ .

The second derivative  $D_f^2$  of a scalar function  $f$  is the derivative of the map  $D_f: U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^1)$ , that is  $D_f^2$  is an element of the space  $\mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^1))$ . If  $D_f^2$  exists and continuous in  $U$ , then we say that  $f$  is twice continuously differentiable in  $U$  and write  $f \in C^2(U)$ . Elements of the linear space  $\mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^1))$  can be identified with ( $\mathbb{R}^1$ -valued) bilinear forms; i.e. with functions  $\varphi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^1$  linear with respect to each argument: if  $\Phi \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^1))$ , then  $\varphi(h_1, h_2) = (\Phi h_1) h_2$ . Therefore, the second derivative  $D_f^2$  can be viewed as a symmetric ( $\mathbb{R}^1$ -valued) bilinear form.

Similarly, if  $f: U \rightarrow \mathbb{R}^m$ , then  $D_f^2$  can be regarded as a symmetric  $\mathbb{R}^m$ -valued bilinear form.

The higher derivatives are defined by recursion with respect to the order  $k$ . If the function  $f$  has continuous derivatives of all order  $k$  in  $U$ , then we say that is  $C^k$ -smooth.

**Problem 3.29.** Let  $f: U \rightarrow \mathbb{R}^m$  be a  $C^k$ -smooth mapping. Identify the  $k$ -th derivative  $D_f^k$  with a symmetric  $k$ -linear  $\mathbb{R}^m$ -valued form; i.e. a mapping  $\underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k \text{ times}} \rightarrow \mathbb{R}^m$  which is linear with respect to each argument

(when the others are fixed) and symmetric.

Hint: use induction with respect to  $k$ .



## 4 Inverse Function Theorem

In this lecture, we address the following problem. Let  $f$  be a  $C^1$ -mapping. Consider equation  $f(x) = y$ . When it can be (locally) inverted? what are the properties of the inverse mapping  $f^{-1}$ ? We expect that locally  $f$  behaves similarly to its derivative  $D_f$ ; i.e. if  $D_f(a)$  is invertible, then  $f$  maps in a one-to-one way a neighbourhood of  $a$  onto a neighbourhood of  $b = f(a)$ , that the inverse map is differentiable at  $b$ , and  $D_{f^{-1}}(b) \cdot D_f(a) = I$  (the identity map). The main result of this lecture (which is the first serious theorem in this course!) confirms that prediction. However, first, we will prove the following simple

**Claim 4.1.** *Let  $U \subset \mathbb{R}^n$  be an open set,  $f: U \rightarrow \mathbb{R}^m$  be a differentiable function with a differentiable inverse. Then  $m = n$ .*

*Proof:* We have  $x = f^{-1} \circ f(x)$ , and the chain rule is applicable:

$$D_{f^{-1}}(f(x)) \cdot D_f(x) = I_n$$

(identity map). Recall that  $D_f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ . Then a result from the Linear Algebra course insists that  $n \leq m$ . (Recall the result!). By symmetry,  $m \leq n$ . Thus  $m = n$ , completing the proof.  $\square$

### 4.1 The Theorem

Our standing assumptions are:  $U \subset \mathbb{R}^n$  is a domain,  $a \in U$ ,  $f: U \rightarrow \mathbb{R}^n$  is a  $C^1$ -mapping.

**Theorem 4.2.** *Suppose that the linear map  $D_f(a)$  is invertible. Then*

- there are a neighbourhood  $U = U_a$  of  $a$  and a neighbourhood  $V = V_b$  of  $b = f(a)$  such that  $f|_U$  is a one-to-one mapping, and  $f(U) = V$ ;*
- the inverse map  $g: V \rightarrow U$  is a  $C^1$ -mapping, and  $D_g(b) \cdot D_f(a) = I_n$ .*

**Examples:**

- $(n = 1)$  The function of one variable

$$f(t) = \begin{cases} t + 2t^2 \sin(1/t) & t \neq 0, \\ 0 & t = 0 \end{cases}$$

is differentiable, and the derivative  $f'(t)$  is bounded everywhere. However, this function is not one-to-one on any neighbourhood of the origin. Here, the  $C^1$ -assumption is violated.

2. ( $n = 2$ ) The mapping  $f$  defined as

$$\begin{cases} x_1 = e^x \cos y, \\ y_1 = e^x \sin y \end{cases}$$

meets conditions of the Inverse Function Theorem.  $f$  maps  $\mathbb{R}^2$  onto  $\mathbb{R}^2 \setminus \{0\}$ , but for any  $w \in \mathbb{R}^2 \setminus \{0\}$ , the equation  $f(z) = w$  has infinitely many solutions. This shows that the IFT can be applied only locally.

## 4.2 Continuity of the inversion of linear operators

We shall prove the linear algebra claim that we need in the proof of the IFT. By  $GL_n$  we denote the group of all invertible linear mappings from  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ .

**Claim 4.3.** 1. Suppose  $A \in GL_n$ , and  $B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  is such that

$$\|B - A\| < \frac{1}{\|A^{-1}\|},$$

then  $B \in GL_n$  as well.

2. The mapping  $A \mapsto A^{-1}$  is continuous in the operator norm.

The second assertion says that if a sequence of operators  $B_k$  converges to  $A \in GL_n$  in the operator norm, then according to the item 1,  $B_k \in GL_n$  for sufficiently large  $k$ , and  $B_k^{-1}$  also converges to  $A^{-1}$ .

*Proof of the claim:*

1. Set

$$\alpha = \frac{1}{\|A^{-1}\|}, \quad \beta = \|B - A\|.$$

Then

$$|Bx| \geq |Ax| - |(B - A)x| \geq |Ax| - \beta|x|.$$

Since,

$$|x| = |A^{-1}Ax| \leq \alpha^{-1}|Ax|,$$

we get  $|Ax| \geq \alpha|x|$ , and then  $|Bx| \geq (\alpha - \beta)|x| > 0$  for all  $x \in \mathbb{R}^n \setminus \{0\}$ . Thus,  $B$  is invertible.

2. Start with the identity

$$B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1},$$

then

$$\|B^{-1} - A^{-1}\| \leq \|B^{-1}\| \cdot \|A - B\| \cdot \|A^{-1}\|.$$

We already know from the first part that  $|Bx| \geq (\alpha - \beta)|x|$ , or

$$(\alpha - \beta)|B^{-1}y| \leq |BB^{-1}y| = |y|,$$

that is,

$$\|B^{-1}\| \leq \frac{1}{\alpha - \beta},$$

and finally

$$\|B^{-1} - A^{-1}\| \leq \frac{\beta}{\alpha(\alpha - \beta)}.$$

Therefore,  $\|B^{-1} - A^{-1}\|$  can be made arbitrary small, if  $\beta = \|B - A\|$  is small enough (and  $\alpha$  is fixed).  $\square$

### 4.3 Proof of the IFT

The proof will be split into several parts.

Set

$$A = D_f(a), \quad \lambda = \frac{1}{4\|A^{-1}\|},$$

and choose a sufficiently small ball  $B$  centered at  $a$  such that

$$\sup_{x \in B} \|D_f(x) - A\| \leq \lambda.$$

(Why this is possible?) This choice guarantees that  $D_f(x)$  is invertible everywhere in  $B$ .

#### 4.3.1 The map $f$ is one-to-one in $B$ .

Let  $x, x+h \in B$ . We shall estimate from below the distance  $|f(x+h) - f(x)|$ .

We need the following

**Claim 4.4.**

$$|f(x+h) - f(x) - Ah| \leq \frac{1}{2}|Ah|.$$

*Proof:* introduce the  $\mathbb{R}^n$ -valued function of one variable

$$\varphi(t) \stackrel{\text{def}}{=} f(x+th) - tAh.$$

Then  $\varphi'(t) = D_f(x+th)h - Ah$ , and

$$\|\varphi'(t)\| \leq \|D_f(x+th) - A\| \cdot |h| \leq \lambda \cdot |h| = \frac{1}{4\|A^{-1}\|} \cdot |h| \leq \frac{1}{4}|Ah|.$$

Therefore,

$$\|\varphi(1) - \varphi(0)\| = \left\| \int_0^1 \varphi'(t) dt \right\| \leq \int_0^1 \|\varphi'(t)\| dt \leq \frac{1}{4}|Ah|,$$

proving the claim.  $\square$

Now,

$$\begin{aligned} |f(x+h) - f(x)| &\geq |Ah| - |f(x+h) - f(x) - Ah| \stackrel{4.4}{\geq} \frac{1}{2}|Ah| \\ &\geq \frac{1}{2\|A^{-1}\|} \cdot |h| = 2\lambda|h|, \end{aligned}$$

that is,  $f$  is one-to-one on the ball  $B$ .

We shall record the estimate

$$(4.5) \quad |f(x+h) - f(x)| \geq 2\lambda|h|$$

for future references.

### 4.3.2 Surjectivity

We shall show that  $f(B)$  is an open set. This will be the neighbourhood  $V$  of  $b$ , where the inverse map  $f^{-1}$  exists.

Take  $y_0 \in f(B)$ ,  $y_0 = f(x_0)$  ( $x_0 \in B$ ), and take a ball  $B' = B(x_0, r)$  such that  $\overline{B'} \subset B$ .

**Claim 4.6.**

$$B(y_0, \lambda r) \subset f(B').$$

Of course, this claim yields that the image  $f(B)$  is an open set.

*Proof of the claim:* Take any  $y$  such that  $|y - y_0| < \lambda r$ . We are looking for the point  $x^* \in B'$  such that  $y = f(x^*)$ . It is natural to try to find  $x^*$  by minimizing the function  $\varphi(x) = |y - f(x)|^2$  over the closed ball  $\overline{B'}$ .

First of all, show that the minimum is not attained on the boundary sphere. Observe that  $\varphi(x_0) = |y - f(x_0)|^2 < (\lambda r)^2$  (due to the choice of  $y$ ). On the other hand, if  $x \in S' = \partial B'$ , then

$$\begin{aligned} \sqrt{\varphi(x)} &= |f(x) - y| > |f(x) - y| + |f(x_0) - y| - \lambda r \\ &\geq |f(x) - f(x_0)| - \lambda r \stackrel{(4.5)}{\geq} 2\lambda|x - x_0| - \lambda r = 2\lambda r - \lambda r = \lambda r, \end{aligned}$$

whence  $\varphi(x) > (\lambda r)^2$ . Thus,  $\varphi$  does not achieve its minimum on the boundary of  $B'$ .

Let  $x^* \in B'$  be the minimum point of  $\varphi$ , then  $D_\varphi(x^*) = 0$ .  
Differentiating  $\varphi(x) = (y - f(x), y - f(x))$ , we get

$$D_\varphi(x)h = -2\langle D_f(x)h, y - f(x) \rangle,$$

and we arrive at the equation

$$\langle D_f(x^*)h, y - f(x^*) \rangle = 0, \quad \text{for any } h \in \mathbb{R}^n.$$

Since the linear map  $D_f(x)$  is invertible everywhere on  $B'$ , and in particular at  $x^*$ , the range of  $D_f(x^*)$  coincides with  $\mathbb{R}^n$ ; i.e.,  $\{D_f(x^*)h\}_{h \in \mathbb{R}^n} = \mathbb{R}^n$ . Since  $y - f(x^*)$  belongs to the orthogonal complement to this set, we get  $y - f(x^*) = 0$ , that is  $y = f(x^*)$ , proving Claim 4.6.  $\square$

### 4.3.3 Continuous differentiability of the inverse map

Let  $g = f^{-1}$ . It remains to show, that  $g \in C^1(V)$ , where  $V = f(B)$ , and that  $D_g = D_f^{-1}$ .

As in the one-dimensional case, first, we shall see that  $g$  is a continuous map, then we check that  $g$  is differentiable and  $D_g = D_f^{-1}$ , and then that the derivative  $D_g$  is continuous on  $V$ .

Let  $y, y + k \in V = f(B)$ . We need to estimate the norm of  $h = g(y + k) - g(y)$ . Set  $x = g(y)$ . Then

$$f(x + h) - f(x) = f(g(y + k)) - y = y + k - y = k,$$

and by estimate (4.5)

$$|k| = |f(x + h) - f(x)| \geq 2\lambda|h| = 2\lambda|g(y + k) - g(y)|.$$

This gives us continuity of  $g$ .

Now, let  $L = D_f^{-1}$ . Since

$$k = f(x + h) - f(x) = D_f(x)h + r_x(h),$$

we get  $Lk = h + Lr_x(h)$ , or

$$g(y + k) - g(y) - Lk = h - Lk = -Lr_x(h).$$

We shall estimate the norm of this expression. Since  $f$  is differentiable,  $|r_x(h)| \leq \epsilon|h|$ , provided that  $|k|$  (and therefore  $|h|$ ) is sufficiently small. Hence

$$|Lr_x(h)| \leq \|L\| \cdot |r_x(h)| \leq \epsilon\|L\| \cdot |h| \stackrel{(4.5)}{\leq} \epsilon\|L\| \cdot \frac{|k|}{2\lambda}.$$

This shows that

$$\lim_{k \rightarrow 0} \frac{|Lr_x(h)|}{|k|} = 0.$$

Hence,  $g$  is differentiable at  $y$ , and its derivative is

$$D_g(y) = L = D_f(x)^{-1}.$$

It remains to check the continuity of the map  $y \mapsto D_g(y)$ . Let us look again at the formula we've obtained:

$$D_g(y) = (D_f(g(y)))^{-1}.$$

The RHS is a composition of three mappings:  $y \mapsto g(y)$ ,  $a \mapsto D_f(a)$ , and  $L \mapsto L^{-1}$ . The first two are continuous. The third map  $L \mapsto L^{-1}$  is also continuous (item 2. of Claim 4.3). This does the job and finishes off the long proof of the Inverse Function Theorem.  $\square$

**Exercise 4.7.** The mapping  $f$  maps the coefficients  $a_1, a_2, a_3$  of the equation  $x^3 + a_1x^2 + a_2x + a_3 = 0$  into its roots  $x_1 \leq x_2 \leq x_3$ . Prove that  $f$  is a  $C^1$ -mapping defined in a neighbourhood of the point  $a_1 = -3, a_2 = 2$  and  $a_3 = 0$ , and compute Jacobian of  $f$  at this point.

**Exercise 4.8.** The mapping  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  maps  $x_1, x_2, \dots, x_n$  into the coefficients  $a_1, a_2, \dots, a_n$  of the polynomial  $(x - x_1)(x - x_2) \cdot \dots \cdot (x - x_n) = x^n + a_1x^{n-1} + \dots + a_n$ .

(i) Prove that the rank of  $D_f$  equals the number of *distinct* values among  $x_1, \dots, x_n$ .

(ii) Compute the Jacobian of  $f$ .

Hint: start with the cases  $n = 2$  and  $n = 3$ .

**Exercise 4.9.** Let  $f$  be a  $C^1$ -mapping. Show that the rank of  $D_f$  is an *upper semi-continuous function*, i.e.,  $\text{rank}D_f(x) \geq \text{rank}D_f(x_0)$  in a neighbourhood of the point  $x_0$ .

In the next two lectures, we shall show how powerful the Inverse Function Theorem is. We exhibit several of its important consequences:

- the open mapping theorem;
- the implicit function theorem;
- the Lagrange multipliers method.

## 5 Open Mapping Theorem and Lagrange Multipliers

### 5.1 Open Mapping Theorem

**Definition 5.1.** The mapping  $f: U \rightarrow \mathbb{R}^m$  is called *open* if for any open subset  $U' \subset U$  the image  $f(U')$  is also open.

**Definition 5.2.** The  $C^1$ -mapping  $f$  is *regular*, if, for all  $a \in U$ ,  $\text{rank } D_f(a) = m$ .

Recall that, for  $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ ,  $\text{rank } L = \dim(L\mathbb{R}^n)$  (in other words, the maximal number of linear independent columns in the matrix representation of  $L$ ).

**Theorem 5.3.** *Regular mappings are open.*

We shall prove a local version of this result which says:

$$(5.4) \quad \text{rank } D_f(a) = m \implies f(a) \in \text{int } f(U).$$

Of course, the Open Mapping Theorem follows from (5.4). If  $m = n$ , then (5.4) is a part of the Inverse Function Theorem. Now, we shall reduce the general case to this special one.

First take a linear map  $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$  such that  $D_f(a) \cdot T$  is invertible. Such a map  $T$  exists. Indeed, since  $\text{rank } D_f(a) = m$ , we have  $\dim \text{Ker } D_f(a) = n - m$ . Let  $Y$  be the orthogonal complement to  $\text{Ker } D_f(a)$  in  $\mathbb{R}^n$ ,  $\dim Y = m$ . As we know from the Linear Algebra course, the mapping  $D_f(a)$  is one-to-one on the subspace  $Y$ . Choose any one-to-one linear mapping  $T: \mathbb{R}^m \rightarrow Y$ , then  $D_f(a) \cdot T$  is invertible.

Now, consider the function  $g(x) \stackrel{\text{def}}{=} f(a + Tx)$ . Its derivative at the origin  $D_g(0) = D_f(a) \cdot T$  is invertible. Thus by the Inverse Function Theorem, there is a neighbourhood  $V$  of the point  $g(0) = f(a)$  which belongs to  $f(U)$ . That is,  $f(a)$  lies in the interior of the image  $f(U)$ .  $\square$

### 5.2 Lagrange Multipliers

In the courses Hedva 1 and Hedva 2 you've learnt how to find extrema of functions of one and several variables. Here, we learn how to find the extremal values of functions when the variables are subjects to additional restrictions.

Let  $U \subset \mathbb{R}^n$  be an open set, the functions  $f, g_1, \dots, g_k$  are  $C^1$  (scalar) functions defined on  $U$ ,

$$M = \{x \in U: g_1(x) = \dots = g_k(x) = 0\}.$$

We want to find extremal values of  $f$  when  $x \in U$  is subject to additional restrictions given by  $x \in M$ . Such extremal values are called *conditional*.

**Theorem 5.5.** *Let  $a \in M$  be a conditional extremum of  $f$ . Suppose that the vectors  $\nabla g_1(a), \dots, \nabla g_k(a)$  are linearly independent vectors. Then the linear span of these vectors contains the vector  $\nabla f(a)$ .*

In other words, there exist constants  $\lambda_1, \dots, \lambda_k$  such that

$$\nabla f(a) = \sum_{j=1}^k \lambda_j \nabla g_j(a).$$

*Proof:* Define the map  $H: U \rightarrow \mathbb{R}^{k+1}$  by

$$H = \begin{pmatrix} f \\ g_1 \\ \dots \\ g_k \end{pmatrix}.$$

This is a  $C^1$ -map, and  $\text{rank}(D_H(a)) \geq k$  (since the vectors  $\nabla g_1(a), \dots, \nabla g_k(a)$  are linearly independent). Assume that  $a$  is a conditional minimum of  $f$ :

$$f(x) \geq f(a) \quad x \in M \cap U_a,$$

where  $U_a$  is a neighbourhood of  $a$ . If  $\text{rank}(D_H(a)) = k+1$ , then by the Open Mapping Theorem

$$\begin{pmatrix} f(a) \\ 0 \\ \dots \\ 0 \end{pmatrix} = H(a) \in \text{int } H(U_a).$$

But then there exist  $t < f(a)$  and  $x \in U_a$  such that

$$H(x) = \begin{pmatrix} t \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

In other words, there is  $x \in M$  such that  $f(x) = t < f(a)$ . Contradiction!

Hence  $\text{rank } D_H(a) = k$ , and the vector  $\nabla f(a)$  belongs to the linear span of the vectors  $\nabla g_1(a), \dots, \nabla g_k(a)$ .  $\square$

Now, we bring several applications of the Lagrange multipliers technique.



### 5.2.1 Geometrical extremal problems

We start with simple geometrical problems taken from the ‘high-school analytic geometry’:

**Find the distance from the origin to the affine hyperplane  $\sum \alpha_i x_i = c$  in  $\mathbb{R}^n$ .** *Solution:* Here, we minimize the function  $f(x) = \sum x_i^2$  under condition  $g(x, y, z) = \sum \alpha_i x_i - c = 0$ . In this case, the Lagrange equations are

$$\begin{cases} 2x_i = \lambda \alpha_i, & 1 \leq i \leq n, \\ \sum \alpha_i x_i = c. \end{cases}$$

We get

$$c = \frac{\lambda}{2} \sum \alpha_i^2,$$

or

$$\lambda = \frac{2c}{\sum \alpha_i^2}.$$

Substituting this value into the first  $n$  of the Lagrange equations, we get the coordinates of the point where  $f$  attains the conditional maximum:

$$x_i = \frac{c \alpha_i}{\sum \alpha_i^2}, \quad 1 \leq i \leq n.$$

The distance is

$$\sqrt{\sum x_i^2} = \frac{|c|}{\sqrt{\sum \alpha_i^2}}.$$

**Exercise 5.6.** Find the point on the line

$$\begin{cases} \alpha x + \beta y + \gamma z = c \\ x + y + z = 1 \end{cases}$$

the closest to the origin.

**Exercise 5.7.** The closed curve  $\Gamma \subset \mathbb{R}^3$  is defined as the intersection of the ellipsoid

$$\sum_{j=1}^3 \frac{x_j^2}{a_j^2} = 1,$$

and the plane

$$\sum_{j=1}^3 A_j x_j = 0.$$

Find the points on  $\Gamma$  which are the closest and the most distant from the origin.

**Isoperimetry for Euclidean triangles** By  $A$  we denote the area, and by  $L$  the length. The Dido isoperimetric inequality says that, for any plane figure  $G$ ,

$$A(G) \leq \frac{L(\partial G)^2}{4\pi},$$

and equality is attained for discs only. For triangles, the estimate can be improved:

**Theorem 5.8** (Heron). *For any plane triangle  $\Delta$ ,*

$$(5.9) \quad A(\Delta) \leq \frac{L(\partial\Delta)^2}{12\sqrt{3}},$$

*and the equality sign attains for the equilateral triangles and only for them.*

In other words, among all triangles with the given perimeter, the equilateral one has the largest area.

*Solution:* is based on the Heron formula that relates the area  $A$  and length of the sides  $x$ ,  $y$  and  $z$ :

$$A^2 = \frac{L}{2} \cdot \left(\frac{L}{2} - x\right) \cdot \left(\frac{L}{2} - y\right) \cdot \left(\frac{L}{2} - z\right).$$

Set  $L = 2s$ . Then we need to maximize the function

$$f(x, y, z) = s(s-x)(s-y)(s-z)$$

under condition

$$g(x, y, z) = x + y + z - 2s = 0.$$

Of course, we have additional restrictions

$$x, y, z > 0, \quad x + y > z, \quad x + z > y, \quad y + z > x,$$

which define the domain  $U$  in the space  $(x, y, z)$ . (Draw this domain!) On the boundary of this domain (when the inequalities turn to the equations), the function  $f$  identically vanishes. Thus,  $f$  attains its maximal value inside  $U$  and we can use the Lagrange multipliers.

The Lagrange equations are

$$\begin{cases} -s(s-y)(s-z) = \lambda \\ -s(s-x)(s-z) = \lambda \\ -s(s-x)(s-y) = \lambda \\ x + y + z = 2s \end{cases}$$

The first three equations give us

$$(s - y)(s - z) = (s - x)(s - z) = (s - x)(s - y),$$

whence

$$x = y = z = \frac{2}{3}s,$$

and  $A^2 = s \cdot (s/3)^3$ . The result follows.  $\square$

### 5.2.2 The linear algebra problems

Here, we look at two Linear Algebra problems.

**Extrema of quadratic forms** We are looking for the maximal and minimal values of the symmetric quadratic form

$$f(x) = \sum_{i,j=1}^n a_{ij}x_i x_j \quad a_{ij} = a_{ji},$$

on the unit sphere

$$\sum_{i=1}^n x_i^2 = 1.$$

In this case,  $f(x) = (Ax, x)$ , where  $A$  is a symmetric linear operator with the matrix coefficients  $a_{ij}$ . Thus  $\nabla f(x) = 2Ax$ . Furthermore,  $g(x) = \sum_i x_i^2 - 1$ , and  $\nabla g(x) = 2x$ . Therefore, the Lagrange equations take the form

$$\begin{cases} 2Ax = 2\lambda x \\ (x, x) = 1. \end{cases}$$

Hence,  $\lambda$  is the eigenvalue of  $A$ , and the maximum of the form is the largest eigenvalue, the minimum of the form is the smallest eigenvalue.

**The operator norm** As a corollary, we compute the (operator) norm of a linear operator  $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ . By definition,

$$\|L\| = \max_{|x|=1} |Lx|.$$

Thus, we need to maximize the function  $f(x) = |Lx|^2 = (Lx, Lx)$  under additional condition  $|x|^2 = 1$ .

Observe that  $f(x) = (Lx, Lx) = (L^*Lx, x)$ . Hence, by the previous paragraph,  $\|L\|^2$  equals the maximal eigenvalue of the symmetric matrix  $L^*L$ .

**Exercise 5.10.** Let  $L$  be an invertible linear operator. Find  $\|L^{-1}\|$ .

### 5.2.3 Inequalities

Lagrange multipliers are very useful in proving inequalities. Here are several examples.

**The Hölder Inequality** Let  $1 < p < \infty$ . Then

$$(5.11) \quad \left| \sum x_i \cdot y_i \right| \leq \left\{ \sum |x_i|^p \right\}^{1/p} \cdot \left\{ \sum |y_i|^q \right\}^{1/q},$$

where  $q$  is ‘the dual exponent’ to  $p$ :  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof:* We assume that all  $x_i$ ’s and  $y_i$ ’s are non-negative. Since Hölder’s inequality is homogeneous with respect to multiplication of all  $x_i$  by the same positive number, we assume that  $\sum x_i^p = 1$ . Given  $y \in \mathbb{R}^n$  with non-negative coordinates, define the function  $f(x) = \sum x_i y_i$ . That is, for a compact set  $K = \{x \in \mathbb{R}^n : x_i \geq 0, \sum x_i^p = 1\}$ , we want to prove that  $\max_K f \leq \{\sum y_i^q\}^{1/q}$ . We use induction with respect to the number  $n$  of variables. For  $n = 1$ , we have  $K = \{1\}$ , and there is nothing to prove.

For an arbitrary  $n \geq 2$ , we look at the extremum of  $f$  on  $K$ . We assume that all  $y_i$  are positive (otherwise, the actual number of variables is reduced and we can use the induction assumption). Note that the Lagrange multipliers technique can be applied only on the set  $K_0 = \{x \in \mathbb{R}^n : x_i > 0, \sum x_i^p = 1\}$ . (why?) However, the rest  $K \setminus K_0$  consists of  $x$ ’s such that at least one of the coordinates  $x_i$  vanishes and  $\sum x_i^p = 1$ . Hence, by the assumption of the induction  $\max_{K \setminus K_0} f < \{\sum y_i^q\}^{1/q}$ . Now, using the Lagrange method, we shall find that the conditional extremum of  $f$  under assumptions  $g(x) = \sum x_i^p - 1 = 0$  equals  $\{\sum y_i^q\}^{1/q}$ . Hence, this is the conditional maximum. This will prove Hölder’s inequality (and also will show that it cannot be improved).

The Lagrange equations have the form

$$y_i = \lambda p x_i^{p-1}, \quad 1 \leq i \leq n,$$

$$\sum x_i^p = 1.$$

To simplify notations, set  $\nu = \lambda p$ . Then  $x_i = (y_i/\nu)^{\frac{1}{p-1}}$ ,  $1 \leq i \leq n$ , whence  $1 = \sum (y_i/\nu)^{\frac{p}{p-1}}$ , or  $\nu^{\frac{p}{p-1}} = \sum y_i^{\frac{p}{p-1}}$ . We get

$$x_j = (y_j/\nu)^{\frac{1}{p-1}} = y_j^{\frac{1}{p-1}} \cdot \left\{ \sum y_i^{\frac{p}{p-1}} \right\}^{-1/p}, \quad 1 \leq j \leq n,$$

then

$$f(x) = \sum x_j y_j = \sum y_j^{1+\frac{1}{p-1}} \cdot \left\{ \sum y_i^{\frac{p}{p-1}} \right\}^{-1/p} = \left\{ \sum y_j^q \right\}^{1-\frac{1}{p}} = \left\{ \sum y_j^q \right\}^{\frac{1}{q}}$$

(recall that  $\frac{p}{p-1} = q$ ). □

We proved Hölder's inequality in the case of finitely many variables  $\{x_i\}$  and  $\{y_i\}$ . Note that it persists in the case of countable many variables  $x_i$  and  $y_i$ . In this case, it means that if two series  $\sum |x_i|^p$  and  $\sum |y_i|^q$  converge (and  $q$  is dual to  $p$ ), then the series  $\sum x_i y_i$  also converges and inequality (5.11) holds.

**Exercise 5.12.** Prove that, for  $x_i > 0$ ,

$$\frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n} \leq \frac{x_1 + x_2 + \dots + x_n}{n}.$$

The equality sign attains only in the case when all  $x_i$ 's are equal.

Hint: to get the first inequality, minimize  $x_1 \cdot x_2 \cdot \dots \cdot x_n$  under assumption that all  $x_i$ 's are positive and  $\sum_i x_i^{-1} = 1$ . To get the second inequality, maximize  $x_1 \cdot x_2 \cdot \dots \cdot x_n$  under assumption that all  $x_i$ 's are positive and  $\sum_i x_i = 1$ .

**Exercise 5.13.** Find the maximum of the function  $f(x, y, z) = x^a y^b z^c$  ( $a, b, c > 0$ ), where  $x, y$  and  $z$  are positive, and  $x^k + y^k + z^k = 1$  ( $k > 0$ ).

The following inequality is essentially more involved. It's first proof used Lagrange multipliers, though later Pólya and Carleson found direct proofs:

**Problem 5.14** (Carleman). Suppose  $c_i > 0$ . Then

$$\sum_n \{c_1 \cdot \dots \cdot c_n\}^{1/n} < e \sum_k c_k.$$

For those who like inequalities, there is an excellent classical book: "Inequalities" by Hardy, Littlewood and Pólya. Our Library also has a recent book by Steele "The Cauchy-Schwarz master class" which looks good.

## 6 Implicit Function Theorem

### 6.1 Curves in the plane

Start with a motivation. Assume we have a curve in the plane  $\mathbb{R}^2$  defined implicitly by

$$(6.1) \quad f(x, y) = 0,$$

where  $f$  is a smooth function. We want to solve this equation; i.e. to find a smooth function  $x = g(y)$  such that

$$(6.2) \quad f(g(y), y) \equiv 0,$$

or a smooth function  $y = h(x)$  such that

$$f(x, h(x)) \equiv 0.$$

After a minute reflection, we come to the conclusion that there is at least one obstacle for this: differentiating identity (6.2), we get

$$f'_x \cdot g' + f'_y = 0,$$

or

$$g'(y) = -\frac{f'_y(g(y), y)}{f'_x(g(y), y)}.$$

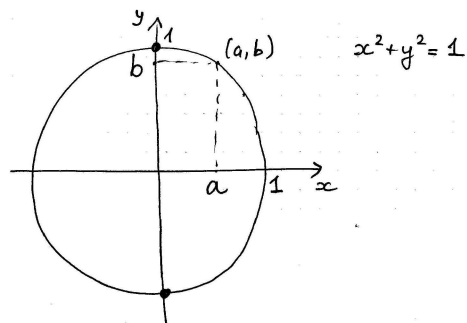
That is the function  $g(y)$  does not exist at the points  $y$  such that simultaneously  $f(x, y) = 0$  and  $f'_x(x, y) = 0$ . Similarly, the function  $h(x)$  does not exist at the points  $x$  such that simultaneously  $f(x, y) = 0$  and  $f'_y(x, y) = 0$ .

To fix the idea, we will be after the function  $x = g(y)$ . The simplest example is the unit circle:  $f(x, y) = x^2 + y^2 - 1$ . We see that the function  $g(y)$  does not exist at the points  $(0, \pm 1)$ . Now, assume that we've excluded these points. Then, there are two solutions  $x = g_{\pm}(y) = \pm\sqrt{1 - y^2}$ , and we need to specify which sign to take. We can do this simply by choosing one point  $(a, b)$  ( $a \neq 0$ ) which automatically determines the whole branch: After we fixed the point  $(a, b)$  we can uniquely determine the smooth function  $x = g(y)$  we were looking for. It is defined in a neighbourhood of  $b$  and  $g(b) = a$ .

Another example is the famous 'FOLIUM CARTESII'. This is a curve in  $\mathbb{R}^2$  defined by the equation

$$f(x, y) = x^3 + y^3 - 3axy = 0,$$

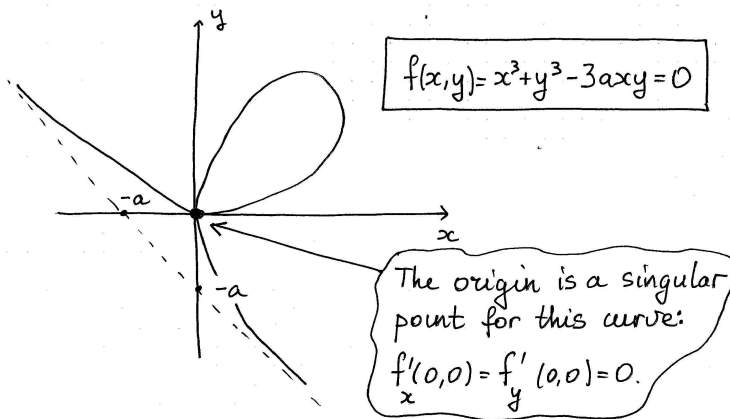
$a$  is a positive parameter.



$$f(x, y) = x^2 + y^2 - 1.$$

In a neighbourhood of the point  $b$  ( $\neq \pm 1$ ) there exists the unique solution  $x(y) = \sqrt{1 - y^2}$  such that  $x(b) = a$ .

### Descartes' Leaf (The Folium Cartesii)



The point  $(0, 0)$  is a *singular point*, since at this point  $f'_x = f'_y = 0$  and we cannot “resolve” the equation  $f(x, y) = 0$ . There are two other points on the folium:  $(a^{2^{1/3}}, a^{2^{2/3}})$  and  $(a^{2^{2/3}}, a^{2^{1/3}})$ ; at the first one  $f'_x$  vanishes, at the second one  $f'_y$  vanishes.

So now, we can formulate the two-dimension result that we hope to get<sup>6</sup>:

**Theorem 6.3** (2D version). *Let  $U \subset \mathbb{R}^2$  be a domain, and  $f \in C^1(U, \mathbb{R}^2)$ . Let  $(a, b) \in U$  be such a point that  $f(a, b) = 0$ , and  $f'_x(a, b) \neq 0$ . Then there exists a neighbourhood  $W$  of  $b$  and a unique function  $g \in C^1(W)$  such that  $g(b) = a$ , and  $f(g(y), y) \equiv 0$  in  $W$ .*

**Exercise 6.4.** The following equations have unique solutions for  $y = y(x)$  near the points indicated:  $x \cos xy = 0$ ,  $(1, \pi/2)$ , and  $xy + \log xy = 1$ ,  $(1, 1)$ . Prove that in both cases the function  $y(x)$  is convex.

**Exercise 6.5.** Find the maximum and the minimum of the function  $y$  that satisfies the equation  $x^2 + xy + y^2 = 27$ .

## 6.2 The theorem

Here are our standing assumptions:

$U \subset \mathbb{R}^n \times \mathbb{R}^m$  is an open set, we denote points of  $U$  by  $(x, y)$ ,  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^m$ ;

$f: U \rightarrow \mathbb{R}^n$  is a  $C^1$ -mapping;

$(a, b) \in U$  is such a point that  $f(a, b) = 0$ .

We consider equation

$$(6.6) \quad f(x, y) = 0.$$

This is a system of  $n$  equations with  $n + m$  variables. We want to solve this system; i.e. to express  $x$  in terms of  $y$ . More formally, we are looking for a neighbourhood  $W$  of  $b$  and for a  $C^1$  mapping  $g: W \rightarrow \mathbb{R}^n$  such that  $a = g(b)$  and

$$f(g(y), y) \equiv 0, \quad y \in W.$$

By  $f'_x$  we denote the derivative of the mapping  $f$  with respect to the variable  $x \in \mathbb{R}^n$  when the other variable  $y$  is fixed,  $f'_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ , and by  $f'_y$  the derivative of  $f$  with respect to  $y$  when  $x$  is fixed,  $f'_y \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ .

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<sup>6</sup>It is a special case of a general result proven below by applying the inverse function theorem, though this special case can be proven by elementary means you know from Hedva-1



Before stating and proving the theorem, let us try to guess the result “linearizing the problem”. We have

$$\begin{aligned} f(x, y) &= f(a, b) + D_f(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix} + \langle \text{remainder} \rangle \\ &= f(a, b) + f'_x(a, b)(x - a) + f'_y(a, b)(y - b) + \langle \text{remainder} \rangle. \end{aligned}$$

Discarding the remainder, to get the function  $x = g(y)$ , we have

$$\partial_x f(a, b)(x - a) + \partial_y f(a, b)(y - b) = 0.$$

If  $\partial_x f(a, b)$  is invertible, then we get

$$x = a - [\partial_x f(a, b)]^{-1} \partial_y f(a, b)(y - b).$$

The left hand side is the unique function  $g$  we are looking for.

Now, the theorem follows:

**Theorem 6.7.** *Suppose that the linear operator  $f'_x(a, b)$  is invertible. Then there is a neighbourhood  $W$  of  $b$  and a unique  $C^1$ -mapping  $g: W \rightarrow \mathbb{R}^n$  such that  $a = g(b)$ , and*

$$(6.8) \quad f(g(y), y) \equiv 0,$$

for any  $y \in W$ .

**Remark** The derivative of the mapping  $g$  is given by

$$(6.9) \quad g'(y) = -[f'_x(g(y), y)]^{-1} \cdot f'_y(g(y), y)$$

We get this differentiating equations (6.8) by  $y$ .

**Exercise 6.10.** If, in assumptions of the implicit function theorem, the function  $f$  belongs to  $C^k(U)$  ( $k \geq 2$ ), then the function  $g$  also belongs to  $C^k(W)$ .

Hint: use induction with respect to  $k$ .

### 6.3 Proof of the implicit function theorem

The idea is not difficult. To explain it, consider the simplest case  $m = n = 1$ . For each  $y$  in a neighbourhood of the point  $b$ , we are looking for the unique  $C^1$ -solution  $x(y)$  of the equation  $f(x, y) = 0$  such that  $x(b) = a$ . Instead, we consider a more general system of two equations

$$\begin{cases} f(x, y) = \xi, \\ y = \eta, \end{cases}$$

with respect to the variables  $(x, y)$ . The right-hand side belongs to a neighbourhood of the point  $(0, b)$ , the solution should belong to a neighbourhood of the point  $(a, b)$ . The Inverse Function Theorem says that there exists a unique  $C^1$ -solution  $x = \varphi(\xi, \eta)$ ,  $y = \psi(\xi, \eta)$  to this system. Clearly,  $y(\xi, \eta) = \eta$ . Now, recalling that we are interested only in a special case  $\xi = 0$ , we get the  $C^1$ -function  $x(y) := \varphi(0, y)$  such that  $f(x(y), y) = 0$ , and  $x(b) = 0$ .

To make this argument formal, define the mapping

$$F(x, y) = \begin{pmatrix} f(x, y) \\ y \end{pmatrix} : U \rightarrow \mathbb{R}^{n+m}.$$

Its derivative at the point  $(a, b)$  acts as follows

$$D_F(a, b) : \begin{pmatrix} h \\ k \end{pmatrix} \rightarrow \begin{pmatrix} D_f \begin{pmatrix} h \\ k \end{pmatrix} \\ k \end{pmatrix}.$$

In the block-matrix form,

$$D_F = \begin{pmatrix} \partial_x f & \partial_y f \\ 0 & I \end{pmatrix}.$$

Since  $\partial_x f(a, b)$  is invertible, the linear operator  $D_F(a, b)$  is also invertible. More formally, if

$$\begin{cases} D_f \begin{pmatrix} h \\ k \end{pmatrix} = 0 \\ k = 0, \end{cases}$$

then  $D_f \begin{pmatrix} h \\ 0 \end{pmatrix} = 0$ , and by our assumption about the operator  $\partial_x f(a, b)$ ,  $h = 0$ .

By the Inverse Function Theorem, there are the neighbourhoods

$$(a, b) \in \tilde{U} \subset \mathbb{R}^{n+m}, \quad (0, b) \in \tilde{V} \subset \mathbb{R}^{n+m},$$

and the  $C^1$ -mapping  $\Phi: \tilde{V} \rightarrow \tilde{U}$  such that  $\Phi(0, b) = (a, b)$ , and  $F \circ \Phi$  is the identity map. We have

$$\Phi(\xi, \eta) = (\varphi(\xi, \eta), \eta), \quad (\xi, \eta) \in \tilde{V},$$

where  $\varphi: \tilde{V} \rightarrow \mathbb{R}^n$  is a  $C^1$ -function. In other words,

$$f(\varphi(\xi, \eta), \eta) \equiv \xi, \quad \text{for any } (\xi, \eta) \in \tilde{V}.$$

Now, define the neighbourhood of  $b$ :  $W = \{\eta: (0, \eta) \in \tilde{V}\}$ , and the  $C^1$ -mapping  $g(y) = \varphi(0, y)$ ,  $y \in W$ . Then we have

$$f(g(y), y) \equiv 0 \quad \text{for } y \in W,$$

and  $g(b) = \varphi(0, b) = a$ .

The mapping  $g$  with these properties is unique. Indeed, if  $f(x, y) = f(x', y)$  for  $(x, y), (x', y) \in \tilde{V}$ , then  $F(x, y) = F(x', y)$ , and since  $F$  is one-to-one  $x = x'$ . This completes the proof of the Implicit Function Theorem.  $\square$

If you feel that this proof is too complicated, I strongly suggest, first, to work out all its details in the case  $n = m = 1$ , and only when this special case will be clear to pass to the general case.

**Exercise 6.11.** Let  $f: U \rightarrow \mathbb{R}^1$  be a  $C^1$ -function on an open set  $U$ , such that  $\frac{\partial f}{\partial x_j} \neq 0$  in  $U$  for all  $j = 1, 2, \dots, n$ . Then the equation

$$f(x_1, \dots, x_n) = 0$$

locally defines  $n$  functions  $x_1(x_2, \dots, x_n), x_2(x_1, x_3, \dots, x_n), \dots, x_n(x_1, \dots, x_{n-1})$ . Find the product

$$\frac{\partial x_1}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_3} \cdot \dots \cdot \frac{\partial x_{n-1}}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_1}.$$

## 7 Null-sets

It is not a simple task to understand *what is the volume?* and *which sets have the volume?* In our course, we'll be able to answer this question only partially. It is much simpler to understand *which sets have zero volume.* These sets play a very important role in analysis and its applications.

### 7.1 Definition

Let  $Q = I_1 \times I_2 \times \dots \times I_n \subset \mathbb{R}^n$  be a brick ( $I_k$  are the intervals, open, semi-open, or closed). Its *volume* equals

$$v(Q) = \prod_{k=1}^n |I_k|.$$

**Definition 7.1.** The set  $E \subset \mathbb{R}^n$  is a *null-set* if for each  $\epsilon > 0$  there exist open bricks  $Q_j$  such that  $E \subset \cup_j Q_j$ , and  $\sum_j v(Q_j) < \epsilon$ .

The null-sets are also called the zero measure sets, and the *negligible sets*. By  $\mathcal{N}$  we denote the set of all null-sets. Here are some useful properties of the null-sets:

1. If  $E \in \mathcal{N}$  and  $F \subset E$ , then  $F \in \mathcal{N}$ .
2. The countable sets are null-sets. Indeed, if  $E = \{a_m\}_{m \in \mathbb{N}}$ , then for each  $m$ , we find a cube  $Q_m$  that contains  $a_m$  and such that  $v(Q_m) < \epsilon 2^{-m}$ .
3. The countable union of null-sets is a null-set. The proof is similar: if  $E = \cup E_m$ , then we cover  $E_m \subset \cup_j Q_{jm}$  in such a way that  $\sum_j v(Q_{jm}) < \epsilon 2^{-m}$ .
4. If  $E$  is a compact set in  $\mathbb{R}^n$ , then for each  $\epsilon > 0$  there is a *finite covering*  $E \subset \cup_j Q_j$  such that  $\sum_j v(Q_j) < \epsilon$ .

Note, that the number of the bricks  $Q_j$  in the covering of a null-set  $E$  can be infinite. For instance, if  $E = \mathbb{Q} \cap [0, 1]$  is a set of all rational points in  $[0, 1]$ , then there is no finite covering with small total volume. (Why?)

Here are several exercises which help to get used to the new notion:

**Exercise 7.2.** Prove or disprove: if  $E \in \mathcal{N}$ , then the closure of  $E$  is also a null-set,  $\bar{E} \in \mathcal{N}$ .

**Exercise 7.3.** Denote by  $Q_*$  the *projection* of the set  $Q \subset \mathbb{R}^n$  onto the hyperplane  $\mathbb{R}^{n-1}$ ; i.e.  $Q_* = \{x \in \mathbb{R}^{n-1} : \exists y \in \mathbb{R}^1 (x, y) \in Q\}$ . Show that if  $Q_*$  is a null-set, then  $Q$  is also a null-set. Is the converse true?

**Exercise 7.4.** Let  $A \subset \mathbb{R}^n$ , and  $f: A \rightarrow \mathbb{R}^n$  be a Lipschitz map; i.e.  $|f(x) - f(y)| \leq M|x - y|$  for any  $x, y \in A$ . Show that if  $E \subset A$  is a null-set, then  $fE$  is a null-set as well.

**Exercise 7.5.** Suppose  $U \subset \mathbb{R}^n$  is an open set,  $f \in C^1(U, \mathbb{R}^n)$ . If  $E \subset U$  is a null set, then  $fE$  is a null set as well.

Hint: any open subset of  $\mathbb{R}^n$  can be exhausted from inside by finite unions of closed bricks.

The both exercises yield that the image of a null-set under a linear transformation  $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  is again a null-set. We see that the notion of null-set is independent on the choice of the coordinates in  $\mathbb{R}^n$ .

## 7.2 Examples of null-sets

**Theorem 7.6.** Let  $Q \subset \mathbb{R}^n$  be a closed brick, and  $f$  be a continuous function on  $Q$ . Then the graph  $\Gamma_f = \{(x, f(x)) : x \in Q\}$  of  $f$  is a null set in  $\mathbb{R}^{n+1}$ .

*Proof:* Fix  $\epsilon > 0$  and choose  $\delta > 0$  such that  $|x_1 - x_2| < \delta$  yields  $|f(x_1) - f(x_2)| < \epsilon$ . Let  $\Pi$  be a partition of  $Q$  onto closed bricks, the partition is chosen so fine that the diameter of each brick from  $\Pi$  is  $< \delta$ .

For any brick  $S$  from  $\Pi$ , the *oscillation* of  $f$  on  $S$  is less than  $\epsilon$ :

$$M_f(S) - m_f(S) < \epsilon, \quad \text{where } M_f(S) = \max_S f, \quad m_f(S) = \min_S f.$$

The graph  $\Gamma_f$  is covered by the  $n+1$ -dimensional bricks  $\tilde{S} = S \times [m_f(S), M_f(S)]$ , and

$$v_{n+1}(\tilde{S}) = v_n(S) \cdot (M_f(S) - m_f(S)) < \epsilon v_n(S).$$

Then

$$\sum_{\tilde{S}} v_{n+1}(\tilde{S}) < \epsilon \sum_S v_n(S) = \epsilon v_n(Q),$$

and we are done. □

In the proof, we silently used that for any finite partition  $\Pi$  of the brick  $Q$  onto bricks  $S$ :

$$\sum_{S \in \Pi} v(S) = v(Q).$$

**Exercise 7.7.** Prove this!

The theorem we proved persists for the graphs of continuous functions defined on open subsets of  $\mathbb{R}^n$ .

**Exercise 7.8.** Prove this!

Hint: again the same idea: any open subset of  $\mathbb{R}^n$  can be exhausted from inside by finite unions of closed bricks.

**Exercise 7.9.** Let  $f: I \rightarrow \mathbb{R}^1$  be an increasing function,  $I$  be an interval. Show that the discontinuity set of  $f$  is a null set.

Hint: show that for each  $j \in \mathbb{N}$ , the set of points  $x \in I$  such that  $f(x+0) - f(x-0) \geq 1/j$  is a null-set.

Another important example is provided by

**Theorem 7.10** (Sard). *Let  $U$  be an open set in  $\mathbb{R}^n$  and  $f$  is a non-constant  $C^1$ -function on  $U$ . Let  $B_f = \{x \in U: D_f(x) = 0\}$  be the set of critical points of  $f$ , and  $C_f = f(B_f)$  be the set of critical values of  $f$ . Then  $C_f$  is a null-set.*

Note that the set  $B_f$  of critical points does not have to be a null-set.

*Proof of Sard's theorem in the one-dimensional case:* First, we assume that  $f \in C^1(J, \mathbb{R}^1)$ , where  $J \subset \mathbb{R}^1$  is a closed bounded interval. Fix an arbitrary  $\epsilon > 0$ . Since  $f'$  is uniformly continuous on  $J$ , we can split  $J$  onto small closed sub-intervals  $I$  with disjoint unions such that  $|f'(x) - f'(y)| \leq \epsilon$  for  $x, y \in I$ .

Consider only those sub-intervals  $I$  that contain at least one critical point of  $f$ . If  $f'(\xi) = 0$  for  $\xi \in I$ , then, by the choice of  $I$ ,  $\max_I |f'| \leq \epsilon$ , whence

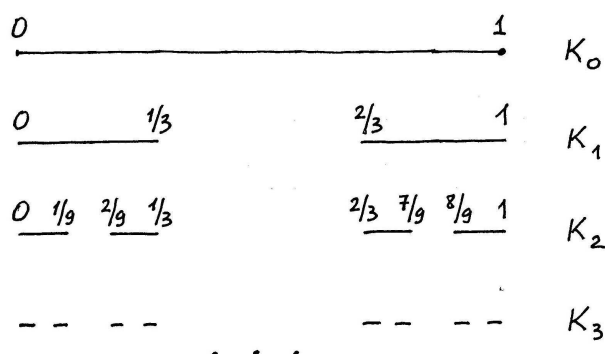
$$\max_I f - \min_I f = f(\beta) - f(\alpha) = \int_{\alpha}^{\beta} f' \leq \int_{\alpha}^{\beta} |f'| \leq \epsilon |I|.$$

In other words, the image  $f(I)$  is contained in an interval of length  $\leq \epsilon |I|$ . Thereby, the critical set  $C_f$  is covered by a system of intervals of length  $\leq \epsilon \sum |I| \leq \epsilon |J|$ .

Now, if  $f$  is a  $C^1$ -function on an open set  $U \subset \mathbb{R}^1$ , then we exhaust  $U$  by finite unions of closed bounded intervals and apply the special case proved above.  $\square$

**Problem 7.11.** Prove Sard's theorem in the multi-dimensional case.

An excellent reference to this problem is Milnor's book "Topology from the differentiable viewpoint".



Construction of the Cantor set  $K = \bigcap_{j \geq 0} K_j$

### 7.3 Cantor-type sets

#### 7.3.1 Classical Cantor set

We construct a nested sequence of compacts  $K_0 \supset K_1 \supset \dots \supset K_m \supset \dots$ .

First step:  $K_0 = [0, 1]$ .

Second step: we remove from  $K_0$  the middle third, i.e.  $K_1 = K_0 \setminus (\frac{1}{3}, \frac{2}{3})$ . Then  $K_1$  consists of two closed intervals of equal length, and  $|K_1| = \frac{2}{3}$ .

After the  $m$ -th step, we obtain a compact  $K_m$  which is a union of  $2^m$  equal intervals of total length  $|K_m| = (\frac{2}{3})^m$ ;  $K_0 \supset K_1 \supset \dots \supset K_m$ . On the  $m + 1$ -st step, we remove from each of these intervals the middle third, and obtain the compact  $K_{m+1}$ . And so on.

Then we set  $K = \bigcap_m K_m$ . This is a compact null-set called *the classical Cantor set*.

**Exercise 7.12.** Show that

1. the set  $\Omega = [0, 1] \setminus K$  is an open set;
2.  $\partial\Omega = K$ ;
3.  $K$  has no isolated points.
4.  $K$  coincides with the set of points  $x \in [0, 1]$  which have no digit 1 in their ternary representation.

**Exercise 7.13.** Consider the set of points  $x \in [0, 1]$  which have no digits 7 in the decimal expansion. Whether this is a null-set?

### 7.3.2 Cantor set with “variable steps”

The Cantor construction has various modifications. For example, on each step, we can remove not the middle third of each component but an interval of a different length that varies from step to step.

On the  $m$ -th step we remove from each closed interval  $I$  the middle open interval of the length  $\epsilon_m|I|$  (in the previous construction,  $\epsilon_m = \frac{1}{3}$  for all  $m$ ). Then the length of the compact  $K_m$  we obtain on the  $m$ -th step is

$$|K_m| = (1 - \epsilon_m)|K_{m-1}| = \dots = \prod_{j=1}^m (1 - \epsilon_j).$$

Now we need to exercise:

**Exercise 7.14.** Let  $0 < \epsilon_j < 1$ . Then

$$\lim_{m \rightarrow \infty} \prod_{j=1}^m (1 - \epsilon_j) = 0$$

if and only if

$$\sum_j \epsilon_j = +\infty.$$

We see that the limiting Cantor-type set  $K = \bigcap_m K_m$  is a null-set if and only if  $\sum_j \epsilon_j = +\infty$ . If  $\epsilon_j$  decay so fast that the series  $\sum \epsilon_j$  converges, then Cantor-type set  $K$  is not a null-set. So that, we obtain an open set  $\Omega = [0, 1] \setminus K$  with a huge boundary  $\partial\Omega = K$  which is not a null set.

### 7.3.3 Cantor-type sets in $\mathbb{R}^2$

If we perform the same construction in the unit square (on each step, one square is replaced by 4 equal smaller sub-squares on the corners of the original square), then the open complement  $[0, 1]^2 \setminus K$  is connected, and we obtain a *domain* in the unit square with a large boundary which is not a null-set.

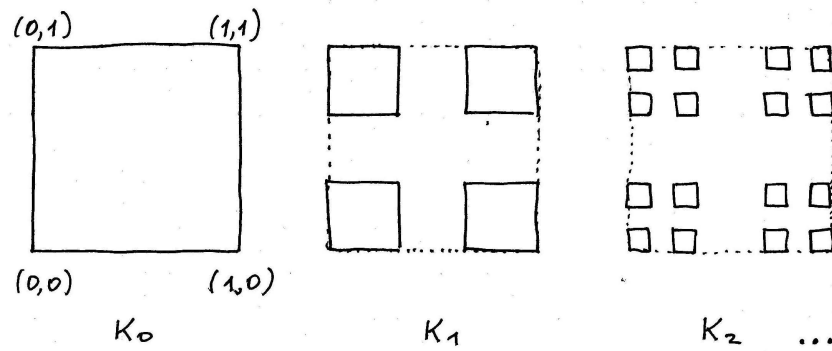
This time, we get a nested sequence of compacts  $K_m$  in the unit square, each  $K_m$  consists of  $4^m$  equal closed squares, and the area of  $K_m$  is

$$\prod_{j=1}^m (1 - \epsilon_j)^2.$$

Then we set  $K = \bigcap_m K_m$ .

**Exercise 7.15.** Check that





Construction of the 2-dimensional

Cantor set  $K = \bigcap_{j \geq 0} K_j$

1.  $K$  is a compact set with empty interior and without isolated points;
2.  $K$  is a null-set if and only if  $\sum \epsilon_j = +\infty$ ;
3.  $\Omega = (0, 1)^2 \setminus K$  is a domain,  $\partial\Omega \supset K$ .

Thus, we've constructed a domain in  $\mathbb{R}^2$  whose boundary is not a null-set.

## 8 Multiple Integrals

### 8.1 Integration over the bricks

#### 8.1.1 Darboux sums

Let  $Q = I_1 \times I_2 \times \dots \times I_n$  be an  $n$ -dimensional brick,  $f: Q \rightarrow \mathbb{R}^1$  a bounded function on  $Q$ .

Construction of the Darboux sums is practically the same as in the one-dimensional case. Let  $\Pi$  be a partition of the brick  $Q$ ,  $S$  a brick from the partition  $\Pi$ . We set

$$M_f(S) = \sup_S f, \quad m_f(S) = \inf_S f.$$

Then the *upper and lower Darboux sums* (of the function  $f$  with respect to the partition  $\Pi$ ) are defined as follows:

$$U(f, \Pi) = \sum_S M_f(S)v(S), \quad L(f, \Pi) = \sum_S m_f(S)v(S).$$

The proofs of the following properties are the same as in the one-dimensional case:

1. if the partition  $\Pi'$  is *finer*<sup>7</sup>, than the partition  $\Pi$ , then

$$L(f, \Pi) \leq L(f, \Pi'), \quad U(f, \Pi') \leq U(f, \Pi);$$

2. for each partitions  $\Pi_1$  and  $\Pi_2$ ,

$$L(f, \Pi_1) \leq U(f, \Pi_2).$$

**Exercise 8.1.** Prove these properties!

#### 8.1.2 The fundamental definition

**Definition 8.2.** The function  $f: Q \rightarrow \mathbb{R}^1$  is (Riemann) *integrable* over  $Q$  if  $f$  is bounded, and

$$\sup_{\Pi} L(f, \Pi) = \inf_{\Pi} U(f, \Pi).$$

The common value for this supremum and infimum is called the integral of  $f$  over the cube  $Q$ , and denoted by

$$\int_Q f = \int_Q f(x) dx.$$

---

<sup>7</sup>This means that each brick from  $\Pi'$  is a sub-brick of a brick from  $\Pi$ .

By  $\mathcal{R}(Q)$  we denote the class of all Riemann integrable functions on  $Q$ .

As in the one-dimensional case, we get

**Claim 8.3.** *The function  $f$  is integrable iff for each  $\epsilon > 0$  exists the partition  $\Pi$  of  $Q$  such that*

$$U(f, \Pi) - L(f, \Pi) < \epsilon.$$

Now, we are able to state *explicitly* which functions are Riemann-integrable<sup>8</sup>.

## 8.2 Lebesgue Theorem

The result is formulated in terms of the size of the *discontinuity set* of the function  $f$ . This set should not be too large. The quantitative way to say that the function  $f$  is discontinuous at the point  $x$  is to measure its oscillation at  $x$ :

$$\text{osc}_f(x) = \limsup_{y \rightarrow x} f(y) - \liminf_{y \rightarrow x} f(y) \left( =: \lim_{\delta \rightarrow 0} \sup_{|y-x| < \delta} f(y) - \lim_{\delta \rightarrow 0} \inf_{|y-x| < \delta} f(y) \right).$$

**Exercise 8.4.** The function  $f$  is continuous at  $x$  iff  $\text{osc}_f(x) = 0$ .

That is, the discontinuity set of  $f$  is  $B_f = \{x : \text{osc}_f(x) > 0\}$ .

**Exercise 8.5** (properties of the discontinuity set). Prove that the sets  $B_{f+g}$ ,  $B_{f \cdot g}$  and  $B_{\max(f,g)}$  are contained in  $B_f \cup B_g$ . Prove that  $B_{|f|} \subset B_f$ .

**Theorem 8.6** (Lebesgue). *The bounded function  $f: Q \rightarrow \mathbb{R}^1$  is Riemann-integrable if and only if  $B_f$  is a null set.*

We start with

**Claim 8.7.** *Let  $f: Q \rightarrow \mathbb{R}^1$  be a bounded function on a closed brick  $Q$ . Then the set  $B_{f,\epsilon} = \{x \in Q : \text{osc}_f(x) \geq \epsilon\}$  is compact.*

*Proof:* We'll show that the set  $\mathbb{R}^n \setminus B_{f,\epsilon}$  is open. A minute reflection shows that it suffices to check that if  $x \in Q$  and  $\text{osc}_f(x) < \epsilon$ , then there exists a neighbourhood of  $x$ , where still  $\text{osc}_f < \epsilon$ .

Choose  $\delta > 0$  such that

$$\sup_{|y-x| < \delta} f(y) - \inf_{|y-x| < \delta} f(y) < \epsilon.$$

---

<sup>8</sup>Probably, the next result is new for you also in the one-dimensional case.

Then, for any  $z$  from the ball  $|z - x| < \frac{\delta}{2}$ ,

$$\operatorname{osc}_f(z) \leq \sup_{|z-y|<\delta/2} f(y) - \inf_{|z-y|<\delta/2} f(y) \leq \sup_{|y-x|<\delta} f(y) - \inf_{|y-x|<\delta} f(y) < \epsilon,$$

and we are done.  $\square$

*Proof of the theorem:* First, assume that  $B_f$  is null-set. We need to build a partition  $\Pi$  of  $Q$  such that  $U(f, \Pi) - L(f, \Pi)$  is as small as we wish.

We choose an arbitrarily small  $\epsilon > 0$  and consider the compact set  $B_{f,\epsilon}$ . Since this is a null-set,  $B_{f,\epsilon}$  can be covered by finitely many open bricks  $\{S_j\}$  with the sum of volumes less than  $\epsilon$ .

Let  $\Pi$  be a partition of  $Q$ . By  $R$  we denote the closed bricks from this partition. If some  $R$  intersects the set  $B_{f,\epsilon}$ , we choose the brick  $R$  so small that it contains in the corresponding brick  $S_j$ <sup>9</sup>. Then we have an alternative: each closed brick  $R$  from this partition is either a sub-brick of one of the bricks  $\overline{S_j}$  (in this case, we say that  $R \in (\text{I})$ ), or is disjoint with the set  $B_{f,\epsilon}$  (then  $R \in (\text{II})$ ). Clearly,

$$U(f, \Pi) - L(f, \Pi) = \sum_{R \in \Pi} \operatorname{osc}_f(R)v(R) = \left( \sum_{R \in (\text{I})} + \sum_{R \in (\text{II})} \right) \operatorname{osc}_f(R)v(R).$$

We estimate separately these two sums.

The first sum is small since the total volume of the bricks  $S_j$  is small:

$$\sum_{R \in (\text{I})} \operatorname{osc}_f(R)v(R) \leq 2\|f\|_\infty \cdot \sum_j v(S_j) < 2\|f\|_\infty \cdot \epsilon,$$

here  $\|f\|_\infty = \sup_Q |f|$ .

If  $R \in (\text{II})$ , then

$$\operatorname{osc}_f(x) < \epsilon, \quad \forall x \in R.$$

Splitting, if needed, the closed brick  $R$  onto finitely many smaller closed bricks, we assume that  $\operatorname{osc}_f(R) = \sup_R f - \inf_R f < \epsilon$ . (Why this is possible?) Now, we see that the second sum is also small:

$$\sum_{R \in (\text{II})} \operatorname{osc}_f(R)v(R) < \epsilon \cdot \sum_{R \in \Pi} v(R) \leq \epsilon v(Q).$$

This proves the result in one direction.

**Question 8.8.** How the proof uses compactness of the sets  $B_{f,\epsilon}$ ?

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<sup>9</sup>Otherwise, we just refine it.

Now, we prove the converse. Assume that  $f$  is Riemann-integrable on  $Q$ . Since  $B_f = \bigcup_j B_{f,1/j}$ , it suffices to show that each  $B_{f,1/j}$  is a null-set.

Given  $\epsilon > 0$ , choose a partition  $\Pi$  of  $Q$  such that  $U(f, \Pi) - L(f, \Pi) < \epsilon/j$ . Let  $\{R^*\}$  be those of the bricks from this partition that intersect the set  $B_{f,1/j}$ . Then, automatically,  $\text{osc}_f(R^*) \geq 1/j$  for each brick  $R^*$ . Summing over these bricks, we get

$$\sum_{R^*} v(R^*) \leq j \cdot \sum_{R^*} \text{osc}_f(R^*)v(R^*) \leq j \cdot \sum_{R \in \Pi} \text{osc}_f(R)v(R) < j \cdot \frac{\epsilon}{j} = \epsilon.$$

That is,  $B_{f,1/j} \subset \bigcup R^*$  is a null-set, and  $B_f$  is a null-set as well. We are done.  $\square$

To better understand this proof, I recommend to check it with all details in the one-dimensional case.

### 8.3 Properties of the Riemann integral

8.9. The constant function is integrable, and

$$\int_Q c = cv(Q).$$

8.10. If the functions  $f$  and  $g$  are integrable, then any linear combination is integrable, and

$$\int_Q (\alpha f + \beta g) = \alpha \int_Q f + \beta \int_Q g.$$

8.11. If  $f$  is an integrable function, then

$$m_f(Q)v(Q) \leq \int_Q f \leq M_f(Q)v(Q).$$

Hence if  $f$  is a non-negative integrable function, then  $\int_Q f \geq 0$ . If  $f$  and  $g$  are integrable functions and  $f \geq g$ , then  $\int_Q f \geq \int_Q g$ .

If the function  $f$  is integrable, then  $|f|$  is also integrable, and

$$\left| \int_Q f \right| \leq \int_Q |f|.$$

8.12. If  $f$  is an integrable function, and  $f = 0$ , except of a null-set, then

$$\int_Q f = 0.$$

Hint: for any partition  $\Pi$  of  $Q$ ,  $L(f, \Pi) \leq 0$  (since in each cube  $S$  of  $\Pi$  there is at least one point where  $f = 0$ ), while  $U(f, \Pi) \geq 0$  (for a similar reason).

**Corollary 8.13.** If two integrable functions  $f_1, f_2$  coincide a.e.<sup>10</sup>, then their integrals coincide:

$$\int_Q f_1 = \int_Q f_2.$$

## 8.4 Integrals over bounded sets with negligible boundaries

Now, we extend the definition of the Riemann integral to any bounded sets  $E$  with a “negligible boundary”  $\partial E$ .

For any set  $E \subset \mathbb{R}^n$ , we denote by

$$\mathbb{1}_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E \end{cases}$$

its *indicator-function*.<sup>11</sup> This function is continuous on the interior and exterior of  $E$  and is discontinuous on the boundary  $\partial E$ .

Let  $Q$  be a brick in  $\mathbb{R}^n$ ,  $E \subset Q$ ,  $f: Q \rightarrow \mathbb{R}^1$  be a bounded function on  $Q$ .

**Definition 8.14.**

$$\int_E f := \int_Q f \cdot \mathbb{1}_E.$$

Of course, to make this definition the “correct” one needs to check several things.

- We’d like to know that the function  $f \cdot \mathbb{1}_E$  is integrable.

**Claim 8.15.** *The function  $\mathbb{1}_E$  is integrable iff the boundary  $\partial E$  is a null-set.*

**Claim 8.16.** *If the functions  $f$  and  $g$  are integrable, then their product  $f \cdot g$  is integrable as well.*

The both claims are obvious corollaries to the Lebesgue theorem (the second one uses that  $B_{f \cdot g} \subset B_f \cup B_g$ ). From now on, writing  $\int_E f$ , we always assume that the boundary  $\partial E$  is a null-set.

<sup>10</sup>i.e. everywhere, except of a null set, a.e. = almost everywhere

<sup>11</sup>It is also called the characteristic function of  $E$ .

- We also would like to know that the value of the integral

$$\int_E f$$

does not depend on the choice of the cube  $Q \supset E$  and on the values of  $f$  on  $Q \setminus E$ .

**Exercise 8.17.** Check this!

Now, we define the class of Riemann-integrable functions  $\mathcal{R}(E)$  on bounded subsets  $E \subset \mathbb{R}^n$  with negligible boundary  $\partial E \in \mathcal{N}$ . The bounded function  $f: E \rightarrow \mathbb{R}^1$  is *Riemann-integrable* on  $E$ , if the function  $f \cdot \mathbb{1}_E$  is integrable on some cube  $Q \supset E$ . This definition does not depend on the choice of the brick  $Q$ .

**Exercise 8.18.** If  $E$  is a compact null-set, then any bounded function on  $E$  is integrable, and

$$\int_E f = 0.$$

Next, for any bounded set  $E$ ,

$$\int_{\text{int}(E)} f = \int_E f = \int_{\bar{E}} f.$$

**Exercise 8.19.** Let  $f$  be a non-negative Riemann integrable function on a cube  $Q$ . If

$$\int_Q f = 0,$$

then  $f = 0$  a.e..

Hint: if  $x$  is a continuity point of  $f$ , then  $f(x) = 0$ .

All the properties of Riemann integrals given in the previous section persist for integrals over bounded sets  $E$  with  $\partial E \in \mathcal{N}$ . (Check!)

## 8.5 Jordan volume

**Definition 8.20.** The bounded set  $E \subset \mathbb{R}^n$  is called *Jordan-measurable* ( $E \in \mathcal{J}$ ) if  $\partial E$  is a null-set. In this case, the *volume* (or *content*) of  $E$  is

$$v(E) = \int_E \mathbb{1}.$$

This definition is rather restricted. For example, there are open sets and even domains which are not Jordan-measurable.

**Exercise 8.21.** Give example of a domain and a compact set which are not Jordan-measurable.

You will learn in the courses “Functions of real variables” and “Measure theory”, how Lebesgue fixed that problem. There is another drawback: our definition, at least formally, depends on the choice of coordinates (since we started with the bricks). This will be fixed quite soon.

**Exercise 8.22.** Check that if  $v(E) = 0$ , then  $E$  is a null-set. The converse statement is wrong.

It follows from Exercise 8.18 that if the set  $E$  is Jordan-measurable, then the sets  $\bar{E}$  and  $\text{int}E$  are also Jordan-measurable and

$$v(\text{int}(E)) = v(E) = v(\bar{E}).$$

**Exercise 8.23.** A bounded set  $E$  is Jordan measurable, iff given  $\epsilon > 0$  there exists a partition  $\Pi$  of a cube  $Q \supset E$  such that

$$\sum_{S \in \Pi_+} v(S) - \sum_{S \in \Pi_-} v(S) < \epsilon,$$

where  $\Pi_+ = \{S \in \Pi: S \cap E \neq \emptyset\}$ ,  $\Pi_- = \{S \in \Pi: S \subset E\}$ .

That is, the Jordan sets are exactly those sets whose volume can be well approximated from “inside” and from “outside” by volumes of *elementary sets*, that is by volumes of finite unions of bricks:<sup>12</sup>

$$v(E) = \sup_{S \in \Pi_-} v(S) = \inf_{S \in \Pi_+} v(S).$$

Note that if  $E_1$  and  $E_2$  are the Jordan sets, then the sets  $E_1 \cap E_2$  and  $E_1 \cup E_2$  are Jordan as well, and

$$v(E_1 \cup E_2) + v(E_1 \cap E_2) = v(E_1) + v(E_2).$$

---

<sup>12</sup>In the Lebesgue theory, all open sets and all compact sets are measurable. If the set  $\Omega$  is open, then

$$v(\Omega) \stackrel{\text{def}}{=} \sup\{v(S): S \subset \Omega\},$$

where  $S$  is a finite union of bricks, and if the set  $K$  is compact, then

$$v(K) \stackrel{\text{def}}{=} \inf\{v(\Omega): \Omega \supset K\},$$

where  $\Omega$  is open.



This follows from the identity

$$\mathbb{1}_{E_1 \cup E_2} + \mathbb{1}_{E_1 \cap E_2} = \mathbb{1}_{E_1} + \mathbb{1}_{E_2},$$

and linearity of the Riemann integral. Thus, the Jordan volume is a *finitely additive set-function*: if the Jordan sets  $E_1, \dots, E_N$  are disjoint (or, at least, have disjoint interiors), then

$$v\left(\bigcup_{j=1}^N E_j\right) = \sum_{j=1}^N v(E_j).$$

Another property of the Jordan volume is its invariance with respect to the translations of  $\mathbb{R}^n$ : if  $E \in \mathcal{J}$ , then, for any  $c \in \mathbb{R}^n$ ,  $E + c \in \mathcal{J}$  (why?) and  $v(E + c) = v(E)$ . Note (this is VERY IMPORTANT!), that the Jordan volume is determined (up to a positive multiplicative constant) by these two properties:

**Theorem 8.24** (uniqueness of the Jordan volume). *The  $n$ -dimensional Jordan volume is the only function  $v: \mathcal{J} \rightarrow \mathbb{R}_+^1$  satisfying the following conditions:*

- (i) *finite additivity;*
- (ii) *translation invariance;*
- (iii) *normalization  $v(Q) = 1$ , where  $Q = [0, 1]^n$  is the unite cube in  $\mathbb{R}^n$ .*

*Proof:* of this remarkable fact is fairly easy. Let  $v^*$  be another function with the properties (i)—(iii). Then on “dyadic cubes” with the side length  $2^m$  ( $m \in \mathbb{Z}$ ) the function  $v^*$  takes the same value (property (ii)), and hence coincides with  $v$  (property (i)). Thereby,  $v^*$  coincides with  $v$  on all bricks in  $\mathbb{R}^n$  (since any brick in  $\mathbb{R}^n$  can be approximated by a finite union of dyadic cubes), and hence on the whole  $\mathcal{J}$ .  $\square$

The theorem we proved yields another very important fact: invariance of the volume with respect to orthogonal transformations.

**Theorem 8.25.** *Let  $O \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  be an orthogonal transformation. Then for any Jordan set  $E$ , the set  $OE$  is Jordan as well, and  $v(OE) = v(E)$ .*

*Proof:* First, note that if  $E$  is a Jordan set, and  $O$  is an orthogonal transformation of  $\mathbb{R}^n$ , then  $OE$  is also a Jordan set. Indeed, the transformation  $x \mapsto Ox$  maps the boundary to the boundary (since this is an open map) and preserves the class of null-sets (since this map is a Lipschitz one).

Then we consider the function  $v_O: \mathcal{J} \rightarrow \mathbb{R}_+$  acting as follows:  $v_O(E) = v(OE)$ . This is a finitely-additive and translation invariant function. Hence,

by the previous theorem, there is a non-negative constant  $c$  such that, for any  $E \in \mathcal{J}$ ,  $v_O(E) = cv(E)$ . It remains to check that  $c = 1$ .

To this end, consider the unit ball  $\mathbb{B} \subset \mathbb{R}^n$ . Clearly, this is a Jordan set since its boundary (the unit sphere) is a union of two graphs of continuous functions. Obviously,  $O\mathbb{B} = \mathbb{B}$ , whence  $cv(\mathbb{B}) = v_O(\mathbb{B}) = v(\mathbb{B})$ . Since  $\mathbb{B}$  is not a null-set, this yields  $c = 1$ .  $\square$

We finish this lecture with another useful and simple theorem:

**Theorem 8.26** (Mean Value Theorem). *If  $G$  is a domain with negligible boundary  $\partial G \in \mathcal{N}$ , and  $f \in C(G, \mathbb{R}^1)$  and bounded, then  $\exists \xi \in G$  such that*

$$\int_G f = f(\xi)v(G).$$

Indeed, we know that

$$\inf_G f \cdot v(G) \leq \int_G f \leq \sup_G f \cdot v(G).$$

Since the function  $f$  is continuous, for any  $c \in (\inf_U f, \sup_U f)$  there exists  $\xi \in U$  such that  $f(\xi) = c$ . In particular, this holds for  $c = \int_G f / v(G)$ .  $\square$

## 9 Fubini Theorem

In this part, we learn how to reduce multiple integrals to iterated one-dimensional integrals. Start with heuristics. Let  $f$  be a continuous function on a rectangle  $Q = [a, b] \times [c, d]$ . Then

$$(9.1) \quad \iint_Q f(x, y) \, dx dy = \int_a^b \left\{ \int_c^d f(x, y) \, dy \right\} dx = \int_c^d \left\{ \int_a^b f(x, y) \, dx \right\} dy.$$

The integrals on the RHS are called the *iterated integrals*. The idea is simple: consider the Riemann sums with the special choice of points  $\xi_{ij} = (x_i, y_j)$ ,  $\xi_{i,j} \in Q_{i,j} = A_i \times B_j$ , Then

$$\sum_{i,j} f(x_i, y_j) |A_i| |B_j| = \sum_i |A_i| \left( \sum_j f(x_i, y_j) |B_j| \right) = \sum_j |B_j| \left( \sum_i f(x_i, y_j) |A_i| \right).$$

In the limit, we get (9.1).

### 9.1 The statement

Let  $A \subset \mathbb{R}^n$ ,  $B \subset \mathbb{R}^m$  be bricks,  $f \in \mathcal{R}(A \times B)$ , that is the multiple integral

$$(9.2) \quad \iint_{A \times B} f(x, y) \, dx dy$$

exists.

**Theorem 9.3** (Fubini). *The iterated integrals*

$$\int_A dx \left( \int_B f(x, y) \, dy \right), \quad \int_B dy \left( \int_A f(x, y) \, dx \right)$$

*exists and equal the multiple integral (9.2).*

The following example shows that one needs some vigilance applying Fubini's theorem:

**Example 9.4.**

$$\int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3} dy \right\} dx = \frac{1}{2}, \quad \text{but} \quad \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3} dx \right\} dy = -\frac{1}{2}.$$

Let  $Q$  be a brick, and  $g$  a bounded function on  $Q$ . Introduce the *upper and lower Darboux integrals*

$$\overline{\int}_Q g = \inf_{\Pi} U(g, \Pi), \quad \underline{\int}_Q g = \sup_{\Pi} L(g, \Pi).$$

Of course,  $\underline{\int}_Q g \leq \overline{\int}_Q g$ , and  $g$  is integrable iff the upper and the lower integrals coincide and equal the regular one.

Now, we define the function

$$F(x) = \int_B f(x, y) dy.$$

If for some  $x \in A$  the integral  $\int_B f(x, y) dy$  does not exist, then we define  $F(x)$  to be *any* number between  $\underline{\int}_B f(x, y) dy$  and  $\overline{\int}_B f(x, y) dy$ . In the course of the proof, we shall see that that

$$\left\{ x \in A : \int_B f(x, y) dy \neq \overline{\int}_B f(x, y) dy \right\}$$

is a null-set, so, in fact, it is not important how  $F(x)$  was defined on that set.

## 9.2 Proof of Fubini's theorem

Choose partitions  $\Pi_A$  of  $A$  and  $\Pi_B$  of  $B$ , and denote the corresponding partition of  $A \times B$  by  $\Pi = \Pi_A \times \Pi_B$ . If  $S$  is a brick from  $\Pi$ , then  $S = A_i \times B_j$  ( $A_j \subset A$ ,  $B_j \subset B$  are bricks), and  $v_{n+m}(S) = v_n(A_i) \cdot v_m(B_j)$ .

Now, we have

$$\begin{aligned}
L(f, \Pi) &= \sum_{i,j} \left( \inf_{A_i \times B_j} f \right) v_n(A_i) v_m(B_j) \\
&= \sum_i \left( \sum_j \inf_{x \in A_i, y \in B_j} f(x, y) v_m(B_j) \right) v_n(A_i) \\
&\leq \sum_i \inf_{x \in A_i} \left( \sum_j \inf_{y \in B_j} f(x, y) v_m(B_j) \right) v_n(A_i) \\
&\leq \sum_i \inf_{x \in A_i} \left( \int_{\underline{B}} f(x, y) dy \right) v_n(A_i) \\
&\leq \sum_i m_F(A_i) v_n(A_i) \leq \sum_i M_F(A_i) v_n(A_i) \\
&\leq \sum_i \sup_{x \in A_i} \left( \int_{\overline{B}} f(x, y) dy \right) v_n(A_i) \\
&\leq \sum_i \sup_{x \in A_i} \left( \sum_j \sup_{y \in B_j} f(x, y) v_m(B_j) \right) v_n(A_i) \\
&\leq \sum_{i,j} \sup_{A_i \times B_j} f(x, y) v_m(B_j) v_n(A_i) = U(f, \Pi).
\end{aligned}$$

Thus

$$L(f, \Pi) \leq \sum_i m_F(A_i) v_n(A_i) \leq \sum_i M_F(A_i) v_n(A_i) \leq U(f, \Pi).$$

Hence,  $F \in \mathcal{R}(A)$ , and

$$\int_A F dx = \iint_{A \times B} f(x, y) dx dy.$$

We are done. □

### 9.3 Remarks

9.5. If  $f \in \mathcal{R}(A \times B)$ , then the sets

$$\left\{ x \in A : \int_B f(x, y) dy \text{ doesn't exist} \right\}$$

and

$$\left\{ y \in B : \int_A f(x, y) dx \text{ doesn't exist} \right\}$$

are the null-sets.

*Proof:* Consider the function

$$x \mapsto \int_B f(x, y) dy - \int_{\overline{B}} f(x, y) dy.$$

This function is integrable (as the difference of two integrable functions), non-negative, and has zero integral over  $A$ . Hence, the function vanishes everywhere on  $A$ , except of a null-set.  $\square$

**9.6.** Suppose  $f(x, y) = \varphi(x) \cdot \psi(y)$ , where  $\varphi \in \mathcal{R}(A)$ ,  $\psi \in \mathcal{R}(B)$ . Then  $f \in \mathcal{R}(A \times B)$ , and

$$\iint_{A \times B} f = \int_A \varphi \cdot \int_B \psi.$$

*Proof:* The integrability of  $f$  follows from the Lebesgue theorem, the rest follows from the Fubini theorem.  $\square$

In many cases, the domain  $D$  of multiple integration can be represented as

$$D = \{(x, y) \in \mathbb{R}^{n+1} : x \in E, f_1(x) \leq y \leq f_2(x)\},$$

where  $E \subset \mathbb{R}^n$  is a Jordan set, and  $f_1, f_2$  are continuous functions on  $E$ ,  $f_1 \leq f_2$ .

**9.7.** In these assumptions, the set  $D$  is Jordan, and for any continuous function  $g$  on  $D$ ,

$$\int_D g = \int_E dx \left( \int_{f_1(x)}^{f_2(x)} g(x, y) dy \right).$$

*Proof:* To check that  $D$  is Jordan, we look at its boundary  $\partial D$ . It is the union of three sets:

$$\begin{aligned} \partial D = \{(x, y) : x \in \partial E, f_1(x) \leq y \leq f_2(x)\} &\cup \{(x, f_1(x)) : x \in E\} \\ &\cup \{(x, f_2(x)) : x \in E\}. \end{aligned}$$

All three sets are null-sets, thus  $D$  is the null-sets as well.

We take a closed interval  $I \supset f_1(E) \cup f_2(E)$ . Then  $E \times I \supset D$ . Set  $h = g\mathbb{1}_D$ , and apply Fubini's theorem:

$$\begin{aligned} \int_D g &= \iint_{E \times I} h = \int_E dx \left( \int_I h(x, y) dy \right) \\ &= \int_E dx \left( \int_{f_1(x)}^{f_2(x)} g(x, y) dy \right). \end{aligned}$$

□

**Exercise 9.8.** Compute the integral

$$\iiint_T (x_1^2 + x_2^2 + x_3^2) dx_1 dx_2 dx_3,$$

where  $T$  is the “simplex” in  $\mathbb{R}^3$  bounded by the planes  $\{x_1 + x_2 + x_3 = a\}$ ,  $\{x_i = 0\}$ ,  $1 \leq i \leq 3$ .

Answer:  $a^5/20$ .

**Exercise 9.9.** Find the volume of

(i) the intersection of two solid cylinders in  $\mathbb{R}^3$ :  $\{x_1^2 + x_2^2 \leq 1\}$  and  $\{x_1^2 + x_3^2 \leq 1\}$ ;

Answer:  $16/3$ .

(ii) the solid in  $\mathbb{R}^3$  under paraboloid  $\{x_1^2 + x_2^2 - x_3 = 0\}$  and above the square  $[0, 1]^2$ .

Answer:  $2/3$ .

**Exercise 9.10.** Find the integrals

$$\int_{[0,1]^n} (x_1^2 + \dots + x_n^2) dx_1 \dots dx_n, \quad \int_{[0,1]^n} (x_1 + \dots + x_n)^2 dx_1 \dots dx_n.$$

**Exercise 9.11.** Let  $f: \mathbb{R}^1 \rightarrow \mathbb{R}^1$  be a continuous function. Prove that

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n f(x_n) = \int_0^x f(t) \frac{(x-t)^{n-1}}{(n-1)!} dt.$$

**Example 9.12.** Compute the integral

$$\int_{[0,1]^n} \max(x_1, \dots, x_n) dx_1 \dots dx_n.$$

First of all, by symmetry, we assume that  $1 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq 0$ , and multiply the answer by  $n!$ . Then  $\max(x_1, \dots, x_n) = x_1$ , and we get

$$n! \int_0^1 x_1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n = n! \int_0^1 \frac{x_1^n dx_1}{(n-1)!} = \frac{n}{n+1}.$$

**Exercise 9.13.** Compute the integral

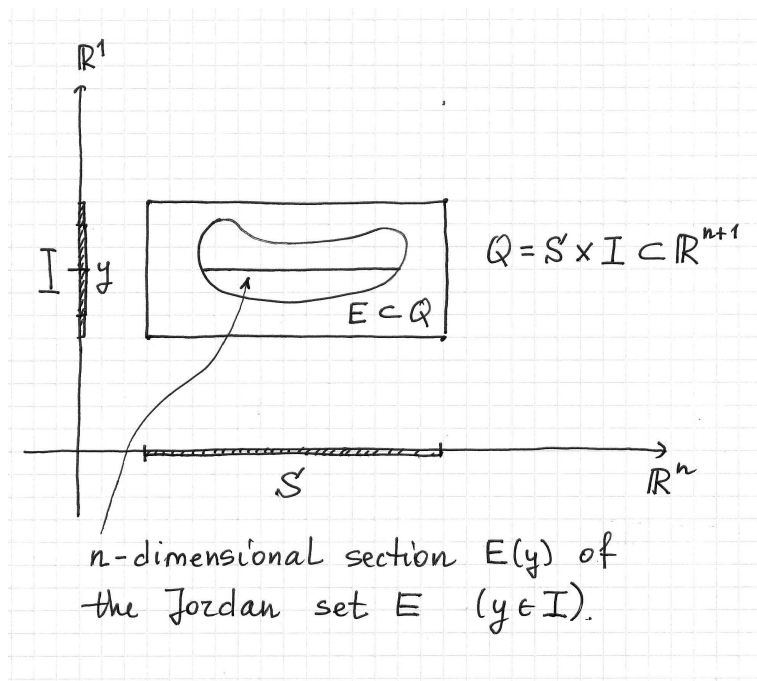
$$\int_{[0,1]^n} \min(x_1, \dots, x_n) dx_1 \dots dx_n.$$

Answer:  $\frac{1}{n+1}$ .

## 9.4 The Cavalieri principle

Intuitively clear, that we can recover the volume of three-dimensional body, integrating the areas of its two-dimensional slices. The next theorem gives the precise statement.

Let  $S \subset \mathbb{R}^n$  be a closed brick,  $I \subset \mathbb{R}^1$  closed interval,  $Q = S \times I$ , and let  $E \subset Q$  be a Jordan set. We denote its  $n$ -dimensional sections by  $E(y) = \{x \in S : (x, y) \in E\}$ .





**Theorem 9.14.** For a.e.  $y \in I$ , the  $n$ -dimensional slice  $E(y)$  is a Jordan set, and

$$v_{n+1}(E) = \int_I v_n(E(y)) dy.$$

*Proof:* Note that  $\mathbb{1}_E(x, y) = \mathbb{1}_{E(y)}(x)$ . By Remark 9.5, we know that for a.e.  $y \in I$  there exists the integral

$$\int_S \mathbb{1}_E(x, y) dx = \int_S \mathbb{1}_{E(y)}(x) dx =: v_n(E(y)).$$

Again, by Fubini,

$$v_{n+1}(E) = \iint_{S \times I} \mathbb{1}_E(x, y) dx dy = \int_I dy \left( \int_S \mathbb{1}_{E(y)}(x) dx \right) = \int_I v_n(E(y)) dy.$$

Done! □

**Corollary 9.15** (Cavalieri). Let  $P, Q \subset \mathbb{R}^3$  be Jordan sets, and

$$P(c) := \{(x, y, z) \in P : z = c\},$$

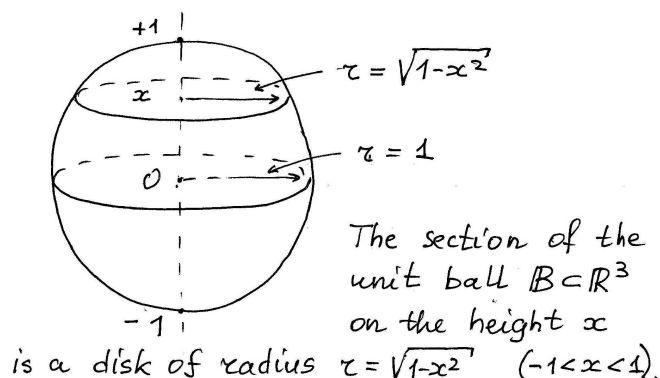
$$Q(c) := \{(x, y, z) \in Q : z = c\},$$

be their two-dimensional horizontal slices. If, for almost every  $c \in \mathbb{R}^1$ ,

$$\text{area}(P(c)) = \text{area}(Q(c)),$$

then  $v(P) = v(Q)$ .

**Example 9.16** (volume of the unit ball in  $\mathbb{R}^3$ ). Inspecting the figure we see



that

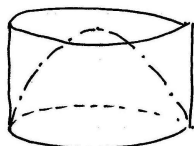
$$\text{vol}(\mathbb{B}) = \int_{-1}^1 \text{area}(\mathbb{D}(\sqrt{1-x^2})) dx.$$

Here  $D_\rho$  is the disc of radius  $\rho$ . The latter integral equals

$$\pi \int_{-1}^1 (1-x^2) dx = \pi \left( 2 - \frac{2}{3} \right) = \frac{4\pi}{3}.$$

Of course, volume of the ball of radius  $r$  in  $\mathbb{R}^3$  equals  $4\pi r^3/3$ .

**Exercise 9.17** (Archimedes). The volumes of an inscribed cone, the half-ball, and a circumscribed cylinder with the same base plane and radius, are in the ratios  $1 : 2 : 3$ .



A cylinder circumscribed  
around the half-ball



A cone inscribed  
into the half-ball

**Exercise 9.18.** Show that the volume of the ball of radius  $r$  in  $\mathbb{R}^n$  equals  $v_n r^n$ . Compute the constant  $v_4$ .

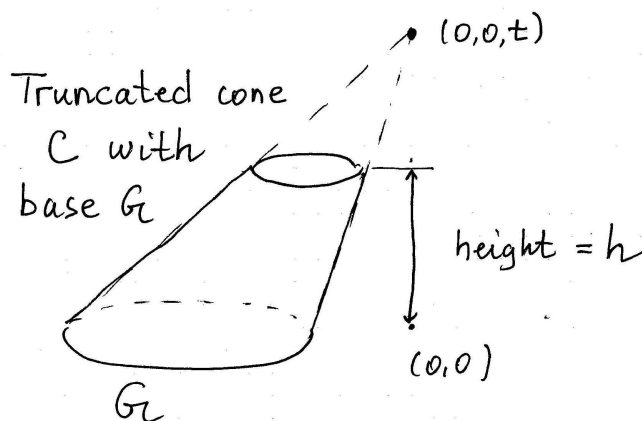
In the same way one can compute the volume of the unit ball in  $\mathbb{R}^n$ . The answer differs in the cases of even and odd dimensions:

$$v_{2k+1} = v_{2k+1}(\mathbb{B}) = 2 \frac{(2\pi)^k}{(2k+1)!!}, \quad v_{2k} = v_{2k}(\mathbb{B}) = \frac{(2\pi)^k}{(2k)!!}.$$

We skip the computation since later we'll find a simpler way to do this.

**Example 9.19** (volume of truncated cone). Let  $G \in \mathbb{R}^2$  be a bounded open set with negligible boundary. Consider the truncated cone with apex  $(0, 0, t)$  and height  $h \leq t$ :

$$C = \left\{ \left( \left( 1 - \frac{s}{t} \right) x, s \right) : x \in G, 0 \leq s \leq h \right\}.$$



Then, by Cavalieri's principle,

$$\begin{aligned} \text{vol}(C) &= \int_0^h \text{area}(G) \left(1 - \frac{s}{t}\right)^2 ds \\ &= \text{area}(G) \left(h - \frac{h^2}{t} + \frac{h^3}{3t^2}\right) = \text{area}(G) \cdot h \cdot \left(1 - \frac{h}{t} + \frac{h^2}{3t^2}\right). \end{aligned}$$

**Exercise 9.20.** Find the volume of the  $n$ -dimensional simplex

$$T = \{x: x_1, \dots, x_n \geq 0, x_1 + \dots + x_n \leq 1\}.$$

Answer:  $\frac{1}{n!}$ .

**Exercise 9.21.** Suppose the function  $f$  depends only on the first coordinate. Then

$$\int_{\mathbb{B}} f(x_1) dx = v_{n-1} \int_{-1}^1 f(x_1) (1 - x_1^2)^{(n-1)/2} dx_1,$$

where  $\mathbb{B}$  is the unit ball in  $\mathbb{R}^n$ , and  $v_{n-1}$  is the volume of the unit ball in  $\mathbb{R}^{n-1}$ .

The next two exercises deal with a very interesting phenomenon of “concentration of high-dimensional volume”.

**Exercise 9.22.** 1. Let  $B_r$  be a ball of radius  $r$  in  $\mathbb{R}^n$ . Compute

$$\frac{v_n(B_r \setminus B_{0.99r})}{v_n(B_r)}$$

for  $n = 3$ ,  $n = 10$ , and  $n = 100$ .

2. Given  $\epsilon > 0$ , the quotient

$$\frac{v_n(B_r \setminus B_{(1-\epsilon)r})}{v_n(B_r)}$$

decays as  $e^{-\epsilon n}$  when  $n \rightarrow \infty$ .

**Exercise 9.23.** 1. Let  $\mathbb{B}$  be the unit ball in  $\mathbb{R}^n$ , and  $P = \{x \in \mathbb{B} : |x_n| < 0.01\}$ . What is larger  $v_n(P)$  or  $v_n(\mathbb{B} \setminus P)$  if  $n$  is sufficiently large?

2. Given  $\epsilon > 0$ , show that the quotient

$$\frac{v_n(\{x \in \mathbb{B} : |x_n| > \epsilon\})}{v_n(\mathbb{B})}$$

tends to zero as  $n \rightarrow \infty$ .

Hint: the quotient equals

$$\frac{\int_{\epsilon}^1 (1-t^2)^{(n-1)/2} dt}{\int_0^1 (1-t^2)^{(n-1)/2} dt}$$

3\*. Find the asymptotic behaviour of that quotient as  $n \rightarrow \infty$ .

## 10 Change of variables

### 10.1 The theorem

Recall the change of variables in one-dimensional Riemann integrals: suppose  $\varphi: [a, b] \rightarrow \mathbb{R}^1$  is a  $C^1$ -injection and  $f: [\varphi(a), \varphi(b)] \rightarrow \mathbb{R}^1$  is a continuous function. Then

$$(10.1) \quad \int_{\varphi(a)}^{\varphi(b)} f = \int_a^b (f \circ \varphi) \varphi'.$$

Before writing the  $n$ -dimensional counterpart of this formula, let us observe a subtle difference in definition of the Riemann integral you've learnt in Hedva-2 and the new one. In Hedva-2, the integral

$$\int_{\alpha}^{\beta} g$$

was defined as the limit of Riemann sums

$$\sum g(\xi_j)(\alpha_{j+1} - \alpha_j),$$

where  $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_N = \beta$  is a partition of the interval  $I = [\alpha, \beta]$ . Our new definition starts with the Riemann sums<sup>13</sup>

$$\sum g(\xi_j)|I_j| = \sum g(\xi_j)|\alpha_{j+1} - \alpha_j|.$$

Therefore, the change of the 'orientation'  $I \rightarrow -I$  changes the sign of the 'old' integral, and does not affect the new one<sup>14</sup>.

Now it is not difficult to guess that in our situation the counterpart of (10.1) looks as follows:

$$\int_{\varphi(I)} f = \int_I (f \circ \varphi) |\varphi'|.$$

**Definition 10.2.** A ( $C^1$ -) diffeomorphism is a bijection  $T$  such that both  $T$  and  $T^{-1}$  are  $C^1$ -mapping.

**Definition 10.3.** The determinant  $J_T = \det D_T$  of the linear operator  $D_T$  is called the *Jacobian* of the mapping  $T$ .

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<sup>13</sup>We have not mentioned Riemann sums at all, and worked with the Darboux sums, but let us discard such a 'detail'.

<sup>14</sup>Later, we'll return to the idea of "oriented integration"

**Theorem 10.4.** *Let  $U \subset \mathbb{R}^n$  be a bounded domain,  $T: U \rightarrow TU$  a diffeomorphism,  $f$  bounded continuous function on  $TU$ . Then for any Jordan set  $\Omega, \bar{\Omega} \subset U$ ,*

$$\int_{T\Omega} f = \int_{\Omega} (f \circ T) |J_T|.$$

Since the proof of this theorem is not short, we have not tried to “optimize” the assumptions. Here is the plan we will follow:

1. First, we will show that for any linear transformation  $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  and any Jordan set  $\Omega \subset \mathbb{R}^n$ ,

$$v(L\Omega) = |\det L|v(\Omega).$$

2. On the next step, we prove the “infinitesimal version” of the theorem which says that

$$\lim_{Q \rightarrow x} \frac{v(TQ)}{v(Q)} = |J_T(x)|.$$

3. On the last step, we introduce the additive set-functions and complete the proof of the theorem.

**Exercise 10.5.** If the set  $\Omega \subset U$  is Jordan,  $\bar{\Omega} \subset U$ , and  $T: U \rightarrow \mathbb{R}^n$  is a diffeomorphism, then the set  $T(\Omega)$  is Jordan as well.

## 10.2 $v(L\Omega) = |\det L|v(\Omega)$

First of all, note that it suffices to prove this only for the standard unit cube  $Q$  in  $\mathbb{R}^n$ . Indeed, define a set-function  $v_L(\Omega) = v(L\Omega)$ . It is finitely additive and translation invariant. Thus, to show that  $v_L = |\det L|v$ , it suffices to check this on the unit cube  $Q$ .

We give two proofs of the identity  $v(LQ) = |\det L|v(Q)$ .

### 10.2.1 Polar decomposition of non-singular operators

The 1-st proof is based on the “polar decomposition” of non-singular (i.e., invertible) linear operators, you’ve probably learnt in Linear Algebra.

First, assume that the operator  $L$  is singular (i.e., is not invertible). Then the image  $LQ$  lies in a proper linear subspace of  $\mathbb{R}^n$ , hence  $v(LQ) = 0$ . Since  $\det(L) = 0$ , there is nothing to prove in this case.

Now, assume that  $L$  is not singular. We use the following

**Claim 10.6** (Linear Algebra). *Any nonsingular linear transformation  $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  is a product of a self-adjoint one and the orthogonal one.*

Question: why this decomposition <sup>15</sup> is called the *polar* one?

Now, we are ready to prove the identity  $v(LQ) = |\det L|v(Q)$  in the case when the operator  $L$  is non-singular. In view of the the polar decomposition, it suffices to check it only for orthogonal and positive operators. For orthogonal operators  $U$ , we already know that  $v(UQ) = v(Q)$ , and since the absolute value of the determinant of  $U$  is one, again, there is nothing to prove. If  $L$  is a positive operator, then, without lost of generality, we assume that it is already diagonal (otherwise, we just change the orthogonal basis in  $\mathbb{R}^n$ , we know that the volume does not depend on the choice of the basis. Then  $LQ = [0, \lambda_1] \times \dots \times [0, \lambda_n]$ , where  $\lambda_i > 0$  are the eigenvalues of  $L$  (recall that all of them are positive, since  $L$  is positive). Thereby,

$$v(LQ) = \prod_{i=1}^n \lambda_i = \det(L)v(Q). \quad \square$$

### 10.2.2 Volume of the parallelepiped in $\mathbb{R}^n$ . Gram determinants

The second proof is based on the computation of the volume of an arbitrary non-degenerate parallelepiped in  $\mathbb{R}^n$ .

Let  $u_1, \dots, u_m$  be vectors in  $\mathbb{R}^n$ . They generate the parallelepiped

$$P = P(u_1, \dots, u_m) = \left\{ x = \sum_{j=1}^m t_j u_j : t_j \in [0, 1] \right\}.$$

It is a subset of the linear subspace  $E_m$  spanned by the vectors  $u_1, \dots, u_m$ . Here, we shall compute the  $m$ -dimensional volume  $v_m(P(u_1, \dots, u_m))$  based on the following rule: if  $P(u_1, \dots, u_{m-1})$  is the ‘base’ of  $P(u_1, \dots, u_m)$ , then

$$v_m(P(u_1, \dots, u_m)) = |y| \cdot v_{m-1}(P(u_1, \dots, u_{m-1})),$$

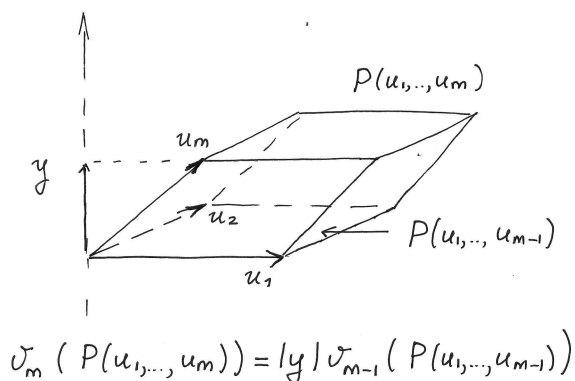
where  $y$  is the orthogonal projection of the vector  $u_m$  on the one-dimensional subspace  $E_m \ominus E_{m-1}$  (that is the ‘height’ of  $P(u_1, \dots, u_m)$ ): Thus, we need to find the length of the vector  $y$ . In fact, you’ve learnt how to do this in the Linear Algebra course.

**Claim 10.7.**

$$|y|^2 = \frac{\Gamma(u_1, \dots, u_m)}{\Gamma(u_1, \dots, u_{m-1})},$$

---

<sup>15</sup>For the sake of completeness, we recall the *proof*: Consider the operator  $LL^*$ . This is a positive operator: for any  $x \in \mathbb{R}^n \setminus \{0\}$ ,  $\langle LL^*x, x \rangle = |Lx|^2 > 0$ . Hence, there exists a (unique) positive square root  $P = \sqrt{LL^*}$ . Set  $U = P^{-1}L$ , then  $UU^* = P^{-1}LL^*P^{-1} = P^{-1}P^2P^{-1} = I$ . That is, the operator  $U$  is orthogonal, and  $L = PU$  is the decomposition we were looking for.  $\square$



where

$$\Gamma(u_1, \dots, u_m) = \begin{vmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \dots & \langle u_1, u_m \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \dots & \langle u_2, u_m \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_m, u_1 \rangle & \langle u_m, u_2 \rangle & \dots & \langle u_m, u_m \rangle \end{vmatrix}.$$

is the Gram determinant.

Note that  $\Gamma(u_1, \dots, u_m) = \det(U^*U)$ , where

$$U = \begin{pmatrix} u_{11} & \dots & u_{m1} \\ \vdots & \ddots & \vdots \\ u_{1n} & \dots & u_{mn} \end{pmatrix}$$

is the matrix whose  $j$ -th column consists of the coordinates of the vector  $u_j$  in the chosen orthonormal basis  $\{e_i\}$  in  $\mathbb{R}^n$ . In other words,  $U$  is the matrix of the linear operator  $L$  such that  $u_i = Le_i$ .

**Corollary 10.8.**

$$v_m(P(u_1, \dots, u_m)) = \sqrt{\Gamma(u_1, \dots, u_m)}.$$



Indeed, by the claim,

$$\begin{aligned}
 v_m^2(P(u_1, \dots, u_m)) &= \frac{\Gamma(u_1, \dots, u_m)}{\Gamma(u_1, \dots, u_{m-1})} \cdot v_{m-1}^2(P(u_1, \dots, u_{m-1})) \\
 &= \dots = \frac{\Gamma(u_1, \dots, u_m)}{\Gamma(u_1, u_1)} \cdot v_1^2(P(u_1)) \\
 &= \frac{\Gamma(u_1, \dots, u_m)}{|u_1|^2} \cdot |u_1|^2 = \Gamma(u_1, \dots, u_m).
 \end{aligned}$$

*Proof of the Claim:* Let

$$u_m = \sum_{j=1}^{m-1} \alpha_j u_j + y.$$

Then

$$\langle u_i, u_m \rangle = \sum_{j=1}^{m-1} \alpha_j \langle u_i, u_j \rangle + \langle u_i, y \rangle.$$

Thus the last column of the determinant is the linear combination of the first  $m - 1$  columns (with the coefficient  $\alpha_j$ ) and the column

$$\begin{pmatrix} \langle u_1, y \rangle \\ \vdots \\ \langle u_{m-1}, u_m \rangle \\ \langle u_m, y \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ |y|^2 \end{pmatrix}.$$

Putting this column on the  $m$ -th place instead of the original one, we get the claim.  $\square$

**Note 10.9.** The proof of the claim also gives the properties of the Gram matrix which we will use later:

$$\Gamma(u_1, \dots, u_m) \geq 0,$$

and

$$\Gamma(u_1, \dots, u_m) = 0$$

if and only if the vectors  $u_1, \dots, u_m$  are linearly dependent.

**Note 10.10.** The proof also reveals the geometric meaning of the claim: assume, we have the vectors  $u_1, \dots, u_m$  and we want to approximate the vector  $v$  by the linear combination of these vectors. In other words, we are looking for

$$\min_{\alpha_1, \dots, \alpha_m} \left\| v - \sum_{j=1}^m \alpha_j u_j \right\|.$$

Then this minimum (i.e. the distance between  $v$  and the linear span of  $u_1, \dots, u_m$ ) equals

$$\frac{\Gamma(u_1, \dots, u_m, v)}{\Gamma(u_1, \dots, u_m)}.$$

**Exercise 10.11.** 1.  $\Gamma(u_1, \dots, u_m) \leq \Gamma(u_1, \dots, u_k) \cdot \Gamma(u_{k+1}, \dots, u_m)$ ,  $1 \leq k \leq m - 1$ .

Hint: first, show that for  $1 \leq j \leq k - 1$ ,

$$\frac{\Gamma(u_j, u_{j+1}, \dots, u_m)}{\Gamma(u_{j+1}, \dots, u_m)} \leq \frac{\Gamma(u_j, u_{j+1}, \dots, u_k)}{\Gamma(u_{j+1}, \dots, u_k)}$$

2.  $\Gamma(u_1, \dots, u_m) \leq |u_1|^2 \cdot \dots \cdot |u_m|^2$ ; that is *the volume of the  $P(u_1, \dots, u_m)$  never exceeds the volume of the brick with the length-sides  $|u_1|, \dots, |u_m|$ .*
3. When the equality sign attains in these inequalities?
4. (Hadamard's inequality) Let

$$A = \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix}.$$

Then

$$A^2 \leq \sum_1^n a_{1k}^2 \cdot \sum_1^n a_{2k}^2 \cdot \dots \cdot \sum_1^n a_{nk}^2.$$

In particular, if  $|a_{ij}| \leq 1$  for all  $i$  and  $j$ , we get  $A \leq n^{n/2}$ .

Return to the volume of  $LQ$ ,  $Q$  the unit cube in  $\mathbb{R}^n$ ,  $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ . Let  $e_1, \dots, e_n$  be the orthonormal basis in  $\mathbb{R}^n$ ,  $u_j = Le_j$ . Then

$$\langle u_j, u_k \rangle = \langle Le_j, Le_k \rangle = \langle L^* Le_j, e_k \rangle,$$

that is,  $\Gamma(u_1, \dots, u_m) = \det(L^*L)$ , and  $v^2(LQ) = (\det(L))^2$ . Thus we got the second proof that  $v(LQ) = |\det(L)|$ .

### 10.3 The infinitesimal version

Let  $T$  be a  $C^1$  diffeomorphism of an open set  $U$ . Intuitively, if a cube  $Q \subset U$  is small, then the mapping  $T$  on  $Q$  is close to its linear part, i.e. to  $T(x) + D_T(x)(x - x_0)$ ,  $x \in Q$ , and  $v(TQ) \approx |\det D_T(x)|v(Q)$ . Recall that the determinant  $\det D_T(x)$  is called the *Jacobian* of  $T$  at  $x$ .

We use notation  $Q \downarrow x$  which means that cubes  $Q$  contain the point  $x$  and  $\text{diam}(Q) \rightarrow 0$ .

**Theorem 10.12.** Let  $T$  be a  $C^1$ -diffeomorphism of an open set  $U$ . Then, for each  $x \in U$ ,

$$\lim_{Q \downarrow x} \frac{v(TQ)}{v(Q)} = |J_T(x)|.$$

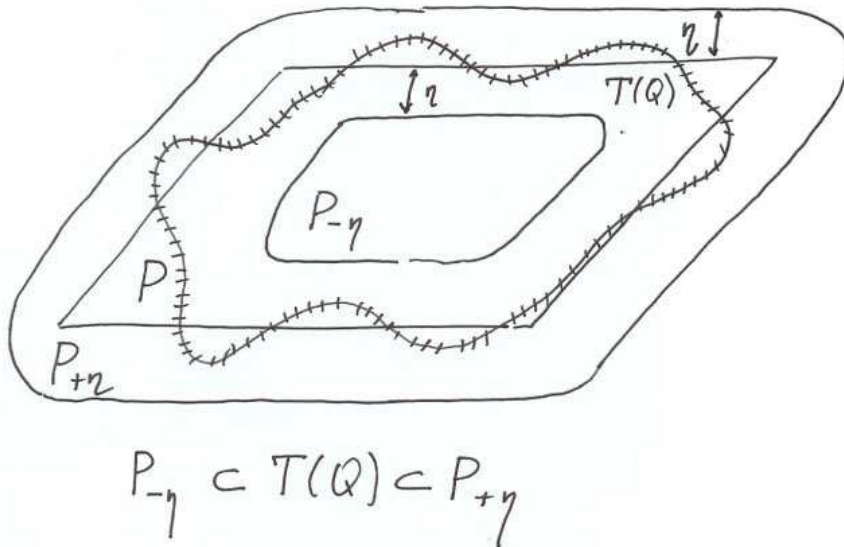
*Proof:* The idea is straightforward. Given  $\epsilon > 0$ , choose  $\delta > 0$  so small that if the diameter of  $Q$  is  $= \delta$ , and  $y \in Q$ , then

$$(10.13) \quad |T(y) - T(x) - D_T(x)(y - x)| < \epsilon|y - x| \leq \epsilon\delta$$

If  $y$  fills the whole cube  $Q$ , then the vector  $T(x) + D_T(x)(y - x)$  fills a parallelepiped  $P$  of volume  $|J_T(x)|v(Q)$ . Next, by (10.13),

$$P_{-\epsilon\delta} \subset T(Q) \subset P_{+\epsilon\delta}.$$

Here we use the following notations:  $A_{+\eta} = \{x: \text{dist}(x, A) \leq \eta\}$  is the  $\eta$ -neighbourhood of the set  $A$ , and  $A_{-\eta} = \{x: \text{dist}(x, \mathbb{R}^n \setminus A) \geq \eta\}$ . It re-



mains to estimate the volumes of the thin layer  $P_{+\epsilon\delta} \setminus P_{-\epsilon\delta}$ . The volume of this layer is majorized by the sum of the volumes of the  $\epsilon\delta$ -neighbourhoods of the faces of  $P$ . The volume of such a neighbourhood is bounded from above by  $2\epsilon \cdot \delta$  times the  $n - 1$ -dimensional volume of the  $n - 1$ -dimensional

$\epsilon\delta$ -neighbourhood of the face. The length-sides of the parallelepiped  $P$  are bounded by  $C(D_T)\delta$ , thus this  $n - 1$ -dimensional  $\epsilon\delta$ -neighbourhood is contained in an  $n - 1$ -dimensional parallelepiped with the length-sides bounded by  $C(D_T, n)\delta$ , and hence the volume of this neighbourhood is

$$\leq C(D_T, n)\delta^{n-1}.$$

We get the estimate for the volume of the thin layer:

$$C(D_T, n)\epsilon\delta \cdot \delta^{n-1} = C(D_T, n)\epsilon \cdot \delta^n.$$

Thus

$$|v(TQ) - v(P)| \leq C(D_T, n)\epsilon \cdot \delta^n,$$

or

$$|v(TQ) - |J_T(x)|v(Q)| \leq C(D_T, n)\epsilon \cdot \delta^n.$$

Recall that the diameter of  $Q$  is  $= \delta$ , that is,  $v(Q) \geq c(n)\delta^n$ . We get

$$\left| \frac{v(TQ)}{v(Q)} - |J_T(x)| \right| \leq C(D_T, n)\epsilon.$$

This completes the proof. □

## 10.4 The additive set-functions

We denote by  $\mathcal{J}$  the collection of all Jordan-measurable subsets of  $\mathbb{R}^n$ ; by  $\mathcal{J}(U)$  we denote the collection of all Jordan-measurable subsets of the open set  $U$ .

**Definition 10.14.** The function  $\mu: \mathcal{J}(U) \rightarrow \mathbb{R}^1$  is called an *additive set-function* if it satisfies the following conditions:

- *additivity:*

$$\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2), \quad \Omega_1, \Omega_2 \in \mathcal{J}(U), \Omega_1 \cap \Omega_2 = \emptyset;$$

- *continuity from below:* if  $\Omega_j \uparrow \Omega$ , then  $\mu(\Omega_j) \rightarrow \mu(\Omega)$ ;
- *differentiability with respect to the cubes:* there exists the “derivative”

$$\mu'(x) = \lim_{Q \downarrow x} \frac{\mu(Q)}{v(Q)}.$$

**Example:**

$$\mu_f(\Omega) = \int_{\Omega} f,$$

where  $f$  is a bounded continuous function on  $U$ .

The additivity is obvious. The continuity from below follows from the boundedness of  $f$ :

$$\left| \int_{\Omega} f - \int_{\Omega_j} f \right| \leq \|f\|_{\infty} \int_{\Omega \setminus \Omega_j} \mathbb{1} = \|f\|_{\infty} (v(\Omega) - v(\Omega_j)).$$

To see that the RHS tends to zero we use the following

**Claim 10.15.** *Let  $\Omega_j \uparrow \Omega$  be an exhaustion of  $\Omega$  by Jordan subsets. Then*

$$\lim_{j \rightarrow \infty} v(\Omega_j) = v(\Omega).$$

*Proof of the claim:* Given  $\epsilon > 0$ , choose a finite union of open cubes  $U \supset \partial\Omega$  such that  $v(U) < \epsilon$ . If  $j$  is large enough, then  $\Omega \setminus \Omega_j \subset U$  (why?). Then

$$v(\Omega) - v(\Omega_j) = v(\Omega \setminus \Omega_j) \leq v(U) < \epsilon.$$

□

The differentiability of the set-function  $\mu_f$  follows by the mean value theorem: for each cube  $Q$  there exists  $\xi \in Q$  such that

$$\mu_f(Q) = f(\xi)v(Q).$$

□

Apparently, this example is “generic”:

**Theorem 10.16.** *Let  $\mu$  be an additive set-function, such that  $\mu'(x)$  is bounded and continuous. Then, for any  $\Omega \in \mathcal{J}(U)$ ,*

$$\mu(\Omega) = \int_{\Omega} \mu'(x) dx.$$

*Proof:* it suffices to prove the theorem in the case  $\mu'(x) \equiv 0$  (then we can apply this special case to the set-function  $\mu - \mu_f$ ,  $f = \mu'$ ). WLOG, we assume that  $\Omega$  is a cube, having the result for the cubes, we easily get the general case by approximating the Jordan sets from below by finite unions of the cubes. We need to show that  $\mu(Q) = 0$  for any cube  $Q$ .

Let the result be wrong; i.e. there exists the cube  $Q_0$  such that  $\mu(Q_0) \neq 0$ , then, for some  $\lambda > 0$ ,  $|\mu(Q_0)| \geq \lambda v(Q_0)$ . Then, using the bisections, we get a nested sequence of cubes  $Q_j$  such that the length sides of  $Q_j$  are twice smaller than those of  $Q_{j-1}$ , and  $|\mu(Q_j)| \geq \lambda v(Q_j)$ . Clearly,  $Q_j \downarrow x$ , and  $\mu'(x) \neq 0$ . Contradiction. □

## 10.5 Proof of the change of variables theorem:

Assume that  $\bar{\Omega} \subset U$  and set

$$\mu(\Omega) \stackrel{\text{def}}{=} \int_{T(\Omega)} f.$$

This is the additive set-function. Indeed, the additivity is obvious. Also, the continuity from below:

$$\mu(\Omega) - \mu(\Omega_j) = \int_{T(\Omega) \setminus T(\Omega_j)} f,$$

the sets  $T(\Omega_j)$  are Jordan (why?) and they exhaust the whole set  $T(\Omega)$ . As above, the differentiability follows by the mean value theorem: for any cube  $Q$ , there exists  $\xi \in Q$  such that

$$\mu(Q) = f(T(\xi))v(TQ),$$

thus

$$\frac{\mu(Q)}{v(Q)} = f(T(\xi)) \frac{v(TQ)}{v(Q)} \rightarrow f(T(x))|J_T(x)|$$

when  $Q \downarrow x$ . Thus,  $\mu'(x) = f(x)|J_T(x)|$ , and we apply the previous theorem.  $\square$

## 10.6 Examples and exercises

**Exercise 10.17** (spherical coordinates in  $\mathbb{R}^3$ ). Consider the map  $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,

$$F(r, \varphi, \theta) = r(\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta).$$

(i) Find and draw the images of the planes

$$r = \text{const}, \varphi = \text{const}, \theta = \text{const},$$

and of the lines

$$(\varphi, \theta) = \text{const}, (r, \theta) = \text{const}, (r, \varphi) = \text{const}.$$

(ii) Prove that  $F$  is surjective but not injective.

(iii) Show that  $J_F = r^2 \sin \theta$ . Find the points  $(r, \varphi, \theta)$ , where  $F$  is regular.

(iv) Let  $V = \mathbb{R}_+ \times (-\pi, \pi) \times (0, \pi)$ . Prove that  $F|_V$  is injective. Find  $U = F(V)$ .

(v) Compute (find the formula) the inverse map  $F^{-1}$  on  $U$ .

**Exercise 10.18.** Compute the integral

$$\iint_{[0,1]^2} \cos^2(\pi(x+y)) \, dx dy.$$

**Exercise 10.19.** Compute the integrals

1.

$$\iiint_{x^2+y^2+z^2 \leq 1} \frac{dx dy dz}{x^2 + y^2 + (z-2)^2},$$

2.

$$\iiint_{x^2+y^2+z^2 \leq 1} \frac{dx dy dz}{x^2 + y^2 + (z-1/2)^2}.$$

Answers:  $\pi(2 - \frac{3}{2} \log 3)$ ,  $\pi(2 + \frac{3}{2} \log 3)$ .

**Exercise 10.20.** Compute the integral

$$\iint \frac{dx dy}{(1+x^2+y^2)^2}$$

taken

(i) over one loop of the lemniscate  $(x^2 + y^2)^2 = (x^2 - y^2)$ ;

(ii) over the triangle with vertices at  $(0, 0)$ ,  $(2, 0)$ ,  $(1, \sqrt{3})$ .

Hint: use polar coordinates.

**Exercise 10.21.** Compute the integral over the four-dimensional unit ball:

$$\iiint \int_{x^2+y^2+u^2+v^2 \leq 1} e^{x^2+y^2-u^2-v^2} \, dx dy du dv.$$

Hint: The integral equals

$$\iint_{x^2+y^2 \leq 1} e^{x^2+y^2} \left( \iint_{u^2+v^2 \leq 1-(x^2+y^2)} e^{-(u^2+v^2)} \, du dv \right) \, dx dy.$$

Then use the polar coordinates.

### 10.6.1 Volume of the ellipsoid in $\mathbb{R}^n$

We start with a special case, and find the volume of the ellipsoid

$$\left\{ x : \sum_{j=1}^n \frac{x_j^2}{a_j^2} \leq 1 \right\}.$$

Changing the variables  $x_j = a_j y_j$ , we easily find that the volume of this ellipsoid equals  $a_1 \cdots a_n v(\mathbb{B})$ , where  $\mathbb{B}$  is the unit ball in  $\mathbb{R}^n$  (we already know its volume).

In particular, if  $n = 2$ , we get that the area of the ellipse  $\left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$  is  $ab\pi$ , and if  $n = 3$ , then the volume of the ellipsoid  $\left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$  is  $\frac{4\pi}{3} abc$ .

Now, consider a general  $n$ -dimensional ellipsoid in  $\mathbb{R}^n$  given by

$$\{x : \langle Ax, x \rangle \leq 1\},$$

where  $A \geq 0$  is a non-negative linear transformation. We make an orthogonal transformation  $x = Oy$  which reduces the matrix of  $A$  to the diagonal form. This transformation does not change the volume of the ellipsoid, and applying the special case considered above, we get the answer:

$$\frac{v(\mathbb{B})}{\sqrt{\lambda_1(A) \cdots \lambda_n(A)}},$$

where  $\lambda_1(A), \dots, \lambda_n(A)$  are the eigenvalues of  $A$ .

**Exercise 10.22.** Compute the integral

$$\iiint |xyz| \, dx dy dz$$

taken over the ellipsoid  $\{x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\}$ .

Answer:  $\frac{a^2 b^2 c^2}{6}$ .

**Exercise 10.23.** Find the volume cut off from the unit ball by the plane  $lx + my + nz = p$ .

### 10.6.2 Volume of the body of revolution in $\mathbb{R}^3$

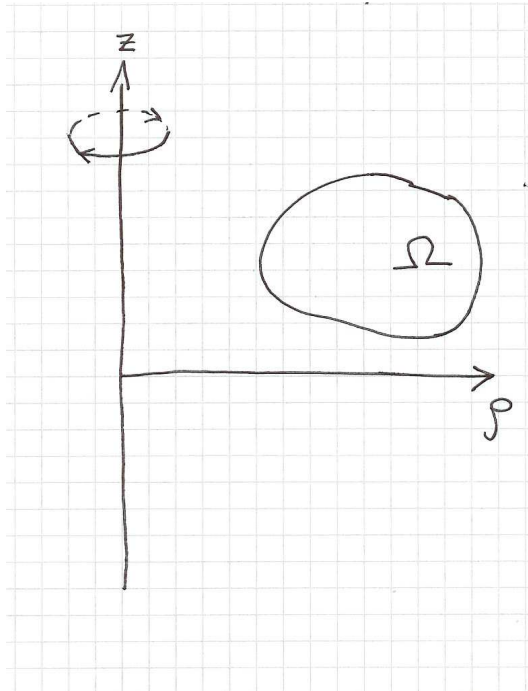
Take a plane domain  $\Omega$  that lies in the right half-plane of the  $(\rho, z)$ -plane, and consider a body of revolution obtained by rotation of the domain  $\Omega$  in  $\mathbb{R}^3$  around the  $z$ -axis; i.e.

$$\tilde{\Omega} = \{(\rho \cos \theta, \rho \sin \theta, z) : (\rho, z) \in \Omega, 0 \leq \theta \leq 2\pi\}.$$

Introduce the cylindrical coordinates:  $x = \rho \cos \theta$ ,  $y = \rho \sin \theta$ ,  $z = z$ . Then  $dx dy dz = \rho \, d\rho \, d\theta \, dz$ , and

$$v(\tilde{\Omega}) = \int_0^{2\pi} d\theta \iint_{\Omega} \rho \, d\rho \, dz = 2\pi \iint_{\Omega} \rho \, d\rho \, dz.$$





Often,  $\Omega$  is defined as  $\{(\rho, z) : a < z < b, 0 < \rho < \rho(z)\}$ . Then

$$\iint_{\Omega} \rho \, d\rho \, dz = \int_a^b dz \int_0^{\rho(z)} \rho \, d\rho = \frac{1}{2} \int_a^b \rho^2(z) \, dz,$$

and we get

$$v(\tilde{\Omega}) = \pi \int_a^b \rho^2(z) \, dz.$$

### 10.6.3 Center of masses and Pappus' theorem

Suppose we have a system of  $N$  material points  $(P_i, m_i)$ ,  $1 \leq i \leq N$ , in  $\mathbb{R}^2$ ,  $P_i = (X_i, Y_i)$  are the points, and  $m_i$  are the masses. The *center of mass* of this system is located at the point

$$P = \frac{\sum m_i P_i}{\sum m_i}.$$

If we have a continuous distribution  $p$  of masses in a plane domain  $\Omega$ , then the total mass of  $\Omega$  is

$$m(\Omega) = \iint_{\Omega} p,$$

and the coordinates of the center of masses are

$$X = \frac{\iint_{\Omega} xp(x, y) dx dy}{m(\Omega)}, \quad Y = \frac{\iint_{\Omega} yp(x, y) dx dy}{m(\Omega)}.$$

The integrals in the numerator are called the *1-st order moments of the distribution*  $p$ . The same formulas hold in  $\mathbb{R}^3$  (and, more generally, in  $\mathbb{R}^n$ ).

**Exercise 10.24.** Show that

1. the mass of the ball of radius  $r$  centered at the origin with density distribution  $p(x, y, z) = x^2y^2z^2$  is

$$M = \frac{4\pi r^9}{945},$$

2. the mass of the ellipsoid  $\{x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\}$  with density distribution  $p(x, y, z) = x^2 + y^2 + z^2$  is

$$M = \frac{4\pi abc}{15} (a^2 + b^2 + c^2).$$

The distribution of masses is *homogenous* if  $p$  is a constant function. Sometimes, the center of masses of a homogeneous distribution is called *centroid*. In this case, we always use normalization  $p \equiv 1$  which, of course, does not affect the position of the centroid.

**Exercise 10.25.** (a) Prove that the centroid of the triangle lies in the intersection of its medians.

(b) Suppose that  $\Omega$  is a finite union of triangles  $\Delta_i$ , and  $P_i$  are the centroids of  $\Delta_i$ . Prove that the centroid  $P$  of  $\Omega$  coincides with the centroid of the system of points  $P_i$  with masses  $m_i = \text{area}(\Delta_i)$ .

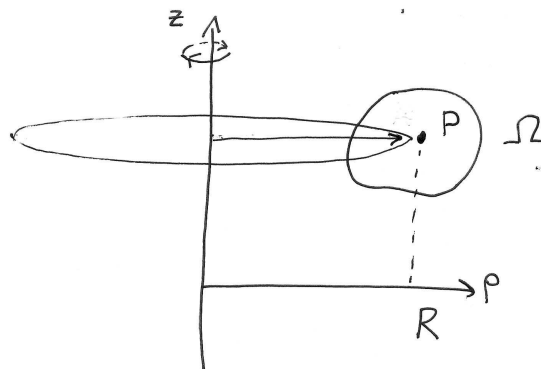
Of course, we can define the center of masses and centroid also for 3- (or  $n$ -) dimensional bodies.

**Exercise 10.26.** Find the centroids of the following bodies in  $\mathbb{R}^3$ :

1. The cone built over the unit disc, the height of the cone is  $h$ .

2. The tetrahedron bounded by the three coordinate planes and the plane  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ .
3. The hemispherical shell  $\{a^2 \leq x^2 + y^2 + z^2 \leq b^2, z \geq 0\}$ .
4. The octant of the ellipsoid  $\{x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1, x, y, z \geq 0\}$ .

We conclude with a beautiful ancient



Circumference described by the centroid of  $\Omega$

**Theorem 10.27** (Pappus). *The volume of the body of revolution  $\tilde{\Omega}$  obtained by rotation of the plane domain  $\Omega$  equals the area of  $\Omega$  times the length of the circle described by the centroid of  $\Omega$ .*

*Proof:* As we know,

$$v(\tilde{\Omega}) = 2\pi \iint_{\Omega} \rho \, d\rho dz = 2\pi \frac{\iint_{\Omega} \rho \, d\rho dz}{\text{area}(\Omega)} \cdot \text{area}(\Omega).$$

Observe that the centroid of  $\Omega$  describes the circle of the radius

$$R = \frac{\iint_{\Omega} \rho \, d\rho dz}{\text{area}(\Omega)}.$$

□

**Example 10.28.** Consider the solid torus  $T$  in  $\mathbb{R}^3$  obtained by rotation of the disc  $\{(\rho, z): (\rho - c)^2 + z^2 \leq r\}$  around the  $z$ -axis. Its volume is

$$v(T) = 2\pi c \cdot \pi r^2 = 2\pi^2 cr^2.$$

# 11 Improper Integrals

The definition of the Riemann integral we gave above has several drawbacks. For instance, it does not allow us to integrate unbounded functions, or consider unbounded domains of integrations. In this lecture, we fix this.

## 11.1 Definition

Let  $\Omega_m \uparrow \Omega$  be an exhaustion of an open set  $\Omega$  by Jordan sets  $\Omega_m$ . We already know that

1. if  $\Omega \in \mathcal{J}$ , then  $v(\Omega_m) \uparrow v(\Omega)$ ;
2. if  $\Omega \in \mathcal{J}$  and  $f \in \mathcal{R}(\Omega)$ , then

$$\int_{\Omega_m} f \rightarrow \int_{\Omega} f.$$

We want to accept the second property as the definition of the integral  $\int_{\Omega} f$  in the cases when the function  $f$  is unbounded, or the domain  $\Omega$  is unbounded<sup>16</sup>. The problem is that the different exhaustions can give the different answers, but we look for the definition of the integral which does not depend on the exhaustion. For example, consider

$$\int_{-\infty}^{\infty} \frac{1+x}{1+x^2} dx.$$

Then

$$\lim_{n \rightarrow \infty} \int_{-n}^n \frac{1+x}{1+x^2} dx = \lim_{n \rightarrow \infty} \arctan x \Big|_{-n}^n + \frac{1}{2} \log(1+x^2) \Big|_{-n}^n = \pi,$$

but

$$\lim_{n \rightarrow \infty} \int_{-n}^{2n} \frac{1+x}{1+x^2} dx = \lim_{n \rightarrow \infty} \arctan x \Big|_{-n}^{2n} + \frac{1}{2} \log(1+x^2) \Big|_{-n}^{2n} = \pi + \log 2.$$

The good news is the following

**Claim 11.1.** *If  $f \geq 0$ , and  $\Omega_m \uparrow \Omega$ ,  $\Omega'_k \uparrow \Omega$  are two exhaustions of an open set  $\Omega$ , then the following limits are equal:*

$$\lim_{m \rightarrow \infty} \int_{\Omega_m} f = \lim_{k \rightarrow \infty} \int_{\Omega'_k} f.$$

---

<sup>16</sup>One can use the same approach in the case when the domain  $\Omega$  is not Jordan.

*Proof:* Let

$$\int_{\Omega_m} f \rightarrow A, \quad \int_{\Omega'_k} f \rightarrow B, \quad A, B \in [0, +\infty].$$

Then

$$\int_{\Omega'_k} f \stackrel{(ii)}{=} \lim_{m \rightarrow +\infty} \int_{\Omega'_k \cap \Omega_m} f \leq \lim_{m \rightarrow \infty} \int_{\Omega_m} f = A.$$

Hence,  $B \leq A$ . By symmetry,  $A \leq B$ . Done.  $\square$ .

**Example 11.2** (Poisson). Consider the integral

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dx dy.$$

First, let us exhaust the plane by the discs  $\Omega_m = \{x^2 + y^2 \leq m^2\}$ . In this case,

$$\iint_{\Omega_m} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} d\theta \int_0^m e^{-r^2} r dr = \pi(1 - e^{-m^2}) \rightarrow \pi.$$

Now, consider the exhaustion by the squares  $\Omega'_k = \{|x|, |y| \leq k\}$ . We get

$$\iint_{\Omega'_k} e^{-(x^2+y^2)} dx dy = \int_{-k}^k e^{-x^2} dx \cdot \int_{-k}^k e^{-y^2} dy \rightarrow \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right)^2.$$

Juxtaposing the answers, we obtain the celebrated Poisson formula:

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.}$$

$\square$

The corresponding  $n$ -dimensional integral equals:

$$(11.3) \quad \boxed{\int_{\mathbb{R}^n} e^{-(Ax,x)} dx = \frac{\pi^{n/2}}{\sqrt{\det A}},}$$

here  $A$  is a positive linear transformation.

First, observe that

$$(11.4) \quad \int_{\mathbb{R}^n} e^{-|x|^2} dx = \left( \int_{-\infty}^{\infty} e^{-t^2} dt \right)^n = \pi^{n/2}.$$

Also observe that

$$(11.5) \quad \int_{-\infty}^{\infty} e^{-at^2} dt = \sqrt{\frac{\pi}{a}}.$$

If the matrix of  $A$  is diagonal, then (11.3) follows from (11.4) and (11.5). If the matrix is not diagonal, we make the orthogonal transformation  $x \rightarrow Ox$  which reduced the matrix of  $A$  to the diagonal one.  $\square$

**Example 11.6.** Let  $f(x) = |x|^{-\alpha}$ ,  $\alpha > 0$ . We consider separately two cases:  $\Omega_1 = \mathbb{B}$ , and  $\Omega_2 = \mathbb{R}^n \setminus \mathbb{B}$ .

First, suppose that we integrate over the unit ball  $\mathbb{B}$ . We split the ball into the layers  $C_n = 2^{-k} \leq |x| \leq 2^{1-k}$ ,  $k \geq 1$ . If  $x \in C_k$ , then integrand is between  $2^{\alpha(k-1)}$  and  $2^{\alpha k}$ . Also,  $c_1 2^{-kn} \leq \text{vol}(C_k) \leq c_2 2^{-kn}$  (the constants  $c_1$  and  $c_2$  depend on the dimension  $n$  only). Hence the integral  $\int_{\mathbb{B}} \frac{dx}{|x|^\alpha}$  converges and diverges simultaneously with the series  $\sum_{k \geq 1} 2^{(\alpha-n)k}$ . We see that the integral converges if  $\alpha < n$ , and diverges otherwise.

In the second case, we use a similar decomposition into the layers  $\{2^k \leq |x| \leq 2^{k+1}\}$  and obtain the series  $\sum_{k \geq 1} 2^{(n-\alpha)k}$ . Hence, the second integral converges iff  $\alpha > n$ . In particular, the both integrals *never* converge simultaneously.

One more

**Example 11.7.**

$$\begin{aligned} \iint_{x^2+y^2 < 1} \frac{dx dy}{(1-x^2-y^2)^\alpha} &= \int_0^{2\pi} d\theta \int_0^1 \frac{r dr}{(1-r^2)^\alpha} \\ &= 2\pi \cdot \frac{1}{2} \int_0^1 \frac{ds}{(1-s)^\alpha} = \pi \int_0^1 \frac{dt}{t^\alpha} = \frac{\pi}{1-\alpha}. \end{aligned}$$

Of course, the computation has a sense only if  $\alpha < 1$ .

**Definition 11.8.** If for *any* exhaustion  $\Omega_m \uparrow \Omega$ ,  $f \in \mathcal{R}(\Omega_m)$ , there exists the limit

$$\lim_{m \rightarrow \infty} \int_{\Omega_m} f,$$

that does not depend on the choice of the exhaustion, then we say that the integral

$$\int_{\Omega} f$$

converges and equals to this limit.

Now we give a very useful MAJORANT sufficient condition for convergence:

**Claim 11.9.** *Suppose that  $|f| \leq g$  on  $\Omega$  and that the integral  $\int_{\Omega} g$  converges. Also suppose that for any exhaustion  $\Omega_m \uparrow \Omega$  by Jordan sets,  $f \in \mathcal{R}(\Omega_m)$ . Then the integrals  $\int_{\Omega} f$  and  $\int_{\Omega} |f|$  converge.*

*Proof:* Since  $f \in \mathcal{R}(\Omega_m)$ ,  $|f| \in \mathcal{R}(\Omega_m)$  (the Lebesgue criterium). Fix  $\epsilon > 0$ , and choose large  $m, k$ ,  $m > k$ . Then

$$0 \leq \int_{\Omega_m} |f| - \int_{\Omega_k} |f| = \int_{\Omega_m \setminus \Omega_k} |f| \leq \int_{\Omega_m \setminus \Omega_k} g = \int_{\Omega_m} g - \int_{\Omega_k} g < \epsilon,$$

if  $k$  is sufficiently large. Thus the sequence of integrals  $\left\{ \int_{\Omega_m} |f| \right\}$  converges, and the integral  $\int_{\Omega} |f|$  exists.

Now, set  $f_+ = \max(f, 0)$ ,  $f_- = (-f)_+ = \max(-f, 0)$ , and observe that  $|f| = f_+ + f_-$ , where  $0 \leq f_-, f_+ \leq |f|$ . Hence, there exist the integrals

$$\int_{\Omega} f_{\pm},$$

and therefore there exists the integral

$$\int_{\Omega} f = \int_{\Omega} f_+ - \int_{\Omega} f_-.$$

The proof is complete. □

**Exercise 11.10.** Compute the integral

$$\int_Q \frac{dx}{|x|},$$

where  $Q = [0, 1]^2$  is the unit square in  $\mathbb{R}^2$ .

**Exercise 11.11.** Compute

1.

$$\iint_{\mathbb{R}^2} |ax + by| e^{-(x^2+y^2)/2} dx dy,$$

2.

$$\int_{\mathbb{R}^n} |\langle x, a \rangle|^p e^{-|x|^2} dx, \quad a \in \mathbb{R}^n, \quad p > -1.$$



Hints:

1. Rotating the plane, introduce new coordinates  $(x', y')$  such that  $x' = \frac{ax+by}{\sqrt{a^2+b^2}}$ .
2. The general case is reduced to  $a = (0, \dots, 0, |a|)$ .

**Exercise 11.12.** Prove that

$$\int_{\mathbb{R}^3} \frac{d\xi}{|x - \xi|^2 |y - \xi|^2} = \frac{c}{|x - y|}.$$

Try to find the positive constant  $c$ .

**Exercise 11.13.** For which values of  $p$  and  $q$  the integral

$$\iint_{|x|+|y|\geq 1} \frac{dxdy}{|x|^p + |y|^q}.$$

converges?

**Exercise 11.14.** Find the sign of the integral

$$\iint_{\max(|x|, |y|) \leq 1} \log(x^2 + y^2) dxdy.$$

**Exercise 11.15.** Verify if the integrals

$$\iint_{\mathbb{R}^2} \frac{dxdy}{1 + x^{10}y^{10}} \quad \iint_{\mathbb{R}^2} e^{-(x+y)^4} dxdy$$

converge or diverge?

## 11.2 Useful inequalities

Here are the integral versions of the classical inequalities of Cauchy-Schwarz, Hölder and Minkowski. By  $L^p(\mathbb{R}^n)$  we denote the class of Riemann-integrable functions in  $\mathbb{R}^n$  such that

$$\|f\|_p \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}^n} |f| \right)^{1/p} < \infty.$$

**Exercise 11.16** (Cauchy-Schwarz, Hölder). 1. Suppose  $f, g \in L^2(\mathbb{R}^n)$ . Then

$$\left| \int_{\mathbb{R}^n} f \cdot g \right|^2 \leq \int_{\mathbb{R}^n} |f|^2 \cdot \int_{\mathbb{R}^n} |g|^2.$$

2. Suppose  $f \in L^p(\mathbb{R}^n)$ ,  $g \in L^q(\mathbb{R}^n)$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\left| \int_{\mathbb{R}^n} f \cdot g \right| \leq \left( \int_{\mathbb{R}^n} |f|^p \right)^{1/p} \cdot \left( \int_{\mathbb{R}^n} |g|^q \right)^{1/q}.$$

Hint: use the inequality  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ ,  $a, b \in \mathbb{R}_+$ .

**Exercise 11.17** (Minkowski). If  $f, g \in L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p, \quad 1 \leq p < \infty.$$

Hint: start with  $|a + b|^p \leq |a||a + b|^{p-1} + |b||a + b|^{p-1}$ , then use Hölder's inequality.

### 11.3 The Newton potential

The gravitational force  $F$  exerted by the particle of mass  $\mu$  at point  $\xi$  on a particle of mass  $m$  at point  $x$  is

$$F = -\frac{\gamma m \mu}{|x - \xi|^3} (x - \xi) = \gamma m \nabla \frac{\mu}{|x - \xi|},$$

$\gamma$  is the gravitational constant. This is the celebrated Newton law of gravitation. The function  $U: x \mapsto \frac{\mu}{|x - \xi|}$  is called the Newton (or gravitational) potential. The reason to replace the force  $F$  by the potential  $U$  is simple: it is easier to work with scalar functions than with the vector ones. If the force  $F$  is known, then one can write down the differential equations of motion of the particle (Newton's second law)  $m\ddot{x} = F$ , or

$$\ddot{x} = \gamma \nabla \frac{\mu}{|x - \xi|}.$$

Then one hopes to integrate these equations and to find out where is the particle at time  $t$ .

What happens if we have a system of point masses  $\mu_1, \dots, \mu_N$  at points  $\xi_1, \dots, \xi_N$ ? The forces are to be added, and the corresponding potential is

$$U(x) = \sum_{j=1}^N \frac{\mu_j}{|x - \xi_j|}.$$

Now, suppose that the gravitational masses are distributed with continuous density  $\mu(\xi)$  over a portion  $\Omega$  of the space. Then the Newton potential is defined as

$$U(x) = \int_{\Omega} \frac{\mu(\xi) d\xi}{|\xi - x|}$$

(the integral is a triple one, of course), and the corresponding gravitational force (after normalization  $\gamma = 1$ ,  $m = 1$ ) is again  $F = \nabla U$ .

Let us compute the Newton potential of the homogeneous mass distribution (i.e.,  $\mu(\xi) \equiv 1$  within the ball  $B_R$  of radius  $R$  centered at the origin:

$$U(x) = \int_{B_R} \frac{d\xi}{|x - \xi|}.$$

By symmetry,  $U$  is a radial function, that is it depends only on  $|x|$ .

**Exercise 11.18.** Check this!

Thus, it suffices to compute  $U$  at the point  $x = (0, 0, z)$ ,  $z \geq 0$ . We use the spherical coordinates:  $\xi_1 = r \sin \theta \cos \varphi$ ,  $\xi_2 = r \sin \theta \sin \varphi$ ,  $\xi_3 = r \cos \theta$ . Then

$$U = \int_0^R dr \, 2\pi \int_0^\pi \frac{r^2 \sin \theta \, d\theta}{\sqrt{(z - r \cos \theta)^2 + r^2 \sin^2 \theta}} = \int_0^R dr \, 2\pi \underbrace{\int_0^\pi \frac{r^2 \sin \theta \, d\theta}{\sqrt{z^2 - 2zr \cos \theta + r^2}}}_V.$$

The underbraced expression  $V$  is the Newton potential of the homogeneous sphere of radius  $r$ . We compute  $V$  using the variable

$$t^2 = z^2 - 2zr \cos \theta + r^2.$$

Then  $|z - r| < t < |z + r|$ , and  $t \, dt = zr \sin \theta \, d\theta$ . We get

$$V = 2\pi r^2 \int_{|z-r|}^{|z+r|} \frac{t \, dt}{zr \cdot t} = \frac{2\pi r}{z} (|z + r| - |z - r|) = 4\pi \min\left(\frac{r^2}{z}, r\right).$$

Now, we easily find  $U$  by integration:

$$U = \int_0^R V \, dr.$$

If  $x$  is an external point, i.e.,  $z \geq R$ , then

$$U = 4\pi \int_0^R \frac{r^2}{z} \, dr = \frac{4\pi R^3}{3z}.$$

If  $x$  is located inside the ball, i.e.,  $z < R$ , then

$$U = 4\pi \left( \int_0^z \frac{r^2}{z} \, dr + \int_z^R r \, dr \right) = 4\pi \left( \frac{z^2}{3} + \frac{R^2}{2} - \frac{z^2}{2} \right) = \frac{2\pi}{3} (3R^2 - z^2).$$

Thus,

$$U(x) = \begin{cases} \frac{4\pi R^3}{|x|} & \text{if } |x| \geq R, \\ \frac{2\pi}{3}(3R^2 - |x|^2) & \text{if } |x| < R. \end{cases}$$

Done! □

Observe that  $4\pi R^3/3$  is exactly the total mass of the ball  $B_R$ . That is, together with Newton, we arrived at the conclusion that *the gravitational potential, and hence the gravitational force exerted by the homogeneous ball on a particle is the same as if the whole mass of the ball were all concentrated at its center, if the point is outside the ball*. Of course, you heard about this already in the high-school.

Another important conclusion is that the potential  $V$  of the homogeneous sphere does not depend on the point  $x$  when  $x$  is inside the sphere! Hence, *the gravitational force is zero inside the sphere*. The same is true for the homogeneous shell  $\{\xi: a < |\xi| < b\}$ : there is no gravitational force inside the shell.

**Exercise 11.19.** Check that all the conclusions are true when the mass distribution  $\mu(\xi)$  is radial:  $\mu(\xi) = \mu(\xi')$  if  $|\xi| = |\xi'|$ . I.e., compute the mass of the ball  $\mathbb{B}_R$ , the potential of the ball  $\mathbb{B}_R$ , and the potential of the shell  $\{x: R_1 < |x| < R_2\}$ .

**Exercise 11.20.** 1. Find the potential of the homogeneous solid ellipsoid  $(x^2 + y^2)/b^2 + z^2/c^2 \leq 1$  at its center.

2. Find the potential of the homogeneous solid cone of height  $h$  and radius of the base  $r$  at its vertex.

**Problem 11.21.** Show that at sufficiently large distances the potential of a solid  $S$  is approximated by the potential of a point with the same total mass located at the center of mass of  $S$  with an error less than a constant divided by the square of the distance. The potential itself decays as the distance, so the approximation is good<sup>17</sup>.

## 11.4 The Euler Gamma-function

**Definition 11.22.**

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s > 0.$$

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<sup>17</sup>This estimate is rather straightforward. A more accurate argument shows that the error is of order constant divided by the *cube* of the distance.

We know that  $\Gamma(s+1) = s\Gamma(s)$ ,  $\Gamma(n) = (n-1)!$  (start with  $\Gamma(1) = 1$ ), and

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} dt = 2 \int_0^\infty e^{-x^2} dx = \sqrt{\pi}.$$

**Exercise 11.23.** Find the limits  $\lim_{s \rightarrow 0} s\Gamma(s)$  and  $\lim_{s \rightarrow 0} \frac{\Gamma(\alpha s)}{\Gamma(s)}$ .

There are two remarkable properties of the  $\Gamma$ -function which we'd like to mention here without proof. The first one is the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

that extends the  $\Gamma$ -function to the negative non-integer values of  $s$ . The second one is the celebrated Stirling's asymptotic formula

$$\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} e^{\theta(s)}, \quad 0 < \theta < \frac{1}{12s}$$

The Gamma-function is very useful in computation of integrals.

**Claim 11.24.**

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \alpha, \beta > 0.$$

The left hand side is called the Beta-function, and denoted by  $B(\alpha, \beta)$ .

*Proof:*

$$\Gamma(\alpha) \cdot \Gamma(\beta) = \int_0^\infty \int_0^\infty t_1^{\alpha-1} t_2^{\beta-1} e^{-(t_1+t_2)} dt_1 dt_2.$$

We introduce the new variables  $(u, v)$ :

$$\begin{cases} t_1 &= u(1-v) \\ t_2 &= uv. \end{cases}$$

This is a one-to-one mapping of the 1-st quadrant  $\{t_1, t_2 > 0\}$  onto the semi-strip  $\{u > 0, 0 < v < 1\}$ . This can be seen, for example, from the formulas

$$\begin{cases} t_1 + t_2 &= u \\ t_1/t_2 &= \frac{1}{v} - 1. \end{cases}$$

The Jacobian equals

$$\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u.$$

We obtain

$$\begin{aligned}\Gamma(\alpha) \cdot \Gamma(\beta) &= \int_0^\infty u \, du \int_0^1 dv u^{\alpha-1+\beta-1} (1-v)^{\alpha-1} v^{\beta-1} e^{-u} \\ &= \int_0^\infty u^{\alpha+\beta-1} e^{-u} \, du \cdot \int_0^1 (1-v)^{\alpha-1} v^{\beta-1} \, dv = \Gamma(\alpha + \beta) B(\alpha, \beta).\end{aligned}$$

□

There are many definite integrals that can be expressed via the Gamma-function.

**Example 11.25.** Consider the integral

$$\int_0^{\pi/2} \sin^{\alpha-1} \theta \cos^{\beta-1} \theta \, d\theta.$$

We rewrite it in the form

$$\int_0^{\pi/2} (\sin^2 \theta)^{\alpha/2-1} (\cos^2 \theta)^{\beta/2-1} \sin \theta \cos \theta \, d\theta,$$

and change the variable:

$$\sin^2 \theta = x, \quad dx = 2 \sin \theta \cos \theta \, d\theta.$$

We get

$$\frac{1}{2} B\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\alpha+\beta}{2}\right)}.$$

□

A special case of this formula says that

$$\begin{aligned}\int_0^{\pi/2} \sin^{\alpha-1} \theta \, d\theta &= \int_0^{\pi/2} \cos^{\alpha-1} \theta \, d\theta \\ &= \frac{1}{2} \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}.\end{aligned}$$

**Exercise 11.26.** 1. Check that

$$B(x, x) = 2^{1-2x} B\left(x, \frac{1}{2}\right).$$

2. Deduce the *duplication formula*:

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right).$$

**Exercise 11.27.** Show that

$$\int_0^1 x^4 \sqrt{1-x^2} dx = \frac{\pi}{32},$$

$$\int_0^\infty x^m e^{-x^n} dx = \frac{1}{n} \Gamma\left(\frac{m+1}{n}\right),$$

$$\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, \quad n \in \mathbb{N},$$

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} = \frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}.$$

We mention without proof another very useful formula

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin \pi p}, \quad 0 < p < 1.$$

There is a simple proof that that uses the residues theorem from the complex analysis course. This formula yields that  $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$  (note that  $\int_0^1 t^{p-1}(1-t)^{-p} dt = \int_0^\infty \frac{x^{p-1}}{1+x} dx$ ).

#### 11.4.1 The Dirichlet formula

We start with the Dirichlet formula:

**11.28.**

$$\int \dots \int_{\substack{x_1, \dots, x_n \geq 0, \\ x_1 + \dots + x_n \leq 1}} x_1^{p_1-1} \dots x_n^{p_n-1} dx_1 \dots dx_n = \frac{\Gamma(p_1) \dots \Gamma(p_n)}{\Gamma(p_1 + \dots + p_n + 1)}, \quad p_1, \dots, p_n > 0.$$

*Proof:* we use induction with respect to the dimension  $n$ . For  $n = 1$  the formula is obvious:

$$\int_0^1 x_1^{p_1-1} dx_1 = \frac{1}{p_1} = \frac{\Gamma(p_1)}{\Gamma(p_1 + 1)}.$$

Now, denote the  $n$ -dimensional integral by  $I_n$ , and assume that the result is valid for  $n - 1$ . Then

$$I_n = \int_0^1 x_n^{p_n-1} dx_n \int \dots \int_{\substack{x_1, \dots, x_{n-1} \geq 0 \\ x_1 + \dots + x_{n-1} \leq 1-x_n}} x_1^{p_1-1} \dots x_{n-1}^{p_{n-1}-1} dx_1 \dots dx_{n-1}.$$

To compute the inner integral, we introduce the new variables  $x_1 = (1-x_n)\xi_1$ , ...,  $x_{n-1} = (1-x_n)\xi_{n-1}$ . Then the inner integral equals

$$\begin{aligned} (1-x_n)^{n-1+(p_1-1)+\dots+(p_{n-1}-1)} &\cdot \int \dots \int_{\substack{\xi_1, \dots, \xi_{n-1} \geq 0 \\ \xi_1 + \dots + \xi_{n-1} \leq 1}} \xi_1^{p_1-1} \dots \xi_{n-1}^{p_{n-1}-1} d\xi_1 \dots d\xi_{n-1} \\ &= (1-x_n)^{p_1+\dots+p_{n-1}} I_{n-1}. \end{aligned}$$

Thus,

$$\begin{aligned} I_n &= I_{n-1} \int_0^1 (1-x_n)^{p_1+\dots+p_{n-1}} x_n^{p_n-1} dx_n \\ &= \frac{\Gamma(p_1) \dots \Gamma(p_{n-1})}{\Gamma(p_1 + \dots + p_{n-1} + 1)} \cdot \frac{\Gamma(p_1 + \dots + p_{n-1} + 1) \Gamma(p_n)}{\Gamma(p_1 + \dots + p_n + 1)} = \frac{\Gamma(p_1) \dots \Gamma(p_n)}{\Gamma(p_1 + \dots + p_n + 1)}. \end{aligned}$$

□

There is a seemingly more general formula:

### 11.29.

$$\int \dots \int_{\substack{x_1, \dots, x_n \geq 0, \\ x_1^{\gamma_1} + \dots + x_n^{\gamma_n} \leq 1}} x_1^{p_1-1} \dots x_n^{p_n-1} dx_1 \dots dx_n = \frac{1}{\gamma_1 \dots \gamma_n} \cdot \frac{\Gamma\left(\frac{p_1}{\gamma_1}\right) \dots \Gamma\left(\frac{p_n}{\gamma_n}\right)}{\Gamma\left(\frac{p_1}{\gamma_1} + \dots + \frac{p_n}{\gamma_n} + 1\right)}.$$

It is easily obtained from the previous one by the change of variables  $y_j = x_j^{\gamma_j}$ .

There is a special case which is worth mentioning:  $p_1 = \dots = p_n = 1$ ,  $\gamma_1 = \dots = \gamma_n = p$ :

$$\int \dots \int_{\substack{x_1, \dots, x_n \geq 0 \\ x_1^p + \dots + x_n^p \leq 1}} dx_1 \dots dx_n = \frac{\Gamma^n\left(\frac{1}{p}\right)}{p^n \Gamma\left(\frac{n}{p} + 1\right)}.$$

We've found the volume of the unit ball in the metric  $l^p$ :

$$v_n(\mathbb{B}_p(1)) = \frac{2^n \Gamma^n\left(\frac{1}{p}\right)}{p^n \Gamma\left(\frac{n}{p} + 1\right)}.$$

Of course, if  $p = 2$ , the formula gives us the volume of the standard unit ball:

$$v_n = v_n(\mathbb{B}) = \frac{2\pi^{n/2}}{n \Gamma\left(\frac{n}{2}\right)}.$$

We also see that the volume of the unit ball in the  $L^1$ -metric equals  $\frac{2^n}{n!}$ .

Question: what the formula gives us in the “ $p \rightarrow \infty$  limit”?



**Exercise 11.30.**

$$\begin{aligned} & \int \cdots \int_{\substack{x_1 + \dots + x_n \leq 1 \\ x_1, \dots, x_n \geq 0}} \varphi(x_1 + \dots + x_n) x_1^{p_1 - 1} \cdots x_n^{p_n - 1} dx_1 \cdots dx_n \\ &= \frac{\Gamma(p_1) \cdots \Gamma(p_n)}{\Gamma(p_1 + \dots + p_n)} \int_0^1 \varphi(u) u^{p_1 + \dots + p_n - 1} du. \end{aligned}$$

## 12 Smooth surfaces

We start with smooth (two-dimensional) surfaces in  $\mathbb{R}^3$  and their tangent planes. Then we define and briefly discuss smooth  $k$ -dimensional surfaces in  $\mathbb{R}^n$ ,  $0 \leq k \leq n$ . The cases  $k = 1$  and  $k = n - 1$  correspond to the lines and hyper-surfaces in  $\mathbb{R}^n$ , the extreme cases  $k = 0$  and  $k = n$  correspond to points and domains in  $\mathbb{R}^n$ . In the case  $k = 2$ ,  $n = 3$ , we get (two-dimensional) surfaces in  $\mathbb{R}^3$ . We finish with discussion of normal vectors to hyper-surfaces.

### 12.1 Surfaces in $\mathbb{R}^3$

There are three definitions of smooth surfaces in  $\mathbb{R}^3$ .

#### 12.1.1 Graph of function

The surface  $M \subset \mathbb{R}^3$  is a graph of function  $z = f(x, y)$ . The class of smoothness of the surface  $M$  is defined according to the class of smoothness of  $f$ .

#### 12.1.2 Zero set of a smooth function

The surface  $M \subset \mathbb{R}^3$  can be defined as the zero set of a smooth function:  $M = \{(x, y, z) : F(x, y, z) = 0\}$ .

Locally, this definition is equivalent to the previous one. Obviously, graph of the function  $z = f(x, y)$  can be viewed as the zero set of the function  $z - f(x, y) = 0$ . To move in the opposite direction, we say that the point  $P(x_0, y_0, z_0) \in M$  is called *regular*, if  $F(P) = 0$ , but  $\nabla F(P) \neq 0$ . Wlog, suppose  $F_z(P) \neq 0$ . Then, by Implicit Function Theorem, we can solve equation  $F(x, y, z) = 0$  near the point  $P$ , i.e., we find the function  $z = f(x, y)$  such that  $z_0 = f(x_0, y_0)$ , and  $F(x, y, f(x, y)) \equiv 0$  in a neighbourhood of  $(x_0, y_0)$ .

#### 12.1.3 Parametric surfaces

are defined as the image in  $\mathbb{R}^3$  of a ‘nice’ domain  $G \subset \mathbb{R}^2$  under a  $C^1$ -injection  $r: G \rightarrow \mathbb{R}^3$ . The variables  $(u, v)$  are called the local parameters on the surface  $M$ . Let, in the coordinates,

$$r(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}.$$

The point  $P(r(u_0, v_0)) \in M$  is called *regular* if the matrix of the derivative  $D_r$  has rank two, i.e.

$$\text{rank} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} (u_0, v_0) = 2.$$

**Claim 12.1.** If  $P \in M$  is a regular point, then in a neighbourhood of  $P$  the surface  $M$  is graph of a function.

*Proof:* Suppose, for instance, that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0,$$

i.e. Jacobian of the mapping

$$(12.2) \quad \begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$$

does not vanish at  $(u_0, v_0)$ . Then by the Inverse Function Theorem the mapping (12.2) can be inverted in a neighbourhood of  $(u_0, v_0)$ :

$$\begin{cases} u = u(x, y) & u_0 = u(x_0, y_0), \\ v = v(x, y) & v_0 = v(x_0, y_0). \end{cases}$$

Substituting this into expression for  $z$ , we get

$$z = f(x, y) \stackrel{\text{def}}{=} z(u(x, y), v(x, y)), \quad z_0 = f(x_0, y_0) = z(u_0, v_0).$$

□

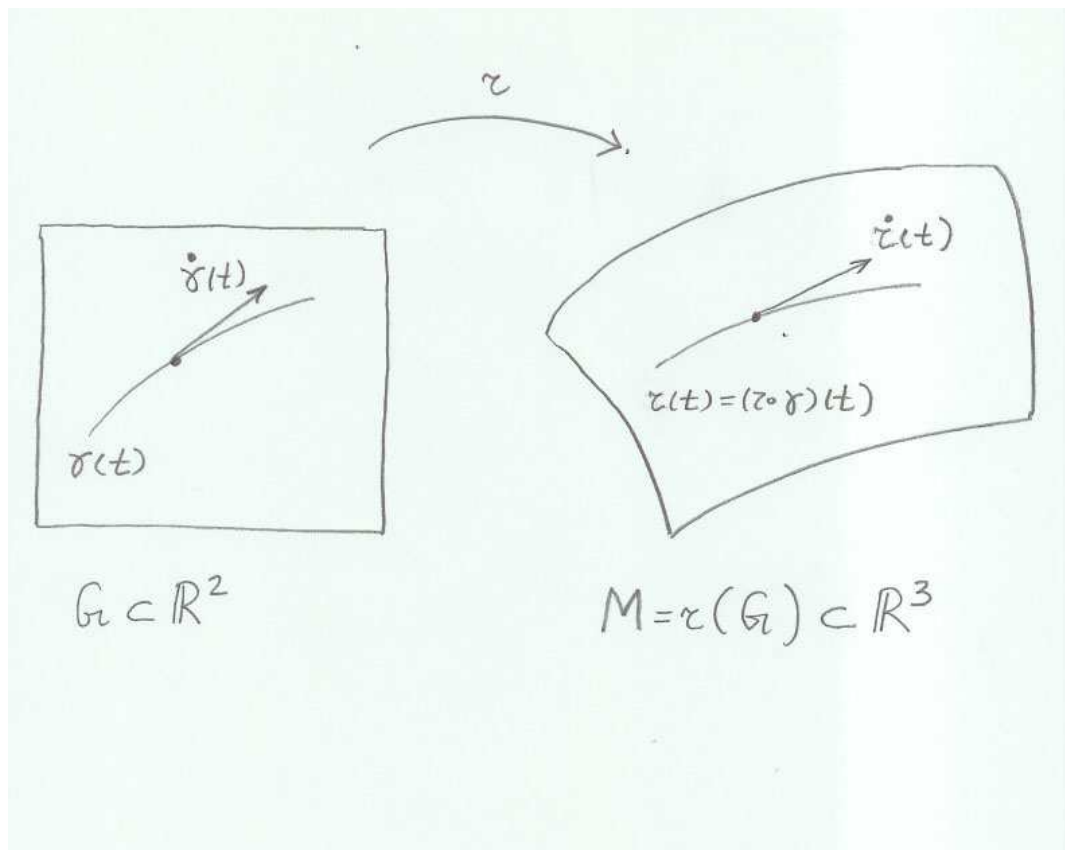
Thus, near regular points all three definitions of surfaces coincide!

#### 12.1.4 The tangent plane

Let  $M$  be a parametric surface,  $G$  domain of the local coordinates  $(u, v)$ , the point  $r(u_0, v_0)$  be regular. Consider a curve  $\gamma(t) = (u(t), v(t)) \subset G$  passing through the point  $(u_0, v_0)$  and its image, the curve  $r(t) = r(u(t), v(t)) \subset M$  passing through the point  $P_0$ : The tangent (= velocity) vector of the curve  $r(t)$  is

$$\dot{r}(t) = r_u \dot{u} + r_v \dot{v}.$$

Since  $(u_0, v_0)$  is a regular point, the vectors  $r_u = (x_u, y_u, z_u)$  and  $r_v = (x_v, y_v, z_v)$  are linearly independent. Thus ANY TANGENT VECTOR TO  $M$



AT  $P_0$  IS A LINEAR COMBINATION OF  $r_u$  AND  $r_v$  (WITH COEFFICIENTS  $\dot{u}$  AND  $\dot{v}$ ), AND THE TANGENT PLANE  $T_{P_0}M$  IS A TWO DIMENSIONAL LINEAR SPACE SPANNED BY THE VECTORS  $r_u$  AND  $r_v$ .

If  $M$  was defined as the zero set of a function  $F$ , then equation of a curve on  $M$  is  $F(x(t), y(t), z(t)) = 0$ . Differentiation by  $t$ , we get

$$F_x \dot{x} + F_y \dot{y} + F_z \dot{z} = 0.$$

If  $\nabla F \neq 0$ , that is, the point is regular, we get equation for the coordinates  $(\xi, \eta, \zeta)$  of the tangent vector:

$$F_x(x_0, y_0, z_0)\xi + F_y(x_0, y_0, z_0)\eta + F_z(x_0, y_0, z_0)\zeta = 0.$$

If we want to think about tangent vectors as of vectors that start at the point  $(x_0, y_0, z_0) \in M$ , then we get the affine plane in  $\mathbb{R}^3$ , its equation is

$$\boxed{F_x(x_0, y_0, z_0)(\xi - x_0) + F_y(x_0, y_0, z_0)(\eta - y_0) + F_z(x_0, y_0, z_0)(\zeta - z_0) = 0.}$$

**Exercise 12.3.** Let  $\Sigma$  be the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

in  $\mathbb{R}^3$ . Find the distance  $p(x, y, z)$  from the origin to the tangent plane of  $\Sigma$  at  $(x, y, z)$ .

### 12.1.5 Examples of surfaces in $\mathbb{R}^3$

**Ellipsoid in  $\mathbb{R}^3$**  is defined by equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

that is, this is the zero set of the quadratic polynomial  $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$ . All points of the ellipsoid are regular. Globally, it cannot be defined as a graph of a function or by only one coordinate map (why?).

The ellipsoid is parameterized by the local coordinates  $x = a \cos \varphi \cos \theta$ ,  $y = b \sin \varphi \cos \theta$ ,  $z = c \sin \theta$ , where  $0 \leq \varphi \leq 2\pi$ ,  $0 \leq \theta \leq \pi$ .

**One sheet hyperboloid** is defined by equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

All points of this surface are regular. It is defined parametrically as  $x = a\sqrt{1 + \frac{z^2}{c^2}} \cos \varphi$ ,  $y = b\sqrt{1 + \frac{z^2}{c^2}} \sin \varphi$ ,  $z = z$ , where  $0 \leq \varphi \leq 2\pi$  and  $-\infty < z < \infty$ .

**Double-sheet hyperboloid (or, elliptic paraboloid)** is defined by equation

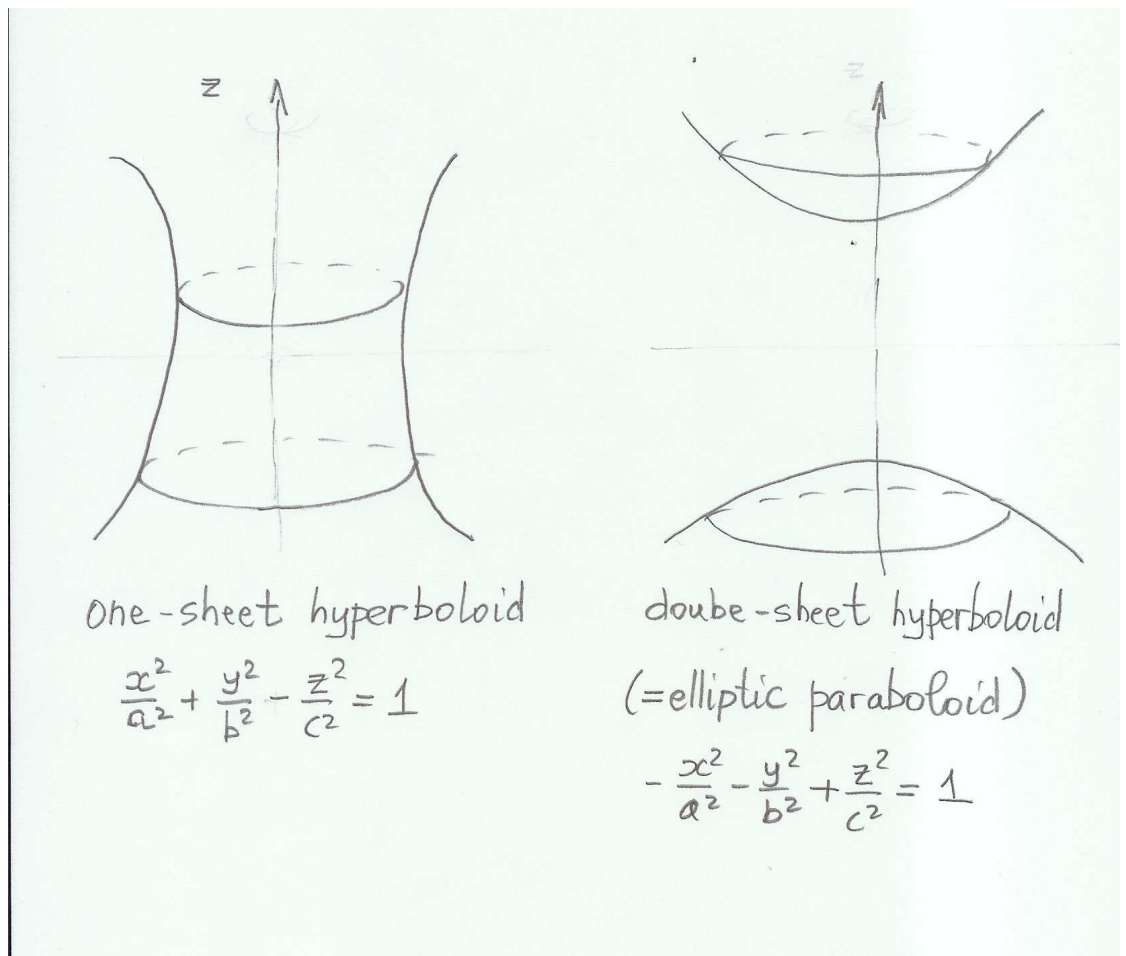
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

All points of this surface are regular. Each sheet is the graph of the function  $z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ .

**Cone**  $C$  is defined by equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

It has a singular point  $(0, 0, 0)$ .  $C \setminus \{0\}$  is a smooth surface.



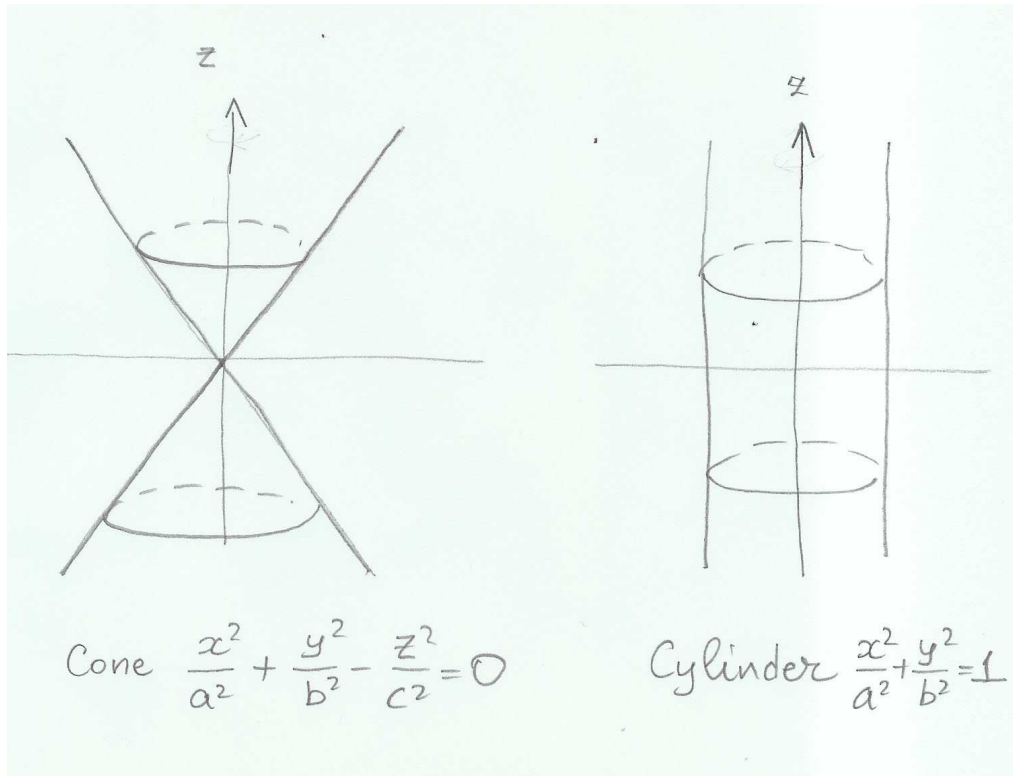
**Cylinder** is defined by equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . All points of the cylinder are regular. It's parametric description is  $x = a \cos \varphi$ ,  $y = b \sin \varphi$ ,  $z = z$ , where  $0 \leq \varphi \leq 2\pi$ ,  $-\infty < z < \infty$ .

**Exercise 12.4.** Build a smooth one-to-one map from the punctured plane  $\{(x, y) : 0 < x^2 + y^2 < \infty\}$  onto the cylinder.

**Surface of revolution** Given a curve  $\gamma: I \rightarrow \mathbb{R}^3$  lying in the half-plane  $\{x > 0, y = 0\}$  we can build the "surface of revolution"  $\Sigma \subset \mathbb{R}^3$  revolving  $\gamma$  around the  $z$ -axis. If  $\gamma(s) = (\gamma_1(s), 0, \gamma_3(s))$ , then  $\Sigma$  is defined by the mapping

$$r(s, t) = (\gamma_1(s) \cos t, \gamma_1(s) \sin t, \gamma_3(s)).$$

The most popular example of surface of revolution is



**Torus** obtained by rotation of the circle of radius  $a$  centered at  $(b, 0, 0)$ ,  $a < b$ . The parametric equations of the torus are  $x = (b + a \cos s) \cos t$ ,  $y = (b + a \cos s) \sin t$ , and  $z = a \sin s$ .

**Helix and helicoid** The helix (or, spiral) is a curve in  $\mathbb{R}^3$  defined by  $t \mapsto (\cos t, \sin t, t)$ ,  $-\infty < t < \infty$ . The helicoid is a surface in  $\mathbb{R}^3$  defined by  $(s, t) \mapsto (s \cos t, s \sin t, t)$ ,  $s > 0$ ,  $-\infty < t < \infty$ .

**Exercise 12.5.** Draw the pictures of helix and helicoid.

## 12.2 Equivalent definitions of $k$ -surfaces in $\mathbb{R}^n$

We give three equivalent definitions of  $k$ -surfaces: as graphs, as zero sets, and as images of open sets. The equivalence will again follow from the Implicit and Inverse Function Theorems.

### 12.2.1 Graphs of functions

A subset  $M \subset \mathbb{R}^n$  ( $M \neq \emptyset$ ) is a *smooth  $k$ -dimensional surface* if for any  $x \in M$  there is a neighbourhood  $U$  of  $x$  such that  $M \cap U$  is the graph of a smooth mapping  $f$  of an open subset  $W \subset \mathbb{R}^k$  into  $\mathbb{R}^{n-k}$ :

$$M \cap U = \{(w, f(w)) : w \in W \subset \mathbb{R}^k\} .$$

Here, we are free to choose which  $n - k$  coordinates in  $\mathbb{R}^n$  are functions of the other  $k$  coordinates. For simplicity, we usually assume that the first  $k$  coordinates are “free” and the last  $n - k$  coordinates are the functions of them.

Observe that the mapping  $r: W \rightarrow \mathbb{R}^n$ ,  $r(w) = (w, f(w))$  is a bijection between  $W$  and  $r(W) = M \cap U$ . The inverse mapping  $r^{-1}: M \cap U \rightarrow W$  is the projection  $(w, f(w)) \mapsto w$ .

According to the class of smoothness of  $f$ , we define the class of smoothness of the surface  $M$ .

#### Examples:

1. Any open set  $X \subset \mathbb{R}^n$  is a  $C^\infty$ -surface. We take  $W = X$ ,  $f: X \rightarrow \mathbb{R}^0 = \{0\}$ , that is  $f(x) = 0$  for all  $x \in X$ , and identify  $x \in X$  with  $(x, f(x))$ .
2. Any point  $x \in \mathbb{R}^n$  is also a  $C^\infty$ -surface. Why?

### 12.2.2 Zero sets

Let  $U \subset \mathbb{R}^n$  be an open set,  $F: U \rightarrow \mathbb{R}^p$  smooth function,  $0 \leq p \leq n$ ,  $0 \in F(U)$ . Consider the zero set

$$Z_F \stackrel{\text{def}}{=} \{x \in U : F(x) = 0\} .$$

The point  $z \in Z_F$  is *regular* if  $\text{rank} D_F(z) = p$ , i.e., is maximal<sup>18</sup>. If  $z$  is a regular point of  $Z_F$ , then there exists a neighbourhood  $U$  of  $z$  such that  $Z_F \cap U$  is a smooth  $k$ -dimensional surface,  $k = n - p$ .

Indeed, since  $\text{rank} D_F(z) = p$ , we can choose  $p$  coordinates, say,  $x_{n-p+1}, \dots, x_n$ , such that the derivative of  $F$  with respect to these variables is invertible. Then we apply the Implicit Function Theorem and conclude that locally the coordinates  $x_{n-p+1}, \dots, x_n$  are the functions of  $x_1, \dots, x_{n-p}$ .  $\square$

**Example 12.6.** The unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$  is the zero set of the function  $F(x) = |x|^2 - 1$ . Since  $\nabla F = 2x \neq 0$  on  $\mathbb{S}^{n-1}$ , all points of the sphere are regular.

<sup>18</sup>Otherwise,  $z$  is a *singular* point of  $Z_F$ .



### 12.2.3 Parametric surfaces

Suppose  $r: V \rightarrow \mathbb{R}^n$  is a smooth mapping,  $V \subset \mathbb{R}^k$ ,  $0 \leq k \leq n$ , is an open set. The mapping  $r$  is *regular* if  $\text{rank} D_r(v) = k$  at every point  $v \in V$ .

**Exercise 12.7.** If  $\text{rank} D_r(v_0) = k$ , then  $\text{rank} D_r(v) = k$  in a neighbourhood of  $v_0$ .

(Of course, this is just a special case of the Exercise 4.9.)

We say that  $M \subset \mathbb{R}^n$  is a smooth  $k$ -dimensional surface if for any  $x \in M$  there exist a neighbourhood  $U$  of  $x$ , an open set  $V \subset \mathbb{R}^k$ , and a regular bijection  $r: V \rightarrow M \cap U$ .

Formally, this definition is more general than the first one, that uses graphs. Let us check that they are actually coincide. Fix a point  $a = r(b)$ . Since  $D_r$  has maximal rank  $k$ , we can choose  $k$  coordinate functions, say

$r_1(v), \dots, r_k(v)$ , such that the corresponding mapping  $r^* = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}$  of  $V$  into

$\mathbb{R}^k$  has the invertible derivative  $D_{r^*}(v)$ . Then, applying the Inverse Function

Theorem, we get a neighbourhood  $W$  of the point  $a^* = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}$ , and the

inverse mapping  $h = (r^*)^{-1}: W \rightarrow V$ . Substituting this mapping into the

functions  $r_{k+1}(v), \dots, r_n(v)$ , we get the mapping  $f = \begin{pmatrix} r_{k+1} \\ \vdots \\ r_n \end{pmatrix} \circ h: W \rightarrow \mathbb{R}^n$

such that locally, in a neighbourhood of the point  $a$ ,  $M$  is the graph of  $f$ . (Why?)  $\square$

The coordinates  $v_1, \dots, v_k$  are called *the local (curvilinear) coordinates* on  $M \cap U$ , the function  $r$  is called *the coordinate patch*, or *the map*. The maps can “overlap”: suppose that there are two maps  $r_1$  and  $r_2$  such that  $r_1(V_1) \cap r_2(V_2) \neq \emptyset$ . Then we can define the function  $r_2^{-1} \circ r_1$ .

**Exercise 12.8.** The composition map  $r_2^{-1} \circ r_1$  is a  $C^1$ -diffeomorphism between the open sets  $r_1^{-1}(r_1(V_1) \cap r_2(V_2))$  and  $r_2^{-1}(r_1(V_1) \cap r_2(V_2))$ .

**Question 12.9.** What is the minimal number of coordinate maps needed to define the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ ? Justify your answer.

### 12.3 The tangent space

**Definition 12.10.** Suppose  $\gamma: I \rightarrow \mathbb{R}^n$  is a  $C^1$ -curve in  $\mathbb{R}^n$ . Then the vector

$$\dot{\gamma}(t) = \begin{pmatrix} \dot{\gamma}_1(t) \\ \vdots \\ \dot{\gamma}_n(t) \end{pmatrix}$$

is called the *tangent* (or *velocity*) vector of  $\gamma$  at the point  $x = \gamma(t)$ .

**Definition 12.11.** Suppose  $M$  is a  $C^1$ -smooth  $k$ -surface. Then the vector  $v \in \mathbb{R}^n$  is called a *tangent vector* to  $M$  at the point  $x \in M$  if there exists a  $C^1$ -curve  $\gamma: I \rightarrow M$  and  $t \in I$  such that  $\gamma(t) = x$ , and  $\dot{\gamma}(t) = v$ . The set of all tangent vectors to  $M$  at  $x$  is called the *tangent space* of  $M$  at  $x$  and denoted by  $T_x M$ .

Geometrically, we think about elements of the tangent space  $T_x M$  as of vectors that start at the point  $x$ .

To better understand this definition, assume first that  $M$  is an open set in  $\mathbb{R}^n$ , and  $x \in M$ . Then a minute reflection shows that  $T_x M = T_x \mathbb{R}^n \simeq \mathbb{R}^n$ . We know that the derivative of a smooth mapping  $f: M \rightarrow \mathbb{R}^m$  at the point  $x \in M$  is a linear operator  $D_f(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ . Now, we understand that it acts on the tangent spaces:  $D_f(x): T_x M \rightarrow T_{f(x)} f(M)$ . Indeed, if  $\gamma: I \rightarrow M$  is a curve in  $M$  passing through the point  $x = \gamma(t)$ , then  $f \circ \gamma: I \rightarrow \mathbb{R}^m$  is a curve in  $\mathbb{R}^m$  passing through the point  $f(x)$ , and by the chain rule

$$\frac{d(f \circ \gamma)}{dt} = D_f(x) \dot{\gamma}.$$

Now, let us return to tangent spaces of  $k$ -surfaces. Since we have three equivalent definitions of  $k$ -surfaces, we get three different ways to find the tangent space.

First, we assume that  $M$  is defined parametrically. Fix  $x \in M$ . Then we can find a neighbourhood  $U$  of  $x$ , an open set  $V \in \mathbb{R}^k$  and  $C^1$ -bijection  $r: V \rightarrow M \cap U$ ,  $x = r(b)$ . This establishes a one-to-one correspondence between  $C^1$ -curves  $x(t)$  in  $M \cap U$  passing through  $x$ , and  $C^1$ -curves  $v(t)$  in  $V$  passing through  $b$ :  $x(t) = (r \circ v)(t)$ . Differentiating this relation, we get

$$\dot{x}(t) = D_r(v(t)) \dot{v}(t) = \sum_{i=1}^k \dot{v}_i(t) r_{v_i}(b).$$

In other words,  $D_r(b)$  maps  $T_b V$  bijectively onto  $T_x M$ . In particular,  $T_x M$  is a linear subspace of  $\mathbb{R}^n$  of dimension  $k$  generated by  $k$  vectors  $r_{v_1}(b), \dots, r_{v_k}(b)$ .

If  $M \cap U$  is the zero set of the  $C^1$ -mapping  $F: U \rightarrow \mathbb{R}^{n-k}$ , we have  $F(x(t)) \equiv 0$  for any curve  $x(t)$  in  $M$ . Differentiating this by  $t$ , we get  $D_F(x)\dot{x}(t) = 0$ , that is  $\dot{x}(t) \in \ker D_F(x)$ , or  $T_x M \subset \ker D_F(x)$ . Since the both are linear subspaces of  $\mathbb{R}^n$  of dimension  $k$ , we get

$$T_x M = \ker D_F(x).$$

For instance, if  $M$  is a hypersurface (i.e.,  $k = n - 1$ , and  $F$  is a scalar-valued function), then the tangent space  $T_x M$  is the orthogonal complement to the gradient  $\nabla F(x)$ .

## 12.4 Normal vectors to hypersurfaces

Suppose  $M \subset \mathbb{R}^n$  is a hypersurface, then  $\dim T_x M = n - 1$  and  $\dim(T_x M)^\perp = 1$ .

**Definition 12.12.** The unit normal  $N(x)$  to  $M$  at  $x$  is a vector from  $(T_x M)^\perp$  such that  $|N(x)| = 1$ .

Clearly,  $N(x)$  is defined up to its sign<sup>19</sup>. Sometimes, the vector  $N(x)$  is written in the form  $(\cos \theta_1, \dots, \cos \theta_n)$ . The angles  $\theta_1, \dots, \theta_n \in [0, \pi]$  are called the *directional cosines*.

Now, let us look how to find the normal vector.

First, suppose that  $M$  is the zero set of a smooth function  $F$ . Then, as we already know, the gradient of  $F$  is orthogonal to any tangent vector, that is

$$N(x) = \pm \frac{\nabla F(x)}{|\nabla F(x)|}.$$

If  $M$  is a graph of a scalar function  $f(x_1, \dots, x_{n-1})$ , we define  $F(x) = x_n - f(x_1, \dots, x_{n-1})$ . Then  $M$  is the zero set of  $F$ . Notice that

$$\nabla F(x) = \begin{pmatrix} -f_{x_1} \\ \vdots \\ -f_{x_{n-1}} \\ 1 \end{pmatrix},$$

and

$$|\nabla F(x)| = \sqrt{1 + |\nabla f(x)|^2}.$$

The case when  $M$  is defined parametrically is the most involved. Suppose, locally,  $M$  is defined by equation  $x = r(v)$ . The tangent hyperplane  $T_x M$  is

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<sup>19</sup>Later we shall use this to orient the hypersurface  $M$ .

spanned by  $n - 1$  vectors  $r_{v_1}, \dots, r_{v_{n-1}}$ , and we need to learn how to compute the normal vector to this linear span. For this, we need to extend an idea of the vector product from the case of two vectors in  $\mathbb{R}^3$  to the case of  $n - 1$  vectors in  $\mathbb{R}^n$ .

**Cross product of  $n - 1$  vectors in  $\mathbb{R}^n$**  Given  $n - 1$  linearly independent vectors  $R_1, \dots, R_{n-1}$  in  $\mathbb{R}^n$  consider the determinant  $\det(R, R_1, \dots, R_{n-1})$ <sup>20</sup>. This is a linear functional on  $R$ , thus there exists a vector  $Y$  that represents this functional:  $\det(R, R_1, \dots, R_{n-1}) = \langle Y, R \rangle$ . The vector  $Y$  is called the *cross product* of the vectors  $R_1, \dots, R_{n-1}$ , and is denoted by  $Y = R_1 \times \dots \times R_{n-1}$ . It follows from the definition of  $Y$  that  $\langle Y, R_i \rangle = \det(R_i, R_1, \dots, R_{n-1}) = 0$ ,  $1 \leq i \leq n - 1$ .

Fix an orthonormal basis  $\{E_i\}_{1 \leq i \leq n}$  in  $\mathbb{R}^n$ . The coordinates of  $Y$  in this basis are

$$Y_i = \langle Y, E_i \rangle = \det(E_i, R_1, \dots, R_{n-1}),$$

and

$$Y = \sum_{i=1}^n Y_i E_i = \sum_{i=1}^n \det(E_i, R_1, \dots, R_{n-1}) E_i.$$

Thus

$$R_1 \times \dots \times R_{n-1} = \begin{vmatrix} E_1 & E_2 & \dots & E_n \\ R_{1,1} & R_{1,2} & \dots & R_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n-1,1} & R_{n-1,2} & \dots & R_{n-1,n} \end{vmatrix}$$

where  $R_{i,j} = (R_i, E_j)$  is the  $j$ -th coordinate of  $R_i$ . Note that the first column of the determinant consists of vectors whilst the other columns consist of scalars.

In particular, when  $n = 3$ , we arrive to the vector product formula

$$A \times B = \begin{vmatrix} E_1 & E_2 & E_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}.$$

It remains to compute the norm of the cross-product. In fact, we already know how to do this:

**Claim 12.13** (linear algebra).

$$|Y| = \sqrt{\Gamma(R_1, \dots, R_{n-1})},$$

where  $\Gamma(R_1, \dots, R_{n-1})$  is the Gram determinant of the vectors  $R_1, \dots, R_{n-1}$ .

<sup>20</sup>That is the determinant of the matrix which columns consist of these vectors.

*Proof:* We have

$$|Y|^4 = \langle Y, Y \rangle^2 = \det^2(Y, R_1, \dots, R_{n-1}).$$

The RHS is the square of the volume of the parallelepiped  $P(Y, R_1, \dots, R_{n-1})$  spanned by the vectors  $Y, R_1, \dots, R_{n-1}$ . We know that it equals to  $\Gamma(Y, R_1, \dots, R_{n-1})$ . Since  $Y$  is orthogonal to the rest of the vectors,

$$\Gamma(Y, R_1, \dots, R_{n-1}) = |Y|^2 \Gamma(R_1, \dots, R_{n-1}).$$

We are done! □

Returning to our problem, we get the useful formula for unit normal vectors  $N(x)$  to parametric surfaces:

$$N(x) = \pm \frac{r_{v_1} \times \dots \times r_{v_{n-1}}}{\sqrt{\Gamma(r_{v_1}, \dots, r_{v_{n-1}})}}.$$

If  $n = 3$  and  $r = r(u, v)$ , then

$$|r_u \times r_v|^2 = \Gamma(r_u, r_v) = |r_u|^2 |r_v|^2 - (r_u, r_v)^2.$$

In differential geometry there are special notations for the quantities that enter the right-hand side:  $E = |r_u|^2$ ,  $F = (r_u, r_v)$ , and  $G = |r_v|^2$ . Then

$$\sqrt{\Gamma(r_u, r_v)} = \sqrt{E \cdot G - F^2}.$$

### 12.4.1 Juxtaposing two computations

If in a neighbourhood of the point  $\xi \in M$  the hyper-surface  $M$  is defined as the graph of the function:  $M \cap U_\xi = \{(x, f(x)) : x \in G \subset \mathbb{R}^{n-1}\}$ , then

$$r(x) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ f(x_1, \dots, x_{n-1}) \end{pmatrix}$$

and

$$r_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ f_{x_1} \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ f_{x_2} \end{pmatrix}, \quad \dots, \quad r_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ f_{x_{n-1}} \end{pmatrix}.$$

We know that the vectors  $r_1 \times \dots \times r_{n-1}$  and  $\begin{pmatrix} -f_{x_1} \\ \vdots \\ -f_{x_{n-1}} \\ 1 \end{pmatrix}$  belong to the same one dimensional vector space  $(T_\xi M)^\perp$ , that is

$$r_1 \times \dots \times r_{n-1} = C \begin{pmatrix} -f_{x_1} \\ \vdots \\ -f_{x_{n-1}} \\ 1 \end{pmatrix}.$$

It is not difficult to check that  $C = (-1)^{n-1}$ . Indeed, comparing the  $n$ -th components of these two vectors, we see that  $C$  is the  $n$ -th component of the vector  $r_1 \times \dots \times r_{n-1}$ ; i.e.

$$C = (-1)^{n-1} \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix} = (-1)^{n-1}.$$

We arrive at the useful corollary:

**Corollary 12.14.** In the notations as above,

$$\Gamma(r_1, \dots, r_{n-1}) = 1 + |\nabla f|^2.$$

## 13 Surface area and surface integrals

To avoid technicalities, we shall tacitly assume that  $M$  is an “*elementary surface*” defined by one patch  $M = r(U)$ ,  $U \subset \mathbb{R}^k$ , is a good bounded domain (usually,  $U$  is a brick or a ball),  $r$  is a  $C^1$ -bijection. In the “practical computations”, sometimes, we’ll need to consider the surfaces that are not elementary, like the unit sphere in  $\mathbb{R}^n$ . In these cases, we just split the domain of integration into finitely many “elementary parts”, requiring additivity of the area and integrals.<sup>21</sup>

### 13.1 Fundamental definitions

We want to define the  $k$ -area on  $M$  and the integrals of “good” functions on  $M$  over the  $k$ -area. First, let us look at the tangent spaces  $T_u\mathbb{R}^k$  and  $T_xM$ ,  $x = r(u) \in M$ . We know that  $D_r: T_u\mathbb{R}^k \rightarrow T_xM$  is a linear bijection, and  $\sqrt{\Gamma(r_1, \dots, r_k)} = \sqrt{\det(D_r^*(u)D_r(u))}$  is the volume distortion coefficient under the mapping  $D_r$ . (Here,  $r_j = \frac{\partial r}{\partial u_j}$ ,  $1 \leq j \leq k$ , are  $k$  vectors that span  $T_xM$ .) Thus, if  $\Omega \subset U$  is a sufficiently small cube centered at  $u$ , then we expect that the  $k$ -dimensional area of the “distorted cube”  $r(\Omega) \subset M$  is  $\approx \sqrt{\Gamma(r_1, \dots, r_k)} \text{vol}_k(\Omega)$ . Moreover, the smaller  $\Omega$  is, the closer

$$\frac{\text{area}_k(r(\Omega))}{\text{vol}_k(\Omega)}$$

is to  $\sqrt{\Gamma(r_1, \dots, r_k)}$ . Since the  $k$ -area on  $M$  is supposed to be set-additive, we naturally arrive to the following definition

**Definition 13.1** ( $k$ -dimensional area).

$$A_k(r(\Omega)) = \int_{\Omega} \sqrt{\Gamma(r_1, \dots, r_k)} = \int_{\Omega} \sqrt{\det(D_r^*D_r)}.$$

**Definition 13.2** ( $k$ -dimensional surface integral). Suppose  $f$  is a continuous function on  $M$  vanishing outside a compact subset of  $M$ .<sup>22</sup> Then

$$\int_M f dS = \int_U (f \circ r) \sqrt{\Gamma(r_1, \dots, r_k)} = \int_U (f \circ r) \sqrt{\det(D_r^*D_r)}$$

<sup>21</sup>Formally, this case is treated by the “partition of unity”. We will come later to its construction.

<sup>22</sup>Of course, we require too much. If  $M$  is parameterized by a finite brick or ball  $U$ , then it suffices to require that the function  $f \circ r$  is continuous on  $\bar{U}$ .

These definitions do not depend on the choice of parameterization of  $M$ . Suppose  $\rho: V \rightarrow M$  is another  $C^1$ -parameterization. Then  $\varphi = r^{-1} \circ \rho: V \rightarrow U$  is a  $C^1$ -diffeomorphism, and

$$\int_U (f \circ r) \sqrt{\det(D_r^* D_r)} = \int_V \underbrace{(f \circ r) \circ \varphi}_{=f \circ \rho} \sqrt{\det(D_r^* D_r) \circ \varphi} \underbrace{|\det(D_\varphi)|}_{=\sqrt{\det(D_\varphi^*) \det(D_\varphi)}}.$$

Since

$$(D_r \circ \varphi) \cdot D_\varphi = D_\rho$$

(the chain rule), we get

$$\int_V (f \circ \rho) \sqrt{\det(D_\rho^* D_\rho)},$$

as expected. □

In the case  $k = n$  our definition coincides with our previous definitions of multiple integral and the volume. Now, we examine the cases  $k = 1$  (length and integrals over the curves),  $k = 2$  and  $n = 3$  (surface area and surface integrals in  $\mathbb{R}^3$ ), and  $k = n - 1$  (“hyperarea” in  $\mathbb{R}^n$ ).

**Exercise 13.3.** Suppose  $u = \varphi(v)$ ; i.e. the ‘old’ local coordinates  $u_1, \dots, u_k$  are  $C^1$ -functions of the ‘new’ local coordinates  $v_1, \dots, v_k$ . Let  $\rho = r \circ \varphi$ . Then

$$\Gamma(\rho_{v_1}, \dots, \rho_{v_k}) = \Gamma(r_{u_1}, \dots, r_{u_k}) \cdot J_\varphi^2,$$

where

$$J_\varphi = \det D_\varphi = \left| \frac{\partial(u_1, \dots, u_k)}{\partial(v_1, \dots, v_k)} \right|$$

is the Jacobian of  $\varphi$ .

## 13.2 Length and integrals over the curves

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a  $C^1$ -curve,  $\Gamma = \gamma(I)$ . Then the length of  $\Gamma$  is

$$L(\Gamma) = \int_I |\dot{\gamma}|.$$

If  $f: \Gamma \rightarrow \mathbb{R}^1$  is a “good function”, then

$$\int_\Gamma f ds = \int_I (f \circ \gamma)(t) |\dot{\gamma}(t)| dt.$$



This definition of length is consistent with the geometric one. Let for time being  $\gamma$  be a continuous curve, and let  $\Pi$  be a partition of the segment  $I = [a, b]$ :  $a = t_0 < t_1 < \dots < t_N = b$ . By  $\Gamma_\Pi$  we denote the corresponding polygonal line inscribed in  $\Gamma$ . It consists of  $N$  segments  $[\gamma(t_j), \gamma(t_{j+1})]$ ,  $0 \leq j \leq N - 1$ . Its length equals

$$\mathcal{L}(\Gamma_\Pi) = \sum_{j=0}^{N-1} |\gamma(t_{j+1}) - \gamma(t_j)|.$$

**Definition 13.4** (length of the curve).

$$\mathcal{L}(\Gamma) = \sup_{\Pi} \mathcal{L}(\Gamma_\Pi).$$

If  $\mathcal{L}(\Gamma) < \infty$ , then the curve  $\Gamma$  is called *rectifiable*.

**Theorem 13.5** (equivalence of the length definitions). *If  $\Gamma$  is a piece-wise  $C^1$ -curve, then  $\Gamma$  is rectifiable, and  $\mathcal{L}(\Gamma) = \int_I |\dot{\gamma}|$ .*

*Proof:* We split the proof into several simple claims.

1. *If a partition  $\Pi'$  is finer than  $\Pi$ , then  $\mathcal{L}(\Gamma_{\Pi'}) \geq \mathcal{L}(\Gamma_\Pi)$ .* It suffices to check what happens when we add one new point to the partition. In this case, the result follows from the triangle inequality.

2. *Additivity of the length:* Let  $c \in (a, b)$  be an inner point of  $I$ . Set  $\gamma_1 = \gamma|_{[a, c]}$ ,  $\gamma_2 = \gamma|_{[c, b]}$ . Then  $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2)$ .

Indeed, let  $\Pi_1$  be a partition of  $[a, c]$ , and  $\Pi_2$  be a partition of  $[c, b]$ . We denote by  $\Pi = (\Pi_1, \Pi_2)$  the corresponding partition of  $[a, b]$ . Then

$$\mathcal{L}(\Gamma_{1, \Pi_1}) + \mathcal{L}(\Gamma_{2, \Pi_2}) = \mathcal{L}(\Gamma_\Pi) \leq \mathcal{L}(\Gamma).$$

In the opposite direction, let  $\Pi$  be any partition of  $[a, b]$ . We choose partitions  $\Pi_1$  of  $[a, c]$  and  $\Pi_2$  of  $[c, b]$  such that  $(\Pi_1, \Pi_2)$  is finer than  $\Pi$ . Then

$$\mathcal{L}(\Gamma_\Pi) \leq \mathcal{L}(\Gamma_{1, \Pi_1}) + \mathcal{L}(\Gamma_{2, \Pi_2}) \leq \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2).$$

Remark: the argument shows that  $\Gamma$  is rectifiable if and only if the both curves  $\Gamma_1$  and  $\Gamma_2$  are rectifiable.

3. *If  $f: I \rightarrow \mathbb{R}^n$  is a continuous function, then*

$$\left| \int_I f \right| \leq \int_I |f|.$$

This we already know (and used in the proof of the inverse function theorem).

4. If  $\Gamma$  is a  $C^1$ -curve, then

$$\mathcal{L}(\Gamma) \leq \int_I |\dot{\gamma}|.$$

Indeed, for each partition  $\Pi$ ,

$$L(\Gamma_\Pi) = \sum_{j=0}^{N-1} |\gamma(t_{j+1}) - \gamma(t_j)| = \sum_{j=0}^{N-1} \left| \int_{t_j}^{t_{j+1}} \dot{\gamma} \right| \leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} |\dot{\gamma}| = \int_I |\dot{\gamma}|.$$

5. Due to additivity, it suffices to prove the result for  $C^1$ -curves. Given  $t \in [a, b]$ , we set  $l(t) = \mathcal{L}(\gamma|_{[a,t]})$ . Then  $l(t'') - l(t') = \mathcal{L}(\gamma|_{[t',t'']})$ ,

$$|\gamma(t'') - \gamma(t')| \leq l(t'') - l(t') \leq \int_{t'}^{t''} |\dot{\gamma}|,$$

and

$$\frac{|\gamma(t'') - \gamma(t')|}{t'' - t'} \leq \frac{l(t'') - l(t')}{t'' - t'} \leq \frac{1}{t'' - t'} \int_{t'}^{t''} |\dot{\gamma}|.$$

Letting  $t'' - t' \rightarrow 0$ , we see that the function  $l(t)$  is differentiable with respect to  $t$ , and  $\dot{l}(t) = |\dot{\gamma}(t)|$  (and hence  $\dot{l}$  is continuous). Since  $l(a) = 0$ , we get

$$l(t) = \int_a^t \dot{l} = \int_a^t |\dot{\gamma}|.$$

In particular,

$$\mathcal{L}(\Gamma) = l(b) = \int_a^b |\dot{\gamma}|,$$

completing the proof of the theorem.  $\square$

**Example 13.6.** 1. If the curve  $\Gamma \subset \mathbb{R}^2$  is the graph of the function  $y = f(x)$ ,  $a \leq x \leq b$ , then

$$L(\Gamma) = \int_a^b \sqrt{1 + f'^2(x)} dx.$$

2. If the curve  $\Gamma \subset \mathbb{R}^2$  is defined by polar coordinates  $r = r(\theta)$ ,  $\alpha \leq \theta \leq \beta$ , then its length is

$$L(\Gamma) = \int_\alpha^\beta \sqrt{r^2(\theta) + r'^2(\theta)} d\theta.$$

**Exercise 13.7.** Compute the length of the cardioid  $r = 2(1 + \cos \theta)$ ,  $0 \leq \theta \leq 2\pi$ .

**Exercise 13.8.** Compute

$$\int_{\Gamma} xy \, ds,$$

where  $\Gamma = \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x, y \geq 0\}$  is the 1-st quarter of the ellipse.

Answer:  $\frac{ab}{3} \frac{a^2+ab+b^2}{a+b}$ .

**Exercise 13.9.** Suppose  $\gamma: I \rightarrow \mathbb{S}^2$  is a curve on the unit sphere:

$$\gamma_1(t) = \sin \varphi(t) \cos \theta(t), \gamma_2(t) = \sin \varphi(t) \sin \theta(t), \gamma_3(t) = \cos \varphi(t).$$

Show that

$$L(\gamma) = \int_I \sqrt{\dot{\varphi}^2 + \dot{\theta}^2 \sin^2 \varphi}$$

**Exercise 13.10.** Find the coordinates of the center of masses of the homogeneous curves:

(a) arc of the cycloid:

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad 0 \leq t \leq 2\pi.$$

(b) the *boundary* of the spherical triangle

$$x^2 + y^2 + z^2 = a^2, \quad x > 0, y > 0, z > 0.$$

The gravitational force  $F$  induced by the curve  $\Gamma \subset \mathbb{R}^3$  with the density distribution  $\mu$  on a particle with the unit mass at the point  $x$  is

$$F(x) = \int_{\Gamma} \frac{\xi - x}{|\xi - x|^3} \mu(\xi) \, d\xi$$

(for simplicity we equal the gravitational constant  $\gamma$  to one).

**Exercise 13.11.** Find the gravitational force exerted by the homogeneous infinite line in  $\mathbb{R}^3$  ( $\mu \equiv 1$ ) on the particle of unit mass at the distance  $h$  from the line.

Hint: choose convenient coordinate system.

### 13.3 Surface area and surface integrals in $\mathbb{R}^3$

Suppose  $M \subset \mathbb{R}^3$  is a  $C^1$  surface defined by the patch  $r: \Omega \rightarrow \mathbb{R}^3$ ,  $\Omega$  is either the square or the disc. Then, according to our general definition,

$$A(M) = \iint_{\Omega} \sqrt{\Gamma(r_u, r_v)} \, dudv = \iint_{\Omega} |r_u \times r_v| \, dudv.$$

If we wish to integrate the function  $f: M \rightarrow \mathbb{R}^1$  over  $M$ , we set

$$\iint_M f \, dS = \iint_{\Omega} (f \circ r) |r_u \times r_v| \, dudv.$$

If the surface  $M$  is defined as the graph of the function  $z = f(x, y)$ , then

$$|r_x \times r_y| = \sqrt{1 + |\nabla f|^2}.$$

If the surface  $M$  is defined by the equation  $F(x, y, z) = 0$ , and  $z$  can be expressed as a function of  $x$  and  $y$ :  $z = f(x, y)$ , then

$$\sqrt{1 + |\nabla f|^2} = \frac{|\nabla F|}{|F_z|}.$$

### 13.3.1 Examples

**Example 13.12** (area of the unit sphere in  $\mathbb{R}^3$ ). It is very easy to guess the answer:

$$A(\mathbb{S}^2) = \frac{d}{dr} \text{vol}(\mathbb{B}_r) \Big|_{r=1} = \left( \frac{4\pi r^3}{3} \right)'_{r=1} = 4\pi.$$

This computation can be justified. However, instead, we shall compute the area directly. The unit sphere is defined by equation  $F(x, y, z) = 0$ , where  $F(x, y, z) = x^2 + y^2 + z^2 - 1$ . We'll deal with the upper hemi-sphere. Then  $F_z = 2z = 2\sqrt{1 - (x^2 + y^2)}$ ,  $|\nabla F|^2 = (2x)^2 + (2y)^2 + (2z)^2 = 4$ , and

$$\begin{aligned} A(\mathbb{S}^2) &= 2 \iint_{x^2+y^2 < 1} \frac{2 \, dx dy}{2\sqrt{1 - (x^2 + y^2)}} = 2 \int_0^{2\pi} d\theta \int_0^1 \frac{r \, dr}{\sqrt{1 - r^2}} \\ &= 2\pi \int_0^1 \frac{ds}{\sqrt{1 - s}} = 2\pi \int_0^1 \frac{dt}{\sqrt{t}} = 4\pi. \end{aligned}$$

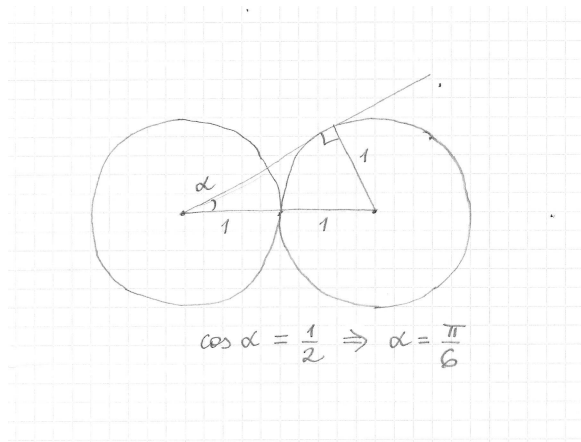
Now, we extend a little bit the previous computation:

**Example 13.13** (area of the spherical cap and the kissing number). Consider the spherical cap

$$S_\psi = \{(\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta) : 0 \leq \varphi < 2\pi, 0 < \theta < \psi\}.$$

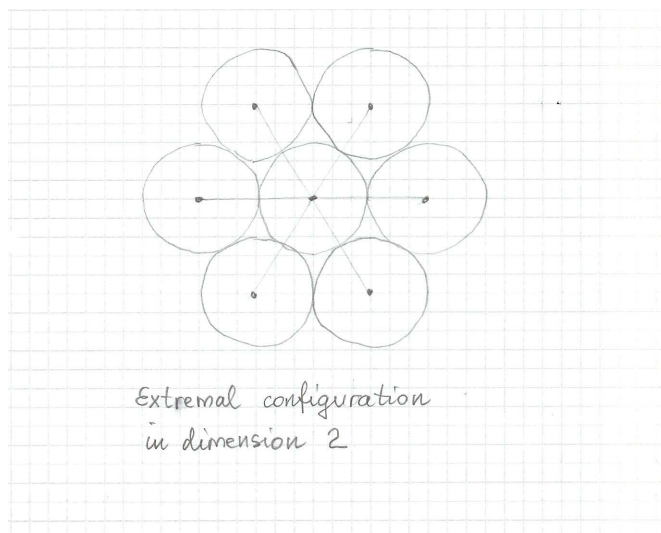
Then  $r(\varphi, \theta) = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$ ,  $r_\varphi = (-\sin \varphi \sin \theta, \cos \varphi \sin \theta, 0)$ ,  $r_\theta = (\cos \varphi \cos \theta, \sin \varphi \cos \theta, -\sin \theta)$ , and  $|r_\varphi \times r_\theta| = \sin \theta$ . Thus

$$A(S_\psi) = 2\pi \int_0^\psi \sin \theta \, d\theta = 2\pi(1 - \cos \psi).$$



Using this computation, we estimate the *kissing number* of the unit spheres in  $\mathbb{R}^3$ , that is the maximal number of the unit spheres that touch a given unit sphere.

Let us start with the plane case: If the circle  $S'$  of radius one touches the circle  $S$  of radius one, that the angle it is seen from the center of  $S$  equals  $\frac{\pi}{3}$ . If  $N(2)$  unit circles touch the unit circle  $S$ , then the sum of the angles they are seen from the center of  $S$  cannot be larger than  $2\pi$ . Hence, the number  $N(2)$  of the circles is  $\leq 6$ . The next figure shows that this bound is sharp.



Now consider the three-dimensional case. If the unit sphere  $S'$  touches the unit sphere  $S$ , then  $S'$  can be placed inside the cone with vertex at the

center of  $S$  and of angle  $\pi/3$  and the area of the spherical cap on  $S$  located inside this cone equals  $2\pi(1 - \cos \pi/6) = \pi(2 - \sqrt{3})$ . If  $N(3)$  unit spheres kiss the given unit sphere  $S$ , then the corresponding spherical caps are disjoint. Thus

$$N(3)\pi(2 - \sqrt{3}) \leq 4\pi,$$

or

$$N(3) \leq 4(2 + \sqrt{3}) < 15.$$

This estimate is not sharp. The sharp bound is  $N(3) = 12$ . It was known already to Newton, though the accurate proof was given only in the 20th Century.<sup>23</sup>

**Exercise 13.14.** Find

$$\iint_{S^2} x_i^2 dS, \quad 1 \leq i \leq 3,$$

without ANY actual computation.

Hint: the integrals do not depend on  $i$ .

**Exercise 13.15** (area surface of the intersection of two cylinders). Compute  $A(\partial K)$ , where  $K = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1, x_1^2 + x_3^2 \leq 1\}$ .

Hint:  $\partial K = \{x_1^2 + x_2^2 \leq 1, x_1^2 + x_3^2 = 1\} \cup \{x_1^2 + x_3^2 \leq 1, x_1^2 + x_2^2 = 1\}$ .

Answer: 16.

**Exercise 13.16.** Find the area of the part of the sphere  $\{x^2 + y^2 + z^2 = R^2\}$  located inside the cylinder  $\{x^2 + y^2 = Rx\}$ . Find the area of the part of the cylinder  $\{x^2 + y^2 = Rx\}$  located inside the sphere  $\{x^2 + y^2 + z^2 = R^2\}$ .

**Example 13.17** (Guldin's rule: area of the surface of revolution in  $R^3$ ). Consider the surface of revolution  $\Sigma$  in  $\mathbb{R}^3$  obtained by rotation of the curve  $z = \varphi(\rho)$ ,  $\alpha \leq \rho \leq \beta$  ( $\alpha > 0$ ) around the  $z$ -axis. Parametric equations of the surface  $\Sigma$  are

$$\begin{cases} x = \rho \cos \theta, \\ y = \rho \sin \theta, \\ z = \varphi(\rho). \end{cases}$$

Here  $0 \leq \theta < 2\pi$ , and  $\alpha \leq \rho \leq \beta$ . Then

$$\begin{aligned} r_\theta &= (-\rho \sin \theta, \rho \cos \theta, 0), & |r_\theta|^2 &= \rho^2, \\ r_\rho &= (\cos \theta, \sin \theta, \varphi'(\rho)), & |r_\rho|^2 &= 1 + \varphi'^2(\rho), \\ (r_\theta, r_\rho) &= 0. \end{aligned}$$

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<sup>23</sup>Try Google.

Hence,

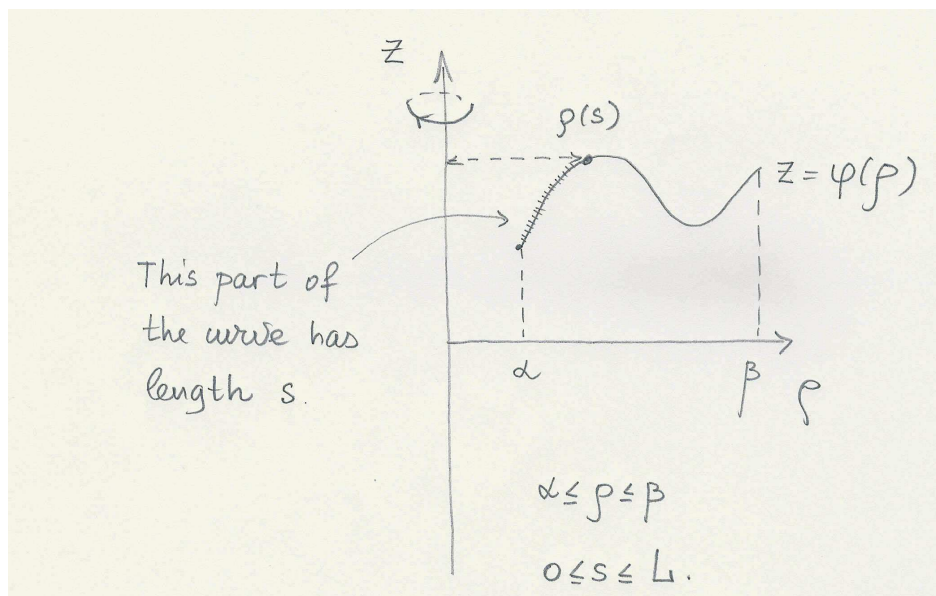
$$A(\Sigma) = 2\pi \int_{\alpha}^{\beta} \rho \sqrt{1 + \varphi'^2(\rho)} d\rho.$$

**Exercise 13.18.** Find the area of the surface  $\Sigma$  obtained by rotation of the curve  $\rho = \rho(z)$ ,  $a \leq z \leq b$  around the  $z$ -axis.

It is convenient to introduce the arc length, as the new parameter on the curve:  $ds = \sqrt{1 + \varphi'^2(\rho)} d\rho$ , and denote by

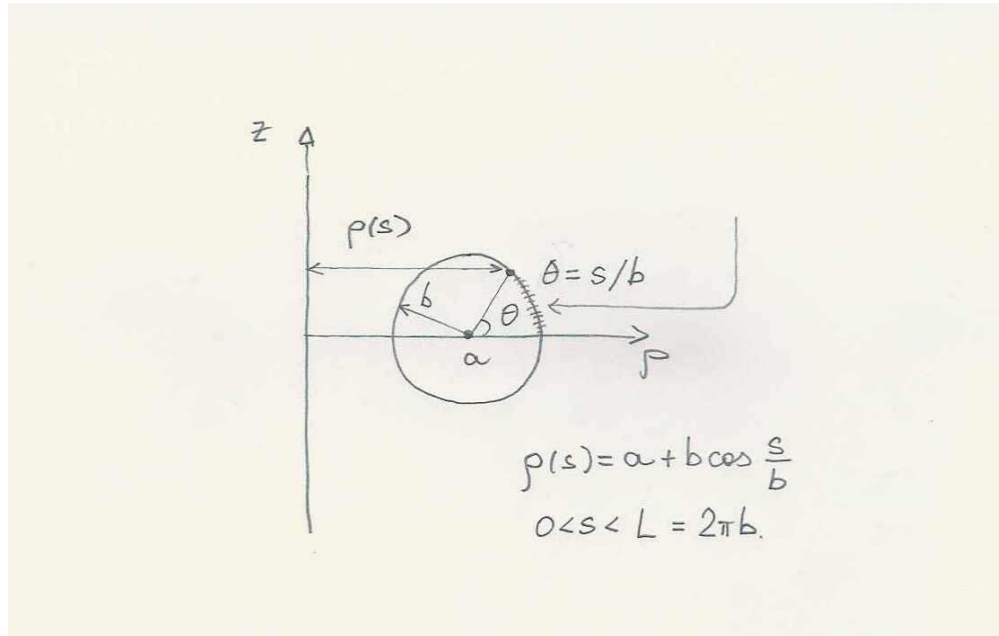
$$L = \int_{\alpha}^{\beta} \sqrt{1 + \varphi'^2(\rho)} d\rho$$

the total length of the curve. Let  $\rho(s)$  be the distance from the  $z$ -axis to the point on the curve that cut the length  $s$  from the beginning of the curve: Then



$$A(\Sigma) = 2\pi \int_0^L \rho(s) ds.$$

**Example 13.19** (area of the surface of the torus). The torus is obtained by rotation of the circle  $z^2 + (\rho - a)^2 = b^2$  ( $a \geq b$ ) around the  $z$ -axis. If  $\theta$  is the



polar angle on that circle, then  $\theta = s/b$ , and  $\rho(s) = a + b \cos(s/b)$ . Then

$$A = 2\pi \int_0^{2\pi b} (a + b \cos(s/b)) ds = 2\pi a \cdot 2\pi b.$$

Remarkably enough, the surface area of the torus equals the product of the lengths of two circles that generate the torus.

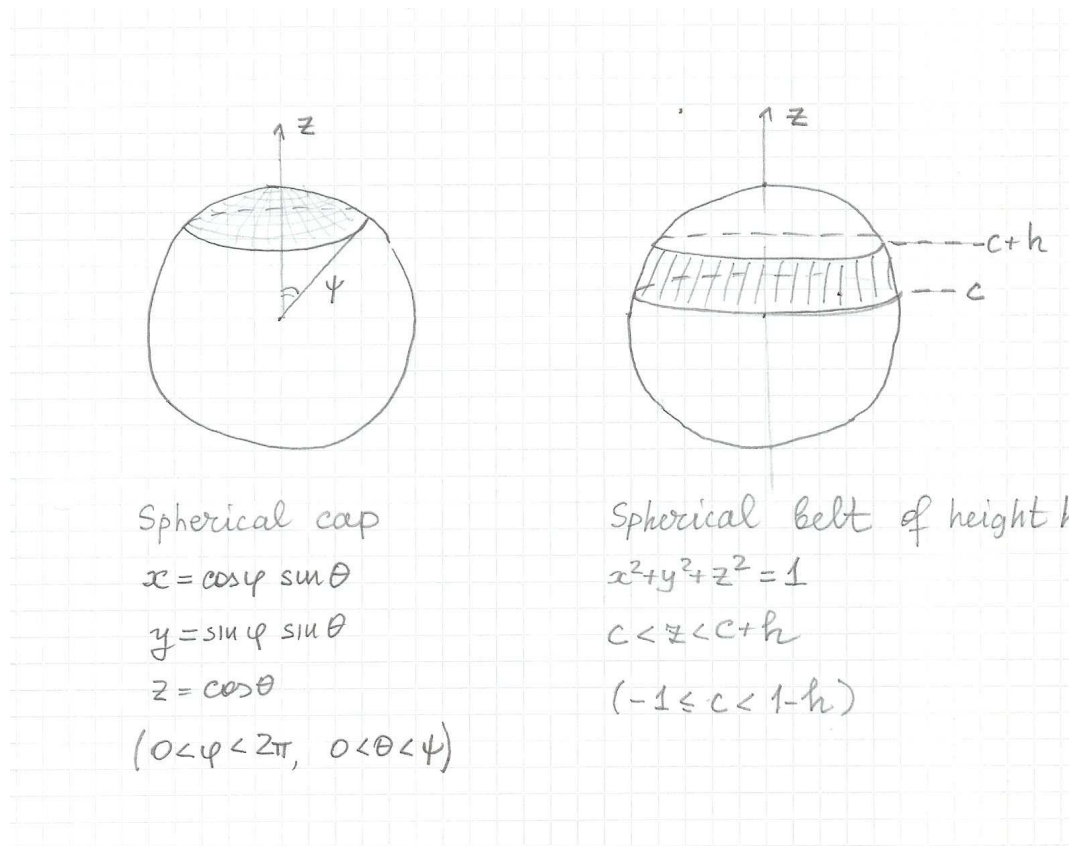
**Exercise 13.20** (Archimedes). Area of the “spherical belt” of height  $h$  on the unit sphere

$$\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, c < x_3 < c + h\}, \quad -1 \leq c \leq 1 - h,$$

equals  $2\pi h$  (and thus does not depend on the position of the belt on the sphere!).

**Exercise 13.21.** Find the volume and the surface area of the solid obtained by rotation of the triangle  $\triangle ABC$  around the side  $AB$ . The length sides  $a = |BC|$ ,  $b = |AC|$ , and the distance  $h$  from the vertex  $C$  to the side  $AB$  are given.





**Exercise 13.22.** The density of the sphere of radius  $R$  is proportional to the distance to the vertical diameter. Find the mass of the sphere, and the center of masses of the upper hemi-sphere.

Answer: the mass equals  $\pi^2 R^3$ , the coordinates of the center of masses are  $(0, 0, \frac{3}{8}R)$ .

**Exercise 13.23.** Find the centroid of the homogeneous conic surface  $z = \sqrt{x^2 + y^2}$ ,  $0 < x^2 + y^2 < 1$ .

Answer:  $(0, 0, \frac{2}{3})$ .

## 13.4 Hyperarea in $\mathbb{R}^n$

### 13.4.1 Some useful formulae

If the hypersurface  $M \subset \mathbb{R}^n$  is defined as graph of the function:  $M = \{x: x_n = \varphi(x_1, \dots, x_{n-1}), (x_1, \dots, x_{n-1}) \in G\}$ , then, according to our compu-

tation, the Gram determinant  $\Gamma$  equals  $1 + |\nabla\varphi|^2$ . Thus

$$A_{n-1}(M) = \int_G \sqrt{1 + |\nabla\varphi|^2} = \int_G \frac{1}{\cos\psi},$$

where  $\psi$  is the angle between the vectors

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\varphi_{x_1} \\ \vdots \\ -\varphi_{x_{n-1}} \\ 1 \end{pmatrix}.$$

**Exercise 13.24.** Suppose  $\Sigma^{n-1} = \{x \in \mathbb{R}^n : x_i \geq 0, \sum_i x_i = 1\}$  is a standard  $n - 1$ -simplex in  $\mathbb{R}^n$ . Then

$$A_{n-1}(\Sigma^{n-1}) = \frac{\sqrt{n}}{(n-1)!}.$$

Similarly, if the hypersurface  $M$  is defined by the equation  $\Phi(x_1, \dots, x_n) = 0$ ,  $\Phi_{x_n} \neq 0$ , then

$$\int_M f dS = \int_G f \frac{|\nabla\Phi|}{|\Phi_{x_n}|} dx_1 \dots dx_{n-1}.$$

Here  $G$  is the “projection” of  $M$  onto the hyperplane  $x_n = 0$ .

Now, suppose that the domain  $V \subset \mathbb{R}^n$  is “covered” by a family of hypersurfaces  $M_c$  defined by

$$\Phi(x_1, \dots, x_n) = c, \quad a < c < b,$$

in such a way that through each point  $x \in V$  there passes one and only one hypersurface  $M_c$ . Suppose that  $\nabla\Phi \neq 0$  in  $V$ . Then

$$(13.25) \quad \int_V f(x) dx = \int_a^b dc \int_{M_c} \frac{f(x)}{|\nabla\Phi(x)|} dS.$$

We also assume for simplicity that  $\Phi_{x_n} \neq 0$  everywhere in  $V$  (the general case can be reduced to this one). Then we replace the coordinates  $x_1, \dots, x_n$  by new coordinates  $x_1, \dots, x_{n-1}, c = \Phi(x_1, \dots, x_n)$ . The corresponding Jacobian is

$$\left| \frac{\partial(x_1, \dots, x_{n-1}, x_n)}{\partial(x_1, \dots, x_{n-1}, c)} \right| = \frac{1}{|\Phi_{x_n}|}$$

(it's faster to compute the Jacobian of the inverse mapping!). Thus,

$$\begin{aligned} \int_V f(x) dx &= \int_a^b dc \int_{M_c} \frac{f(x_1, \dots, x_n)}{|\Phi_{x_n}|} dx_1 \dots dx_{n-1} \\ &= \int_a^b dc \int_{M_c} \frac{f(x_1, \dots, x_n)}{|\nabla\Phi|} \underbrace{\frac{|\nabla\Phi|}{|\Phi_{x_n}|} dx_1 \dots dx_{n-1}}_{=dS} . \end{aligned}$$

proving (13.25). □

### 13.4.2 Integration over spheres

The formula (13.25) is very useful. We apply it to the case when  $M_\rho = \rho\mathbb{S}^{n-1}$  is the sphere of radius  $\rho$  in  $\mathbb{R}^n$ . In this case

$$\Phi(x_1, \dots, x_n) = \sqrt{x_1^2 + \dots + x_n^2}, \quad \Phi_{x_i} = \frac{x_i}{\rho}, \quad |\nabla\Phi| = 1.$$

A minute thought shows that

$$\int_{\rho\mathbb{S}^{n-1}} f(x) dS(x) = \int_{\mathbb{S}^{n-1}} f(\rho y) \rho^{n-1} dS(y).$$

**Exercise 13.26.** Check this!

Thus (13.25) gives us

$$(13.27) \quad \boxed{\int_{r\mathbb{B}} f(x) dx = \int_0^r \rho^{n-1} d\rho \int_{\mathbb{S}^{n-1}} f(\rho y) dS(y)}$$

Differentiating by  $r$ , we arrive at

$$\frac{d}{dr} \int_{r\mathbb{B}} f(x) dx = r^{n-1} \int_{\mathbb{S}^{n-1}} f(ry) dS(y).$$

In particular we find the relation between the volume  $v_n$  of the unit ball in  $\mathbb{R}^n$  and the hyper-area  $\omega_n$  of the unit sphere in  $\mathbb{R}^n$ :

$$\frac{d}{dr} (v_n r^n) = r^{n-1} \omega_n,$$

or

$$\boxed{v_n = \frac{\omega_n}{n}}$$

Now, suppose that  $f$  is the *radial* function; i.e.  $f(x) = h(|x|)$ . Then formula (13.27) gives us

$$(13.28) \quad \int_{r\mathbb{B}} h(|x|) dx = \omega_n \int_0^r h(\rho) \rho^{n-1} d\rho, \quad 0 < r \leq \infty.$$

### 13.4.3 $n - 1$ -area of the unit sphere in $\mathbb{R}^n$

Making use the latter relation, we easily compute the area of the unit sphere and the volume of the unit ball in  $\mathbb{R}^n$ . First, we use (13.28) with  $h(\rho) = e^{-\rho^2}$  and  $r = \infty$ . The left-hand side equals

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \left( \int_{\mathbb{R}^1} e^{-t^2} dt \right)^n = \pi^{n/2}.$$

The integral on the right-hand side equals

$$\int_0^\infty \rho^{n-1} e^{-\rho^2} d\rho = \frac{1}{2} \int_0^\infty t^{(n/2-1)} e^{-t} dt = \frac{1}{2} \Gamma\left(\frac{n}{2}\right).$$

Thus

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

That is,  $\omega_1 = 2$  (explain the meaning!),  $\omega_2 = 2\pi$ ,  $\omega_3 = 4\pi$ ,  $\omega_4 = 2\pi^2$ , ... etc.

From here, we once more find the volume  $v_n$  of the unit ball:

$$v_n(\mathbb{B}) = \frac{\omega_n}{n} = \frac{2(\pi)^{n/2}}{n\Gamma\left(\frac{n}{2}\right)} = \frac{(\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

#### Exercise 13.29.

$$\int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^2)^p} = \pi^{n/2} \frac{\Gamma\left(p - \frac{n}{2}\right)}{\Gamma(p)}$$

In the next two exercises we use the following notations:

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}, \quad \mathbb{S}_+^{n-1} = \{x \in \mathbb{S}^{n-1} : x_i \geq 0, 1 \leq i \leq n\}.$$

#### Exercise 13.30. Compute the integral

$$\int_{\mathbb{S}_+^{n-1}} y_1^{p_1} \dots y_n^{p_n} dS(y).$$

Hint: integrate the function  $f(x) = e^{-|x|^2} x_1^{p_1} \dots x_n^{p_n}$  over  $\mathbb{R}_+^n$ . Check your answer in the special case  $p_1 = \dots = p_n = 0$ .

#### Exercise 13.31. Suppose $a \in \mathbb{R}_+^n$ . Then

$$\int_{\mathbb{S}_+^{n-1}} \frac{dS(y)}{\langle a, y \rangle^{n-1}} = \frac{1}{(n-1)! a_1 \dots a_n}.$$

Hint: integrate  $e^{-\langle a, x \rangle}$  over  $\mathbb{R}_+^n$ .

**Exercise 13.32** (Poisson). Suppose  $f$  is a continuous function of one variable. Then

$$\int_{\mathbb{S}^{n-1}} f(\langle x, y \rangle) dS(y) = \omega_{n-1} \int_{-1}^1 f(|x|t)(1-t^2)^{\frac{n-3}{2}} dt.$$

Hint: due to the symmetry with respect to  $x$ , it suffices to consider the case  $x = (0, \dots, 0, |x|)$ .

## 14 The Divergence Theorem

### 14.1 Vector fields and their fluxes

**Definition 14.1** (Vector fields). Let  $U \subset \mathbb{R}^n$  be an open set. The vector field  $F$  on  $U$  is the mapping

$$\text{point } x \in U \mapsto \text{tangent vector } F(x) \in T_x(\mathbb{R}^n).$$

#### 14.2. Examples:

- the gradient field  $F = \nabla f$ ;
- the velocity field of a flow of fluid or gas:  $\dot{x}(t) = F(x(t))$  ('stationary field'), or more generally,  $\dot{x}(t) = F(t, x(t))$  ('non-stationary field'); the solution  $x(t)$  is called *trajectory* of the field;
- the force field (gravitational, Coulomb, magnetic)

**Definition 14.3** (the flux form).

$$\omega_F(\xi_1, \dots, \xi_{n-1}) = \det(F(x), \xi_1, \dots, \xi_{n-1}), \quad \xi_1, \dots, \xi_{n-1} \in T_x(\mathbb{R}^n),$$

that is, the 'oriented volume' of the parallelepiped  $P(F(x), \xi_1, \dots, \xi_{n-1})$  generated by the vectors  $F(x), \xi_1, \dots, \xi_{n-1}$ .

The flux can be written as  $\langle F(x), N \rangle \text{vol}_{n-1} P(\xi_1, \dots, \xi_{n-1})$ , where  $N$  is the oriented unit normal vector to the parallelepiped  $P(\xi_1, \dots, \xi_{n-1})$ :

$$N = \frac{\xi_1 \times \dots \times \xi_{n-1}}{\sqrt{\Gamma(\xi_1, \dots, \xi_{n-1})}}.$$

If  $F$  is the velocity field of a flow of liquid in  $\mathbb{R}^3$ , then the flux form equals the volume of the liquid that runs through the 'oriented parallelogram'  $P(\xi_1, \xi_2)$  in the unit time.

To define the flux of the vector field through the hyper-surface, we need to choose the unit normal  $N(x)$  at  $x \in M$ , that depends *continuously* on  $x$ <sup>24</sup>. Sometimes, this is impossible, for instance, for the Möbius strip in  $\mathbb{R}^3$ . Such surfaces are called 'non-orientable' and we shall not consider them. From now on, we always assume that there exists a continuous normal vector field  $N(x)$  on  $M$ , it defines the *orientation* of  $M$ .

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<sup>24</sup>If this choice is possible, then there are exactly *two* choices of continuous normal field. Prove this!

**Definition 14.4** (flux through the hyper-surface). Suppose  $M$  is a hyper-surface in  $\mathbb{R}^n$ ,  $N(x)$  is the unit normal to  $M$  at  $x$  that depends continuously on  $x \in M$ ,  $F$  is a continuous vector field on  $M$ . The *flux* of  $F$  through  $M$  equals

$$\text{flux}_F(M) = \int_M \langle F, N \rangle \, dS.$$

If  $M = \partial G$  is a boundary of a ‘good open set’  $G \in \mathbb{R}^n$ , then we always choose the unit *outward* normal  $N(x)$ , that corresponds to the ‘outward flux through  $M$ ’.

How to decide which normal is the ‘outward’ one? If  $G$  is defined as the sublevel set of a  $C^1$ -function, i.e.,  $G = \{x: F(x) < c\}$ , then the outward normal to  $M = \partial G$  coincides with the normalized gradient:  $N = \frac{\nabla F}{|\nabla F|}$ . If locally  $M = \partial G$  is the graph of a  $C^1$ -function, for instance,  $M \cap U = \{x \in U: x_n = f(x_1, \dots, x_{n-1})\}$ , then either  $G \cap U = \{x \in U: x_n < f(x_1, \dots, x_n)\}$ , or  $G \cap U = \{x \in U: x_n > f(x_1, \dots, x_n)\}$ . In the first case, the  $n$ -th component of the outward normal is positive, in the second case, it is negative.

**Exercise 14.5.** Find the flux of the vector field  $F = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$  through the following surfaces  $M$ :

- (a)  $M = \{x^2 + y^2 = a^2, 0 < z < h\}$ , the boundary surface of the cylinder, the normal  $N$  looks ‘outward’ with respect to the cylinder.
- (b)  $M = \{x^2 + y^2 < a^2, z = h\}$ , the top of the same cylinder. The normal  $N$  looks in the  $z$ -direction.
- (c)  $M = \{x^2 + y^2 + z^2 = a^2, x, y, z > 0\}$ . The normal  $N$  looks ‘outward’ with respect to the ball.
- (d)  $M = \{x + y + z = a, x, y, z > 0\}$ , the  $z$ -component of the normal is positive.

The problem we want to look at is as follows: Suppose  $G \subset \mathbb{R}^n$  is a domain with ‘good boundary’,  $F(x)$  is a  $C^1$ -vector field on  $\overline{G}$ . How to compute the outward flux of  $F$  through  $\partial G$ ? There are two key observations which will allow us to guess the right answer.

First, notice that the  $\text{flux}_F(\partial G)$  is the set-additive functions of  $G$ : if  $G = G_1 \cup G_2$ ,  $G_1 \cap G_2 = \emptyset$ , then

$$\text{outward flux}_F(\partial G) = \text{outward flux}_F(\partial G_1) + \text{outward flux}_F(\partial G_2)$$

(the integral over  $\partial G_1 \cap \partial G_2$  is cancelled). Of course, this set-additive function is defined on a rather restrictive class of sets  $G$ .

Now, our intuition<sup>25</sup> suggests us to look at the ‘density’ of this set-additive function.

**Definition 14.6** (divergence of the vector field). Density of flux $_F$  with respect to the cubes is called *divergence* of the vector field  $F$ :

$$\operatorname{div} F(x) = \lim_{Q \downarrow x} \frac{1}{v_n(Q)} \int_{\partial Q} \langle F, N \rangle dS.$$

**Lemma 14.7.**

$$\boxed{\operatorname{div} F = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} .}$$

**Exercise 14.8.** Check that  $\operatorname{div}(hF) = h \operatorname{div} F + \langle \nabla h, F \rangle$  ( $h$  is a scalar function), and  $\operatorname{div}(\nabla f) = \Delta f$  ( $= \sum_{i=1}^n \partial_{ii}^2 f$ , the Laplacian of  $f$ ).

**Notations:** For  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , we set  $\widehat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  (the  $i$ -th coordinate is missing).

*Proof of the Lemma:* This will be a straightforward computation. Fix  $a = (a_1, \dots, a_n)$ , and consider the cube  $Q = \prod_{i=1}^n [a_i, a_i + \epsilon]$ . The boundary  $\partial Q$  is the union of  $2n$  faces:

$$E_i^- = \{x : x_i = a_i, \widehat{x}_i \in \widehat{Q}_i\}, \quad E_i^+ = \{x : x_i = a_i + \epsilon, \widehat{x}_i \in \widehat{Q}_i\},$$

where  $\widehat{Q}_i = \prod_{k \neq i} [a_k, a_k + \epsilon]$ . Then

$$\langle F, N \rangle|_{E_i^-} = -F_i|_{E_i^-}, \quad \langle F, N \rangle|_{E_i^+} = F_i|_{E_i^+},$$

and

$$\begin{aligned} \frac{1}{v_n(Q)} \int_{\partial Q} \langle F, N \rangle dS &= \frac{1}{\epsilon^n} \sum_{i=1}^n \int_{\widehat{Q}_i} [F_i(\widehat{x}_i, a_i + \epsilon) - F_i(\widehat{x}_i, a_i)] d\widehat{x}_i \\ &= \sum_{i=1}^n \frac{1}{v_{n-1}(\widehat{Q}_i)} \int_{\widehat{Q}_i} \frac{F_i(\widehat{x}_i, a_i + \epsilon) - F_i(\widehat{x}_i, a_i)}{\epsilon} d\widehat{x}_i. \end{aligned}$$

We hope that the RHS converges to  $\sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(a)$  as  $\epsilon \downarrow 0$ . To justify the limit transition, we use the fact that  $F$  is a  $C^1$ -vector field:

$$\frac{F_i(\widehat{x}_i, a_i + \epsilon) - F_i(\widehat{x}_i, a_i)}{\epsilon} = \frac{\partial F_i}{\partial x_i}(\widehat{x}_i, a_i + \epsilon \theta_i) = \frac{\partial F_i}{\partial x_i}(a) + o(1),$$

---

<sup>25</sup>worked out during the proof of the change of variables theorem



where  $o(1)$  tends to zero *uniformly* in  $\overline{G}$  when  $\epsilon \downarrow 0$ . Integrating over  $\widehat{Q}_i$  and taking the sum over  $i$ , we get the result.  $\square$

**Warning:** Our computation works only in the Cartesian coordinates. In the Differential Geometry course, you'll learn how to compute the divergence (and other differential operators, like gradient and Laplacian) in other coordinate systems. Meanwhile, we'll use the divergence only in the Cartesian coordinates.

Combining these observations, we expect to get the celebrated result:

**Theorem 14.9** (Lagrange-Gauss-Ostrogradskii). *Suppose  $G \subset \mathbb{R}^n$  is an 'admissible' bounded domain,  $F$  is a  $C^1$ -vector field in  $G$  continuous in  $\overline{G}$ ,  $N$  is the outward unit normal on  $\partial G$ . Then*

$$\int_{\partial G} \langle F, N \rangle dS = \text{outward flux}_F(\partial G) = \int_G \operatorname{div} F = \int_G \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}$$

We will not reveal the formal definition of the class of 'admissible' domains now, it will be given later, when we come to the proof. Right now, we only mention that this class is sufficiently large for all practical purposes. It contains domains defined as the level sets of  $C^1$ -functions  $\{x: F(x) < c\}$ , where  $F \in C^1(\mathbb{R}^n, \mathbb{R}^1)$ ,  $\nabla F \neq 0$ . More generally, this class contains all domains with  $C^1$ -smooth regular boundary. It also contains domains which are the unions of finitely many cubes. In the plane case, any bounded domain whose boundary is a piece-wise smooth regular curve is admissible.

From the divergence theorem we immediately obtain the formula for integration by parts in  $\mathbb{R}^n$ .

**Corollary 14.10.** Suppose  $f$  and  $g$  are  $C^1$ -functions on an admissible domain  $\overline{G}$ . Then, for each  $i$ ,  $1 \leq i \leq n$ ,

$$\boxed{\int_G u_{x_i} v \, dx = \int_{\partial G} uv N_i \, dS - \int_G uv_{x_i} \, dx}$$

Here,  $N_i$  is the  $i$ -th component of the unit outward normal  $N$  to  $\partial G$ .

*Proof:* follows at once from the divergence theorem:

$$\int_{\partial G} uv \cdot N_i \, dS = \int_G (uv)_{x_i} \, dS.$$

**Exercise 14.11.** Prove:

$$\int_G \nabla f \, dx = \int_{\partial G} f N \, dS.$$

(The both integrands are vector-fields).

**Exercise 14.12.** Find the flux of the vector field  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  through the boundary surface of the cone  $\{x^2 + y^2 = z^2, 0 < z < h\}$ , the normal looks outside of the cone, i.e., its  $z$ -component is negative.

Hint: It is simpler, to find the flux through the top  $\{x^2 + y^2 = h^2\}$  of the cone, and then to use the Divergence Theorem.

**Exercise 14.13.** Let  $E = \{x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\}$  be the solid ellipse,  $p(x, y, z)$  be the distance from the origin to the tangent plane to  $\partial E$  at the point  $(x, y, z)$ . Compute the integrals

$$\iint_{\partial E} p \, dS, \quad \iint_{\partial E} \frac{dS}{p}.$$

Answer:  $4\pi abc, \frac{4}{3}\pi abc \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$ .

Hint: one can compute these integrals directly, though the divergence theorem makes the computations much shorter. First, compute  $p$  and  $N$ :

$$p = \frac{1}{\sqrt{x^2/a^4 + y^2/b^4 + z^2/c^4}}, \quad N = p \begin{pmatrix} x/a^2 \\ y/b^2 \\ z/c^2 \end{pmatrix}.$$

Then observe that  $\frac{1}{p} = \langle V, N \rangle$ ,  $V = \begin{pmatrix} x/a^2 \\ y/b^2 \\ z/c^2 \end{pmatrix}$ , and  $p = \langle W, N \rangle$ ,  $W = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ .

**Exercise 14.14.** Let  $F = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$  be the vector field in  $\mathbb{R}^3$ . Show that for any admissible domain  $G$  the flux of  $F$  through  $\partial G$  equals the volume of  $G$ .

**Exercise 14.15.** Let  $G \subset \mathbb{R}^3$  be an admissible domain. Find the surface integrals

$$\iint_{\partial G} N \, dS, \quad \iint_{\partial G} R \times N \, dS.$$

Here  $R(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  is the 'radius-vector'. (The both integrand are vector-functions!).

Answer: the both integrals vanish.

**Exercise 14.16.** Let  $H$  be a homogeneous polynomial in  $\mathbb{R}^3$  of degree  $k$ , that is,  $H(r \cdot) = r^k H(\cdot)$ . Prove that

$$\iint_{\mathbb{S}^2} H \, dS = \frac{1}{k} \iiint_{\mathbb{B}} \Delta H.$$

Hint: use the Euler identity:  $kH = xH_x + yH_y + zH_z$ .

**Exercise 14.17.** Compute the outward flux of the vector field  $F = \begin{pmatrix} x^k \\ y^k \\ z^k \end{pmatrix}$

( $k \geq -1$  is an integer) through the unit sphere.

Answer:

$$\begin{cases} 0, & k = 2m \\ \frac{12\pi}{k+2} & k = 2m - 1. \end{cases}$$

## 14.2 The Gauss Integral

Consider the vector field

$$E(x) = \frac{1}{|x|^3} x, \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

This is the potential field:  $E = \nabla U$ ,  $U(x) = \frac{1}{|x|}$  is the potential of the field  $E$ . The field  $E$  has zero divergence:

$$\operatorname{div} E = \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( \frac{x_i}{|x|^3} \right) = \sum_{i=1}^3 \left( \frac{1}{|x|^3} - 3 \frac{x_i^2}{|x|^5} \right) = 0.$$

(Alternatively, one can check that  $\Delta U = 0$ , i.e., the function  $U$  is harmonic.)

What the divergence theorem tells us about the flux of  $E$ ?

Let  $G$  be an admissible domain,  $0 \notin \partial G$ . If  $0 \notin G$ , then

$$\iint_{\partial G} \langle E, N \rangle \, dS = 0.$$

Now, suppose that  $0 \in G$ . Remove from  $G$  a small ball  $B_\epsilon = \{|x| \leq \epsilon\}$ ,  $G_\epsilon = G \setminus B_\epsilon$ . Then

$$\iint_{\partial G_\epsilon} \langle E, N \rangle \, dS = 0,$$

or

$$\iint_{\partial G} \langle E, N \rangle \, dS = \iint_{S_\epsilon} \langle E, N \rangle \, dS,$$

$$S_\epsilon = \partial B_\epsilon.$$

In our case,

$$\langle E, N \rangle = \frac{\cos \angle(x, N)}{|x|^2}$$

( $\angle(x, N)$  is the angle between the vectors  $x$  and  $N$ .) On  $S_\epsilon$ :

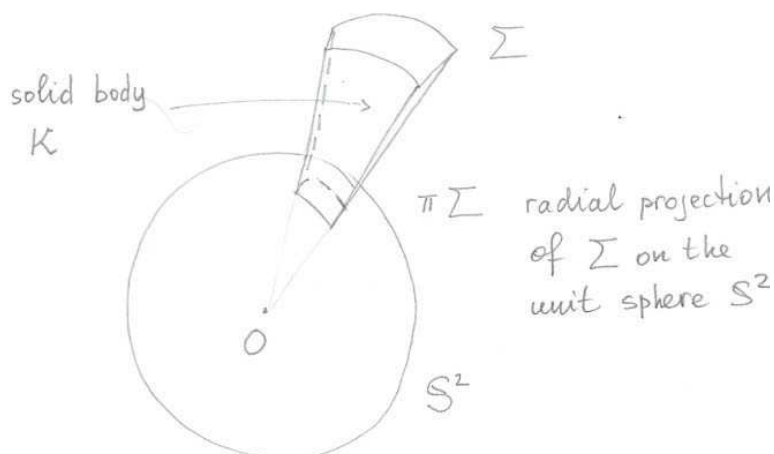
$$\langle E, N \rangle = \frac{1}{|x|^2} = \frac{1}{\epsilon^2}.$$

Thus

$$\iint_{S_\epsilon} \langle E, N \rangle dS = \frac{1}{\epsilon^2} \text{Area}(S_\epsilon) = 4\pi,$$

and we obtain

$$\iint_{\partial G} \frac{\cos \angle(x, N)}{|x|^2} dS(x) = \begin{cases} 4\pi, & 0 \in G \\ 0, & 0 \notin G. \end{cases}$$



What actually did we compute together with Gauss? Suppose  $\Sigma$  is a smooth surface in  $\mathbb{R}^3$  that does not contain the origin. Then the integral

$$\iint_{\Sigma} \frac{\cos \angle(x, N)}{|x|^2} dS$$

represents the *solid angle subtended by  $\Sigma$* . Indeed, let

$$\pi: \Sigma \rightarrow S^2$$

be the radial projection of  $\Sigma$  on the unit sphere  $\mathbb{S}^2$ . We suppose that  $\pi$  is a one-to-one mapping. Consider the solid body  $K$  bounded by  $\Sigma$ ,  $\pi\Sigma$  and the “conic part”. On the conic part of  $\partial K$ , the vectors  $x$  and  $N$  are orthogonal. Thus, the flux of  $E$  through the conic part is zero, and by the divergence theorem, the fluxes through  $\Sigma$  and  $\pi\Sigma$  are equal. On  $\pi\Sigma$ ,  $\langle E, N \rangle = 1$  (see above), hence, the flux of  $E$  through  $\pi\Sigma$  equals the area of  $\pi\Sigma$ .

### 14.3 The Green’s formulas and harmonic functions

The three celebrated Green’s formulas follows at once from the divergence theorem:

#### The 1-st Green formula

$$\boxed{\int_G \Delta u \, dx = \int_{\partial G} \frac{\partial u}{\partial n} \, dS}$$

Here  $\frac{\partial u}{\partial n} = \langle \nabla u, N \rangle$  is the (outward) normal derivative of  $u$ .

*Proof:*  $\Delta u = \operatorname{div}(\nabla u)$ . □

#### The 2-nd Green formula:

$$\boxed{\int_G \langle \nabla u, \nabla v \rangle \, dx = - \int_G u \Delta v \, dx + \int_{\partial G} u \frac{\partial v}{\partial n} \, dS}$$

*Proof:*  $u \frac{\partial v}{\partial n} = \langle u \nabla v, N \rangle$ , and  $\operatorname{div}(u \nabla v) = \langle \nabla u, \nabla v \rangle + u \Delta v$ . □

If  $u = 1$ , we get the first formula (for  $v$ ); if  $v = u$  we get

$$\int_G |\nabla u|^2 \, dx = - \int_G u \Delta u \, dx + \int_{\partial G} u \frac{\partial u}{\partial n} \, ds.$$

The LHS of this formula is called *the Dirichlet integral of  $u$* .

**The 3-rd Green formula** is the symmetrized form of the 2-nd one:

$$\boxed{\int_G (u \Delta v - v \Delta u) \, dx = \int_{\partial G} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dS}$$

**Properties of harmonic functions.** A  $C^2$ -function  $u$  is called *harmonic* if  $\Delta u = 0$ . In the two-dimensional case, harmonic functions are intimately linked with analytic functions. Namely, if  $u$  is harmonic in a domain  $G \in \mathbb{R}^2$ ,

then its *complex gradient*  $u_x - iu_y$  is the holomorphic function in  $G$ , i.e., its real and imaginary parts satisfy the Cauchy-Riemann equations:  $(u_x)_x = (-u_y)_y$ , and  $(u_x)_y = -(-u_y)_x$ . Many properties of harmonic function in plane domains follow from those of analytic functions.

Now, we use the divergence theorem and Green's formulas to prove several fundamental properties of harmonic functions in  $\mathbb{R}^n$ . Suppose  $G \subset \mathbb{R}^n$  is an admissible domain,  $u$  is harmonic in  $G$  with the gradient continuous in  $\overline{G}$ . Since  $\Delta = \operatorname{div} \nabla$ , the gradient vector field  $\nabla u$  has zero divergence. Hence

(i) The flux of the gradient flow of  $u$  across  $\partial G$  is zero:

$$\int_{\partial G} \frac{\partial u}{\partial n} dS = 0.$$

Now, from the second Green's formula, we get

(ii) If  $\frac{\partial u}{\partial n} = 0$  on  $\partial G$ , then  $u$  is a constant function.

and

(iii) If  $u = 0$  on  $\partial G$ , then  $u$  is the zero function.

(iv) **Mean-value property** If  $u$  is harmonic in a ball  $B = B(x_0, r)$  of radius  $r$  centered at  $x_0$ , then

$$u(x_0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B} u dS = \frac{1}{v_n r^n} \int_B u.$$

First of all, we assume that  $x_0 = 0$ . This will simplify the notations. We apply the 3-rd Green identity to the functions  $u$  and  $v(x) = \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}$  in the domain  $G_\epsilon = B \setminus \{|x| \leq \epsilon\}$ . Note that the function  $v$  is harmonic in  $G_\epsilon$  (check this!), and vanishes on the "outer sphere"  $|x| = r$ . The apply the Green's identity we need to compute the normal derivative  $\frac{\partial v}{\partial n} = \langle \nabla v, N \rangle$  on the boundary spheres  $|x| = r$  and  $|x| = \epsilon$ .

Since  $\nabla v = -\frac{n-2}{|x|^n} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ , and  $N = \frac{1}{|x|} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$  (with the minus sign on the

small "inner sphere"  $|x| = \epsilon$ ), we have  $\frac{\partial v}{\partial n} = -\frac{n-2}{|x|^{n-1}}$  (with the plus sign on the small "inner sphere"  $|x| = \epsilon$ ). Thus, the Green's formula gives us

$$-\frac{n-2}{r^{n-1}} \int_{|x|=r} u dS + \frac{n-2}{\epsilon^{n-1}} \int_{|x|=\epsilon} u dS + \int_{|x|=\epsilon} \left( \frac{1}{\epsilon^{n-2}} - \frac{1}{r^{n-2}} \right) \frac{\partial u}{\partial n} dS = 0.$$

It remains to let  $\epsilon \rightarrow 0$  and to check what happens with the second and third surface integrals. Since  $u$  is continuous, the second integral converges to  $(n-2)\omega_n u(0)$  (as above,  $\omega_n$  is the hyper-area of the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ ). Since  $\frac{\partial u}{\partial n}$  is bounded, the third integral converges to zero.

**Exercise 14.18.** Fill the details!

Thus we get the first mean-value formula. The second formula follows from the first one by spherical integration.  $\square$

In the two-dimensional case  $n = 2$ , the same proof works with the function  $v(x) = \log \frac{r}{|x-x_0|}$ , though you can deduce the same from the mean value property of analytic functions.

**Exercise 14.19.** Fill the details!

**(v) Maximum principle** Harmonic functions have no local maxima or minima. More precisely, if for some  $x_0 \in G$ ,  $u(x_0) = \max_{\overline{G}} u$ , then  $u$  is the constant function.

**Exercise 14.20.** Give the proof.

Hint: the set  $M = \{x \in G: u(x) = \max_{\overline{G}} u\}$  is closed and open. Hence, it coincides with  $G$ .

**(vi) Liouville Theorem** If  $u$  is a harmonic function in  $\mathbb{R}^n$  bounded from above (or from below). Then it is a constant function.

Hint: Suppose  $u \geq 0$ . Fix any two points  $x$  and  $y$ . Choose large  $r$  and  $R > |x - y| + r$ . Then by the mean-value property

$$\begin{aligned} u(x) &= \frac{1}{v_n r^n} \int_{B(x,r)} u < \frac{1}{v_n r^n} \int_{B(y,R)} u \\ &= \left(\frac{R}{r}\right)^n \frac{1}{v_n R^n} \int_{B(y,R)} u = \left(\frac{R}{r}\right)^n u(y). \end{aligned}$$

It remains to send  $r \rightarrow \infty$  and  $R \rightarrow \infty$  in such a way that  $R/r \rightarrow 1$ . We get  $u(x) \leq u(y)$ . By the symmetry,  $u(x) = u(y)$ .  $\square$

You will learn more about harmonic functions in the course of partial differential equations.

**Exercise 14.21** (solution of the Poisson equation). Let  $u$  be a  $C^2$ -function that vanishes outside of a compact set in  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $f = \Delta u$ . Show that

$$u(x) = c_n \int_{\mathbb{R}^n} \frac{f(\xi) d\xi}{|x - \xi|^{n-2}}.$$

Compute  $c_n$ . Guess how the corresponding formula looks for  $n = 2$ .

Hint: fix  $x$ , and apply the 3-rd Green formula in  $B(x, R) \setminus B(x, \epsilon)$ , where  $R$  is sufficiently large and  $\epsilon$  is small, to the functions  $u(\xi)$  and  $v(\xi) = |x - \xi|^{2-n}$ . Then let  $\epsilon \rightarrow 0$ .

## 15 Proof of the Divergence Theorem

Here, we'll prove the Divergence Theorem. First, we consider the case when the boundary is smooth. Then we prove a more general version which is sufficient for most of the applications. In the course of the proof we'll meet two important constructions: *'the partition of unity'* and *'the cut-off function'*.

### 15.1 Smooth boundary

**Definition 15.1.** Let  $G \subset \mathbb{R}^n$  be a bounded domain. We say that  $G$  has a *smooth boundary*  $\Gamma = \partial G$ , if

$\forall x \in \Gamma \exists$  ball  $B_x$  centered at  $x$ , and  $\exists C^1$ -function  $g$  such that

$$\Gamma \cap B_x = \{\xi \in B_x : \xi_n = g(\xi_1, \dots, \xi_{n-1})\},$$

$$G \cap B_x = \{\xi \in B_x : \xi_n < g(\xi_1, \dots, \xi_{n-1})\}$$

(after possible re-numeration of the coordinates).

Note that at each boundary point  $x \in \Gamma$  the unit outward normal is well-defined.

We fix a covering  $\mathcal{O}$  of  $\overline{G}$  by the balls  $B_x$ ; if  $x \in \Gamma$ , then their radii  $\rho_x$  are chosen as above, if  $x \in G$ , we always assume that  $\rho_x < \text{dist}(x, \Gamma)$ .

#### 15.1.1 Partition of unity

Suppose  $K \subset \mathbb{R}^n$  is a compact set with a given covering  $\mathcal{O}$  by balls:

$$K \subset \bigcup_{x \in K} B_x.$$

**Definition 15.2.** *Partition of unity on  $K$  subordinated to the covering  $\mathcal{O}$*  is a finite collection of  $C^1$ -functions  $\{\varphi_j\}$  such that

- (i)  $\varphi_j \geq 0$ ;
- (ii)  $\sum_j \varphi_j \equiv 1$  in a neighbourhood of  $K$ ;
- (iii)  $\varphi_j$  vanishes outside of some ball  $\frac{1}{2}B_{x_j}$ .

It is not difficult to construct a partition of unity. Take

$$\psi_x(y) = \begin{cases} (\rho_x^2 - 4|x - y|^2)^2, & |x - y| \leq \frac{1}{2}\rho_x \\ 0, & \text{otherwise} \end{cases}$$



(as above  $\rho_x$  is the radius of the ball  $B_x$ ). Then consider the covering

$$K \subset \bigcup_{x \in K} \frac{1}{4} B_x,$$

and choose a finite sub-covering. The centers of the balls from the sub-covering are  $x_1, \dots, x_M$ . Set

$$\varphi_j(y) \stackrel{\text{def}}{=} \frac{\psi_{x_j}(y)}{\psi_{x_1}(y) + \dots + \psi_{x_M}(y)}$$

( $\varphi_j(y) = 0$  if  $\psi_j$  vanishes at  $y$ ). The properties (i)–(iii) hold. (Check them!)

In the same way, one builds  $C^k$ -partitions of unity.

**Exercise 15.3.** Construct a  $C^\infty$ -partition of unity subordinated to a given covering of the compact set  $K$ .

Hint: use the function

$$h(t) = \begin{cases} \exp(-1/t), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

as the ‘building block’.

### 15.1.2 Integration over $\Gamma$

Recall, that we have not defined yet the integral over the whole  $\Gamma$ , only over  $\Gamma \cap B_x$ . Now, having at hands continuous partitions of unity on  $\Gamma$ , we readily define the integral over  $\Gamma$ : for any  $f \in C(\Gamma)$ , we set

$$\int_{\Gamma} f dS \stackrel{\text{def}}{=} \sum_j \int_{\Gamma} f \varphi_j dS.$$

The integrals on the RHS are defined since on the RHS the integration is local, it is taken only over  $\Gamma \cap B_{x_j}$ . Of course, the continuity of  $f$  can be replaced by a weaker assumption (after all, we have the Lebesgue criterium of the Riemann-integrability), but we will not pursue this.

The definition does not depend on the choice of the partition of unity. Indeed, if  $\{\varphi'_k\}$  is another partition of unity, then, by additivity of the Riemann integral,

$$\int_{\Gamma} f \varphi_j dS = \sum_k \int_{\Gamma} f \varphi_j \varphi'_k dS,$$

so that

$$\begin{aligned} \sum_j \int_{\Gamma} f \varphi_j dS &= \sum_{j,k} \int_{\Gamma} f \varphi_j \varphi'_k dS \\ &= \sum_k \int_{\Gamma} f \varphi'_k \left( \sum_j \varphi_j \right) dS = \sum_k \int_{\Gamma} f \varphi'_k dS. \end{aligned}$$

□

### 15.1.3 The theorem and its proof

Now, we are ready to formulate and then to prove the divergence theorem in the smooth case.

**Theorem 15.4.** *Suppose  $G \subset \mathbb{R}^n$  is a bounded domain with a smooth boundary  $\Gamma$ ,  $N$  is the outward unit normal to  $\Gamma$ , and  $F$  is a  $C^1$ -vector field on  $\overline{G}$ . Then*

$$\int_{\Gamma} \langle F, N \rangle dS = \int_G \operatorname{div} F dx.$$

First, observe that it suffices to prove the result in the special case when the field  $F$  is ‘localized’; i.e. given a covering of  $G$  by balls

$$\overline{G} \subset \bigcup_{x \in \overline{G}} B_x,$$

we can always assume that  $F \equiv 0$  outside of a ball  $B_x$  from this covering. Indeed, we construct a  $C^1$ -partition of unity  $\{\varphi_j\}$  on  $\overline{G}$  subordinated to this covering. Then we apply the special case to the vector fields  $\varphi_j F$ , and add the results.

We consider separately two cases:  $x \in G$  and  $x \in \Gamma$ .

Assume, first, that  $x \in G$ , then  $B_x \subset G$  as well (this was our choice of the radius  $\rho_x$ ). Thus  $F$  vanishes on  $\Gamma$ , and we need to check that the integral of the divergence also vanishes. For each  $i$ , we denote  $\widehat{y}_i = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$ , the  $i$ -th coordinate is missing. Then

$$\int_G \frac{\partial F_i}{\partial y_i} dy = \int d\widehat{y}_i \int \frac{\partial F_i}{\partial y_i} dy_i,$$

to avoid cumbersome notations, we skip the limits of integration.

The inner integral vanishes since  $F_i$  vanishes on the boundary of  $B_x$ , thus

$$\int_G \partial F_i / \partial y_i dy = 0.$$

Hence the divergence integral over  $G$  also vanishes.

Now, consider the second case  $x \in \Gamma$ . Then

$$\Gamma \cap B_x = \{\xi \in B_x : \xi_n = g(\xi_1, \dots, \xi_{n-1})\}.$$

We assume that for any  $x \in \Gamma$  the outward normal  $N(x)$  is not parallel to any of the coordinate axes, i.e.,  $g_{x_i} \neq 0$ ,  $1 \leq i \leq n-1$  (otherwise, we just rotate a bit the coordinate system). Then, for any  $i$ ,  $\Gamma \cap B_x$  can be represented as

$$\Gamma \cap B_x = \{\xi \in B_x : \xi_i = g_i(\widehat{\xi}_i)\}, \quad 1 \leq i \leq n.$$

Then

$$\int_{\Gamma \cap B_x} F_i N_i dS = \int F_i(\widehat{\xi}_i, g_i(\widehat{\xi}_i)) d\widehat{\xi}_i.$$

On the other hand, using Fubini's theorem, we get

$$\int_{G \cap B_x} \frac{\partial F_i}{\partial \xi_i} d\xi = \int d\widehat{\xi}_i \int^{g_i(\widehat{\xi}_i)} \frac{\partial F_i}{\partial \xi_i} d\xi_i$$

(the lower limit in the inner integral is inessential since anyway  $F_i$  vanishes therein). Thus the inner integral equals  $F_i(\widehat{\xi}_i, g_i(\widehat{\xi}_i))$ , and

$$\int_{\Gamma \cap B_x} F_i N_i dS = \int_{G \cap B_x} \frac{\partial F_i}{\partial \xi_i} d\xi.$$

Done! □

## 15.2 'Piece-wise smooth' boundary

Here, we assume that the boundary  $\Gamma = \partial G$  is decomposed into two parts: a 'smooth one' which is large, and a 'bad one' which is small:  $\Gamma = \Gamma_0 \cup K$ , where  $\Gamma_0$  is a finite union (maybe, disconnected) of smooth 'patches', and at each point  $x \in \Gamma_0$  condition 15.1 holds (in particular, the unit outward normal to  $G$  is well-defined at these  $x$ 's),  $K$  is a 'bad' compact set such that

$$(15.5) \quad v_n(K_{+\epsilon}) = o(\epsilon),$$

as  $\epsilon \rightarrow 0$ . Here,  $K_{+\epsilon}$  is an open  $\epsilon$ -neighbourhood of  $K$ . We shall call such sets '( $n-1$ )-negligible compacts'<sup>26</sup>. Note that a finite union of ( $n-1$ )-negligible compacts, is again an ( $n-1$ )-negligible compact.

<sup>26</sup>Look at Google for the notion 'Minkowski dimension' (or 'box dimension').

**Example 15.6** (compact elementary  $(n - 2)$ -surface is  $(n - 1)$ -negligible).  
Suppose

$$\Sigma = \{(y, f_1(y), f_2(y)) : y \in Q\},$$

where  $Q \subset \mathbb{R}^{n-2}$  is a closed cube. Then

$$\Sigma_{+\epsilon} \subset \{(y, x_1, x_2) : y \in Q_{+\epsilon}, |x_i - f_i(y)| < \epsilon, i = 1, 2\}.$$

Hence,

$$v_n(\Sigma_{+\epsilon}) \leq \int_{Q_{+\epsilon}} dy \int_{f_1(y)-\epsilon}^{f_1(y)+\epsilon} dx_1 \int_{f_2(y)-\epsilon}^{f_2(y)+\epsilon} dx_2 = 4\epsilon^2 v_{n-2}(Q_{+\epsilon}) = o(\epsilon),$$

as  $\epsilon \rightarrow 0$ .

**Exercise 15.7.** Show that if the compact  $K \subset \mathbb{R}^n$  is  $(n - 1)$ -negligible, then for each  $\epsilon > 0$  it can be covered by cubes  $Q_j$  such that

$$\sum_j l(Q_j)^{n-1} < \epsilon, \quad (\star)$$

$l(Q_j)$  is the length-side of  $Q_j$ .

**Exercise 15.8.** The *packing number*  $P(K, \epsilon)$  of a compact set  $K \subset \mathbb{R}^n$  is the maximal cardinality of an  $\epsilon$ -separated subset in  $K$ <sup>27</sup>. Show that the compact  $K$  is  $(n - 1)$ -negligible iff  $P(K, \epsilon) = o(\epsilon^{1-n})$  for  $\epsilon \rightarrow 0$ .

The *covering number*  $C(K, \epsilon)$  of  $K$  is the minimal cardinality of a set of balls of radius  $\epsilon$  that cover  $K$  (the centers of these balls, generally speaking, do not belong to  $K$ ). Show that the compact  $K$  is  $(n - 1)$ -negligible iff  $C(K, \epsilon) = o(\epsilon^{1-n})$  for  $\epsilon \rightarrow 0$ .

Now, we formulate the version of the divergence theorem we mean to prove:

**Theorem 15.9.** Suppose  $G \subset \mathbb{R}^n$  is a bounded open set,  $\Gamma = \partial G$  can be decomposed into a smooth part  $\Gamma_0$  and a bad part  $K$  which is  $(n - 1)$ -negligible, and  $F$  is a  $C^1$ -vector field on  $\overline{G}$ . Then

$$(15.10) \quad \int_G \operatorname{div} F \, dx = \int_{\Gamma_0} \langle F, N \rangle \, dS$$

<sup>27</sup>The set  $\{x_i\}$  is called  $\epsilon$ -separated if  $|x_i - x_j| \geq \epsilon$  for  $i \neq j$ .

### 15.2.1 The idea

First, suppose that the vector-field vanishes in a neighbourhood of a bad compact set  $K$ . Then the proof given above works without any changes and gives us (15.10). Now, we approximate the field  $F$  by another one that vanishes in a neighbourhood of  $K$ .

For this, given  $\epsilon > 0$ , we build a ‘ $C^1$ -cut-off function’  $\psi = \psi_\epsilon$  such that

- (i)  $0 \leq \psi \leq 1$  everywhere;
- (ii)  $\psi \equiv 1$  in the  $\epsilon$ -neighbourhood of  $K$ ;
- (iii)  $\psi$  vanishes outside of the  $3\epsilon$ -neighbourhood of  $K$ ;
- (iv) For  $\epsilon \rightarrow 0$ ,

$$\int_{\mathbb{R}^n} \psi = o(\epsilon), \quad \int_{\mathbb{R}^n} |\nabla \psi| = o(1).$$

A bit later, we’ll build such a cut-off function for any compact  $(n - 1)$ -negligible set  $K$  and any  $\epsilon > 0$ . Meanwhile, an instructive exercise, is to consider the case when  $K$  is a point in  $\mathbb{R}^1$  (then the cut-off with these properties does not exist), and in  $\mathbb{R}^2$  (when the cut-off does exist).

Having at hands the cut-off function  $\psi$ , we easily complete the proof as follows. We apply the divergence theorem to the vector field  $(1 - \psi)F$  that vanishes near the bad set  $K$  and get

$$\int_G \operatorname{div} [(1 - \psi)F] \, dx = \int_{\Gamma_0} \langle (1 - \psi)F, N \rangle \, dS.$$

Now, we let  $\epsilon \rightarrow 0$ . Since  $\psi$  vanishes at that limit, we expect to get (15.10). First, look at the LHS. Since

$$\operatorname{div}(1 - \psi)F - \operatorname{div}F = -\langle \nabla \psi, F \rangle - \psi \operatorname{div}F,$$

we need to estimate two integrals over  $G$ . For this, we use that  $F \in C^1(G)$  and the property (iv) of the cut-off  $\psi$ :

$$\left| \int_G \langle \nabla \psi, F \rangle \, dx \right| \leq \max_G |F| \int_{\mathbb{R}^n} |\nabla \psi| \, dx = o(1),$$

and

$$\left| \int_G \psi \operatorname{div}F \, dx \right| \leq \max_G |\operatorname{div}F| \int_{\mathbb{R}^n} \psi \, dx = o(\epsilon).$$

Now, look at the RHS. Here, the situation is even more simple: we can think that  $\Gamma_0$  is an elementary ‘patch’. Since the  $\langle F, N \rangle$  is a bounded function on  $\Gamma_0$ ,

$$\int_{\Gamma_0} \langle \psi F, N \rangle \, dS \rightarrow \int_{\Gamma_0} \langle F, N \rangle \, dS,$$

as  $\epsilon \rightarrow 0$ . This step does not need the property (iv) of the cut-off function. (Fill the details!).

This proves the theorem modulo the construction of the cut-off function.

### 15.2.2 The cut-off function

We'll *smooth* the indicator function  $\mathbb{1}_{K_{+2\epsilon}}$ . For this, we fix a  $C^1$ -function  $\chi$  with the following properties:

- (a)  $\chi$  vanishes outside of the ball  $\frac{1}{2}\mathbb{B}$  of radius  $\frac{1}{2}$  centered at the origin;
- (b)  $\chi$  is non-negative, and  $\chi(0) > 0$ ;
- (c)

$$\int \chi = 1.$$

Clearly such a function exists.

**Exercise 15.11.** Construct a  $C^\infty$ -function with these properties.

We scale this function:  $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ , and finally set

$$(15.12) \quad \psi(x) = \int_{\mathbb{R}^n} \mathbb{1}_{K_{+2\epsilon}}(y)\chi_\epsilon(x-y) dy$$

$$(15.13) \quad = \int_{\mathbb{R}^n} \mathbb{1}_{K_{+2\epsilon}}(x-y)\chi_\epsilon(y) dy.$$

We'll readily check that  $\psi$  satisfies the properties (i)-(iv) stated above.

Clearly,  $\psi$  is non-negative, and

$$\psi(x) \leq \int_{\mathbb{R}^n} \chi_\epsilon(x-y) dy = \int_{\mathbb{R}^n} \chi(y) dy = \epsilon^{-n} \int_{\mathbb{R}^n} \chi(y/\epsilon) dy = 1.$$

We get (i).

If  $x \in K_{+\epsilon}$ , then  $x-y \in K_{+2\epsilon}$  (remember that  $y \in \epsilon\mathbb{B}$  in (15.13)). Hence, for these  $x$ 's,

$$\psi(x) = \int_{\mathbb{R}^n} \chi_\epsilon(y) dy = 1,$$

that is, (ii).

If  $x \notin K_{+3\epsilon}$ , then for the same reason,  $x-y \notin K_{+2\epsilon}$ , and the integrand vanishes. Thus  $\psi(x) = 0$  for such  $x$ 's.

The integral estimates in (iv) are also simple:

$$\int_{\mathbb{R}^n} \psi = \int_{\mathbb{R}^n} \mathbb{1}_{K_{+2\epsilon}} \cdot \int_{\mathbb{R}^n} \chi_\epsilon = o(\epsilon),$$

and

$$\int_{\mathbb{R}^n} |\nabla \psi| \leq \int_{\mathbb{R}^n} \mathbb{1}_{K+2\epsilon} \cdot \underbrace{\int_{\mathbb{R}^n} |\nabla \chi_\epsilon|}_{=\frac{1}{\epsilon} \int_{\mathbb{R}^n} |\nabla \chi|} = o(\epsilon) \cdot O(1/\epsilon) = o(1),$$

completing the argument. □

## 16 Linear differential forms. Line integrals

### 16.1 Work (motivation)

We start with motivation: suppose  $F$  is a force field in  $\mathbb{R}^3$ ,  $\gamma: I \rightarrow \mathbb{R}^3$  piecewise smooth path of motion of particle in the field  $F$ , that is

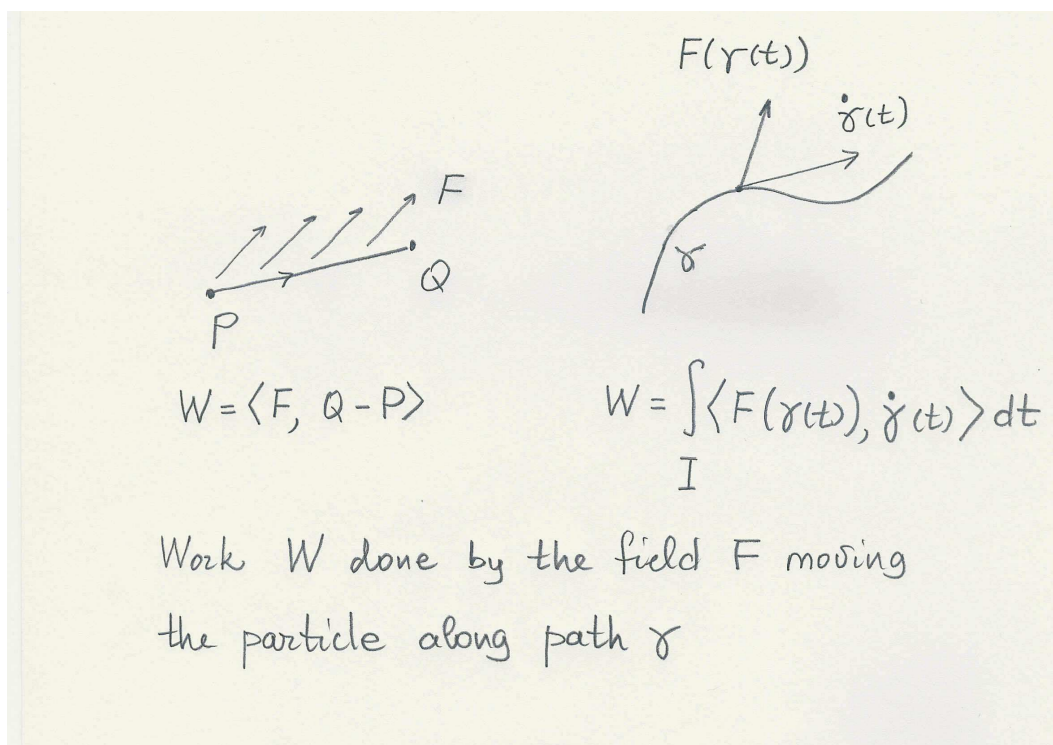
$\gamma(t)$  is a position of the particle at time  $t$ ,

$\dot{\gamma}(t)$  is a velocity of the particle at time  $t$ ,

$F(\gamma(t))$  the force acting on the particle at time  $t$ .

We want to compute the amount of work  $W$  done by the field  $F$  moving the particle along  $\gamma$ .

Let us recall how this problem was solved in the high-school physics. First, suppose that the field  $F$  is constant. If we move the particle along the segment  $[P, Q] \subset \mathbb{R}^3$ , then  $W = \langle F, Q - P \rangle$ .



If the path is not straight and the field is not constant, we consider a partition  $\Pi$  of the segment  $I = [a, b]$ :  $a = t_0 < t_1 < \dots < t_N = b$ , and use the additivity of the work:

$$W(\gamma, F, \Pi) = \sum_{j=1}^N \langle F(\gamma(t_j)), \gamma(t_j) - \gamma(t_{j-1}) \rangle.$$



Rewriting the RHS as

$$\sum_{j=1}^N \left\langle F(\gamma(t_j)), \frac{\gamma(t_j) - \gamma(t_{j-1})}{\Delta t} \right\rangle \Delta t, \quad \Delta t = t_j - t_{j-1},$$

we easily recognize the integral sum. Thus, in the limit  $\Delta t \rightarrow 0$ , we get

$$W = \int_I \langle F(\gamma(t)), \dot{\gamma}(t) \rangle dt.$$

In the coordinates,  $F = (F_x, F_y, F_z)$ ,  $\gamma(t) = (x(t), y(t), z(t))$ , and

$$W = \int_I \left( F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} \right) dt,$$

or symbolically

$$W = \int_{\gamma} F_x dx + F_y dy + F_z dz.$$

The RHS is called *the line integral over the curve  $\gamma$  of the linear differential form  $F_x dx + F_y dy + F_z dz$* . Note that it does not depend on the choice of parameterization of  $\gamma$ !

## 16.2 Linear differential forms. Differentials

Let  $f$  be a  $C^1$ -function in a neighbourhood of  $x \in \mathbb{R}^n$ . Let us recall that its differential<sup>28</sup>  $df_x$  is a linear function on the tangent space  $T_x \mathbb{R}^n$ . Indeed, if  $\gamma$  is a smooth curve passing through  $x$ ,  $\gamma(t) = x$ , then  $f(\gamma(t))$  is a smooth curve passing through  $f(x)$ , and

$$\frac{d}{dt} f(\gamma(t)) = df_x(\dot{\gamma}(t)) \left( = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \dot{\gamma}_i(t) \right).$$

**Definition 16.1.** The set of linear functionals on the tangent space is called the co-tangent space, and denoted by  $(T_x \mathbb{R}^n)^*$ .

Let us fix the orthonormal basis  $e_1, \dots, e_n$  in  $\mathbb{R}^n$ , and hence, in all tangent spaces  $T_x \mathbb{R}^n$ . It induces the dual orthonormal basis in all co-tangent spaces  $(T_x \mathbb{R}^n)^*$ : consider the differential  $dx_k$  of the  $k$ -th coordinate function  $x \mapsto x_k$  in  $\mathbb{R}^n$ , if  $\xi \in T_x \mathbb{R}^n$ , then

$$dx_k(\xi) = \sum_{i=1}^n \frac{\partial x_k}{\partial x_i} \xi_i = \xi_k.$$

<sup>28</sup>For traditional reason, we say here ‘differential’, not ‘derivative’, and write  $df$ .

That is,  $dx_1, \dots, dx_n$  is the orthonormal basis in  $(T_x\mathbb{R}^n)^*$  dual to  $e_1, \dots, e_n$ . Since

$$df_x(\xi) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \xi_i = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i(\xi),$$

we see that the expansion of the differential  $df_x$  in this bases is

$$df_x = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

**Definition 16.2.** The linear differential form is a mapping

$$x \mapsto \omega_x \in (T_x\mathbb{R}^n)^* .$$

We usually assume that the linear form  $\omega_x$  is a  $C^1$ -function of  $x$ .

In the coordinates,

$$\omega_x(\xi) = \omega_x \left( \sum_{i=1}^n \xi_i e_i \right) = \sum_{i=1}^n \omega_x(e_i) \xi_i .$$

Introduce the functions  $a_i(x) \stackrel{\text{def}}{=} \omega_x(e_i)$  (the ‘coefficients’ of  $\omega_x$ ); if  $\omega_x$  is a  $C^1$ -form, then the coefficients are  $C^1$ -functions of  $x$  (and vice versa!). Thus

$$\omega_x = \sum_{i=1}^n a_i(x) dx_i .$$

If  $\omega = df$ , then we call  $f$  *the primitive function*, and  $\omega$  *the differential*. The first natural question: *does any differential form has a primitive function?*

In the one-dimensional situation ( $n = 1$ ) this is true:  $\omega_x(\xi) = a(x) \cdot \xi$ , that is  $\omega_x = df$ , where  $f' = a$  is a primitive function to  $a$ .

Consider the two-dimensional case. If  $\omega = df = a(x, y)dx + b(x, y)dy$ , then

$$\frac{\partial a}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial b}{\partial x} .$$

That is, we get a *necessary* condition for the differential form to have a primitive function:

$$\boxed{\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}}$$

We shall see a bit later, that generally speaking, this condition is not a sufficient one, though in the discs and in the whole  $\mathbb{R}^2$  it is sufficient.

### 16.2.1 Examples

Compute the action of the linear forms  $\omega_1 = dx_1$ ,  $\omega_2 = x_1 dx_2$ ,  $\omega_3 = dr^2$  ( $r^2 = x^2 + y^2$ ) in  $\mathbb{R}^2$  on the vectors  $\xi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in T_0\mathbb{R}^2$ ,  $\xi_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \in T_{(2,2)}\mathbb{R}^2$ , and  $\xi_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in T_{(2,2)}\mathbb{R}^2$ .

The results are given in the following table

	$\xi_1$	$\xi_2$	$\xi_3$
$\omega_1$	0	-1	1
$\omega_2$	0	-2	-2
$\omega_3$	0	-8	0

**Exercise 16.3.** Compute  $\omega_x(\xi)$  if

1.  $\omega = x_2 dx_1$  is the differential form in  $\mathbb{R}^3$ , and  $\xi = (1, 2, 3) \in T_{(3,2,1)}\mathbb{R}^3$ .

Answer:  $\omega(\xi) = 2$ .

2.  $\omega = df$  is the differential form in  $\mathbb{R}^n$ ,  $f = x_1 + 2x_2 + \dots + nx_n$ ,  $\xi = (+1, -1, +1, \dots, (-1)^{n-1}) \in T_x\mathbb{R}^n$ ,  $x = (1, 1, \dots, 1)$ .

Answer:

$$\omega(\xi) = \begin{cases} -m & n = 2m, \\ m + 1 & n = 2m + 1. \end{cases}$$

## 16.3 Line integrals

$U \subset \mathbb{R}^n$  domain,

$\omega_x$  differential form on  $U$ ,

$\gamma: I \rightarrow U$  piece-wise  $C^1$  curve.

**Definition 16.4** (line integral).

$$\int_{\gamma} \omega \stackrel{\text{def}}{=} \int_I \omega_{\gamma(t)} (\dot{\gamma}(t)) dt.$$

In the chosen coordinates, the integral equals

$$\int_I \left( \sum a_i(\gamma(t)) \right) dt = \int_{\gamma} \sum a_i(x) dx_i.$$

### 16.3.1 Examples

1.  $\omega = z dx + x dy + y dz$ ,  $\gamma(t) = (\cos t, \sin t, t)$ ,  $0 \leq t \leq 2\pi$  spiral in  $\mathbb{R}^3$ ,  
 $\dot{\gamma}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$ . Then

$$\omega_{\gamma(t)}(\dot{\gamma}(t)) = t \cdot (-\sin t) + \cos t \cdot \cos t + \sin t \cdot 1,$$

and

$$\begin{aligned} \int_{\gamma} \omega &= \int_0^{2\pi} (-t \sin t + \cos^2 t + \sin t) dt \\ &= \int_0^{2\pi} t d \cos t + \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt = 2\pi + 2\pi \cdot \frac{1}{2} = 3\pi. \end{aligned}$$

**Exercise 16.5.** Find  $\int_{\gamma} \omega$  for  $\omega = x dy - y dx$ , and  $\omega = x dy + y dx$ . The curve  $\gamma$  connects the origin  $O$  with the point  $(1, 1)$ :

$\gamma$  is a segment,

$\gamma$  is a part of the parabola  $\{y = x^2 : 0 \leq x \leq 1\}$ ,

$\gamma$  is a union of two segments: the vertical one going from  $(0, 0)$  to  $(0, 1)$ , and the horizontal one going from  $(0, 1)$  to  $(1, 1)$ .

**Exercise 16.6.** Compute

$$\int_{\gamma} \frac{x dx + y dy + z dz}{\sqrt{x^2 + y^2 + z^2}}$$

where the path  $\gamma$  starts at the sphere  $x^2 + y^2 + z^2 = r_1^2$  and terminates at the sphere  $x^2 + y^2 + z^2 = r_2^2$ .

**Exercise 16.7.** Find

$$\int z dx + x dy + y dz$$

over the parabolic arc

$$x = a(1 - t^2), \quad y = b(1 - t^2), \quad z = t,$$

starting at  $(0, 0, 1)$  and ending at  $(0, 0, -1)$ .

2. Consider a smooth form on  $\mathbb{R}^2 \setminus \{0\}$

$$\omega = \frac{-y dx + x dy}{x^2 + y^2},$$

and integrate it over the unit circle:  $\gamma(t) = (\cos t, \sin t)$ ,  $0 \leq t \leq 2\pi$ . In this case,

$$\omega_{\gamma(t)}(\dot{\gamma}(t)) = -(\sin t) \cdot (-\sin t) + (\cos t) \cdot (\cos t) = 1,$$

so that

$$\int_{\gamma} \omega = \int_0^{2\pi} 1 = 2\pi.$$

Let's have a closer look at this example which is nothing but a two-dimensional version of the Gauss integral (why?). Consider the polar angle function

$$\theta(x, y) = \arctan \frac{y}{x}.$$

If  $(x, y) \neq 0$ , then

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + y^2/x^2} \left( -\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2},$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + y^2/x^2} \left( \frac{1}{x} \right) = \frac{x}{x^2 + y^2},$$

hence  $d\theta = \omega$ . Clearly, the function  $\theta$  cannot be defined continuously in the whole  $\mathbb{R}^2 \setminus \{0\}$ .

In this example,  $\omega = a dx + b dy$ , and the necessary condition  $a_y = b_x$  for the form to have a primitive function holds. On the other hand,  $\omega$  does not have a primitive in  $\mathbb{R}^2 \setminus \{0\}$  by the following version of the Newton-Leibnitz formula:

$$\int_{\gamma} df = \int_a^b \frac{d}{dt} (f \circ \gamma)(t) dt = f(\gamma(b)) - f(\gamma(a)).$$

In particular, if the curve  $\gamma$  is closed (i.e.  $\gamma(a) = \gamma(b)$ ), then

$$\int_{\gamma} df = 0.$$

In our case, we know that the integral of  $\omega$  over the circle does not vanish!

### 16.3.2 Properties of line integrals

- The definition does not depend on the choice of parameterization of the curve  $\gamma$ .

Indeed, suppose  $\gamma: I \rightarrow \mathbb{R}^n$ ,  $\mu = \gamma \circ \varphi$  another parameterization ( $\varphi: J \rightarrow I$   $C^1$ -smooth, orientation preserving bijection). Then, after the change of

variables  $t = \varphi(s)$ , we get

$$\begin{aligned} \int_{\gamma} \omega &= \int_I \omega_{\gamma(t)} (\dot{\gamma}(t)) dt = \int_J \omega_{\mu(s)} (\dot{\gamma}(\varphi(s))) \dot{\varphi}(s) ds \\ &= \int_J \omega_{\mu(s)} (\dot{\gamma}(\varphi(s))) \dot{\varphi}(s) ds = \int_J \omega_{\mu(s)} (\dot{\mu}(s)) ds. \end{aligned}$$

- If  $-\gamma$  the curve traversed in the opposite direction, then

$$\int_{-\gamma} \omega = - \int_{\gamma} \omega.$$

(Prove this!)

- Suppose the starting point of the curve  $\gamma_2$  coincides with the terminating point of the curve  $\gamma_1$ . Denote the ‘composite curve’ by  $\gamma_1 + \gamma_2$ . Then

$$\int_{\gamma_1 + \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

- **Estimate of the line integral**

**Exercise 16.8.** Suppose  $\gamma: [0, 1] \rightarrow \mathbb{R}^n$  is a piece-wise  $C^1$ -path,  $\omega$  is a continuous linear form on  $U \supset \gamma[0, 1]$ . Then

$$\left| \int_{\gamma} \omega \right| \leq \sup_{x \in \gamma[0, 1]} \|\omega_x\| \cdot L(\gamma).$$

- **Polygonal approximation of line integrals.**

**Exercise 16.9.** Suppose  $\gamma: [0, 1] \rightarrow \mathbb{R}^n$  is a piece-wise  $C^1$ -path,  $\omega$  is a continuous linear form on  $U \supset \gamma[0, 1]$ . Then, for any  $\epsilon > 0$ , there exists a polygonal line  $\nu: [0, 1] \rightarrow U$ ,  $\nu(0) = \gamma(0)$ ,  $\nu(1) = \gamma(1)$ , such that

$$\max_{t \in [0, 1]} |\nu(t) - \gamma(t)| < \epsilon,$$

and

$$\left| \int_{\gamma} \omega - \int_{\nu} \omega \right| < \epsilon.$$

•

**Claim 16.10.** Suppose  $\omega$  is a differential form in  $U \subset \mathbb{R}^n$  with continuous coefficients. Then TFAE:

- (i) for any closed curve  $\gamma$ ,  $\int_{\gamma} \omega = 0$ ;
- (ii) if the curves  $\gamma$  and  $\mu$  have the same beginning and the same end, then  $\int_{\gamma} \omega = \int_{\mu} \omega$ ;
- (iii) there exists a  $C^1$ -function  $f \in C^1(U)$  such that  $df = \omega$ .

We shall prove only (ii)  $\Rightarrow$  (iii), the rest is obvious. Fix  $p \in U$ , and set

$$f(x) = \int_{\gamma} \omega,$$

where the path  $\gamma$  starts at  $p$  and terminates at  $x$ . The function  $f$  is well-defined by (ii). Then

$$\frac{f(x_1 + \epsilon, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\epsilon} = \frac{1}{\epsilon} \int_{x_1}^{x_1 + \epsilon} a_1(t, x_2, \dots, x_n) dt$$

( $\omega = \sum a_i dx_i$ ). Thus,  $\partial_{x_1} f = a_1$ , and similarly, for all  $i$ ,  $\partial_{x_i} f = a_i$ . Clearly,  $f \in C^1(U)$ .  $\square$

The next statement is deeper than the previous ones:

• **Poincaré Lemma** If  $U \subset \mathbb{R}^n$  is a ball (or the whole  $\mathbb{R}^n$ ), then condition

$$\frac{\partial a_i}{\partial x_k} = \frac{\partial a_k}{\partial x_i}, \quad 1 \leq i, k \leq n \quad (\star)$$

is *equivalent* to any of conditions (i)-(iii) from Claim 16.10.

The forms satisfying condition  $(\star)$  are called *closed*. That is, in balls and in the whole  $\mathbb{R}^n$  ANY closed form is a differential of a function<sup>29</sup>. We will prove later, that the same result holds in arbitrary *simply connected domains* in  $\mathbb{R}^n$ .

*Proof:* it is not difficult to *guess* the primitive function. WLOG, suppose that  $\omega = \sum a_i dx_i$  is an exact form in the unit ball  $\mathbb{B} \subset \mathbb{R}^n$ . Set

$$g(x) \stackrel{\text{def}}{=} \int_0^1 \sum_{i=1}^n x_i a_i(tx) dt = \sum_{i=1}^n x_i \int_0^1 a_i(tx) dt.$$

<sup>29</sup>such forms are called *exact* ones

Then

$$\begin{aligned}
\frac{\partial g}{\partial x_k} &= \int_0^1 a_k(tx) dt + \sum_{i=1}^n x_i \int_0^1 \frac{\partial a_i}{\partial x_k} t dt \\
&\stackrel{(\star)}{=} \int_0^1 a_k(tx) dt + \sum_{i=1}^n x_i \int_0^1 t \frac{\partial a_k}{\partial x_i} dt \\
&= \int_0^1 a_k(tx) dt + \int_0^1 t \frac{d}{dt} a_k(tx) dt \\
&= \int_0^1 a_k(tx) dt + ta_k(tx) \Big|_{t=0}^{t=1} - \int_0^1 a_k(tx) dt = a_k(x).
\end{aligned}$$

□

If you start to feel that you've learnt something very similar in the Complex Analysis course, then you are right. If  $f = u + iv$  is a complex-valued function, then

$$f dz = u dx - v dy + i(v dx + u dy).$$

**Exercise 16.11.** Deduce from the Poincaré lemma the Cauchy theorem: if  $f = u + iv$  is analytic function in a disc  $D$  (that is,  $f$  is a  $C^1$ -function, and its real and imaginary parts  $u$  and  $v$  satisfy the Cauchy-Riemann equations  $u_x = v_y$ ,  $u_y = -v_x$ ), then for any closed contour  $\gamma \subset D$ ,

$$\int_{\gamma} f(z) dz = 0.$$

Hint: find out when the complex-valued differential form  $f dz$  is closed.

Given a closed form in a disc (or in  $\mathbb{R}^n$ ) it is easy to find its differential by integration:

**Example 16.12.** Let  $\omega = y dx + x dy + 4dz$ . Conditions  $(\star)$  are satisfied. We are looking for the primitive function  $f$ .

$$\frac{\partial f}{\partial x} = y \Rightarrow f(x, y, z) = xy + f_1(y, z),$$

$$\frac{\partial f}{\partial y} = x \Rightarrow x + \frac{\partial f_1}{\partial y} = x \Rightarrow f_1 = f_1(z),$$

$$\frac{\partial f}{\partial z} = 4 \Rightarrow f_1(z) = 4z + \text{Const}.$$

Thus the primitive function is

$$f(x, y, z) = xy + 4z + \text{Const}.$$



**Exercise 16.13.** Check which of the following linear differential forms has a primitive function. If the primitive exists, find it.

$$(4x^3y^3 - 2y^2) dx + (3x^4y^2 - 2xy) dy,$$

$$((x + y + 1)e^x - e^y) dx + (e^x - (x + y + 1)e^y) dy.$$

## 16.4 Vector fields and differential forms

There is a simple *duality* between the differential forms and the vector fields: if  $F$  is a vector field in  $U \subset \mathbb{R}^n$ , then the work form  $\omega_F$  is defined as  $\omega_F(\xi) = \langle F(x), \xi \rangle$ ,  $\xi \in T_x\mathbb{R}^n$ . Having the form  $\omega$ , the same formula we recovers the field  $F$ . The integral of  $\omega_F$  over the curve  $\gamma \subset U$  gives us expression for the work done by the field  $F$  for transportation of a particle along  $\gamma$ :

$$\int_{\gamma} \omega_F = \int_I \langle F(\gamma(t)), \dot{\gamma}(t) \rangle dt.$$

The vector field  $F$  is called *potential* field or *gradient* field if there exists a function  $U$  (called *potential*) such that  $F = \nabla U$ . Equivalently, the work form  $\omega_F = dU$ . For example, the Coulomb and gravitational fields are potential ones. The field  $F$  is called *conservative* if the work done by  $F$  in moving a particle along *any* loop  $\gamma \in U$  is zero. Equivalently, the work done by  $F$  in moving a particle from the point  $x$  to the point  $y$  depends only on  $x$  and  $y$ , and does not depend on the path. In virtue of Claim 16.10, the notions of potential and conservative fields coincide.

**Exercise 16.14.** Check that the vector field in  $\mathbb{R}^3$   $F = \begin{pmatrix} e^x \cos y + yz \\ xz - e^x \sin y \\ xy + z \end{pmatrix}$  is conservative, and find its potential.

**Exercise 16.15.** The vector field  $F$  is called *central-symmetric* (with respect to the origin), if the size of the field is a radial function:  $|F(x)| = f(|x|)$ , and the direction of the field coincides with the one of the ‘point-vector’, i.e.

$$F(x) = f(|x|) \frac{x}{|x|}.$$

Prove that any central-symmetric field in  $\mathbb{R}^n$  is a potential one. Find the radial potential  $U(r)$ ,  $r = |x|$ , in terms of  $f$ .

## 16.5 The ‘arc-length form’ $ds$

Now, we have two notions of integrals over the curves: we learnt ‘non-oriented integrals’ of functions (as a special case of surface integrals) and ‘oriented integrals’ of differential forms. How these two notions are related to each other?

Suppose

$\gamma: I \rightarrow \mathbb{R}^n$  smooth path,

$T(x) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$  unit tangent vector to  $\gamma$  at  $x = \gamma(t)$  (Check that it *does not* depend on the choice of the parameterization of  $\gamma$ !)

**Definition 16.16** (arc-length differential form).

$$ds_x(\xi) = \langle T(x), \xi \rangle, \quad \xi \in T_x \mathbb{R}^n.$$

**WARNING:** *in spite of (traditional) notation  $ds$ , this is not a differential!*

If  $\rho$  is a continuous function on the curve  $\gamma$ , then we can define a new form  $\rho ds$  which has a density  $\rho$  with respect to the form  $ds$ ; i.e.

$$(\rho ds)(\xi) = \rho(x) ds_x(\xi), \quad x = \gamma(t).$$

Then

$$\begin{aligned} \text{‘oriented’} \int_{\gamma} \rho ds &= \int_I \rho(\gamma(t)) (ds)_{\gamma(t)}(\dot{\gamma}(t)) dt \\ &= \int_I \rho(\gamma(t)) \langle T(\gamma(t)), \dot{\gamma}(t) \rangle dt \\ &= \int_I \rho(\gamma(t)) |\dot{\gamma}(t)| dt = \text{‘non-oriented’} \int_{\gamma} \rho ds, \end{aligned}$$

that is our definitions coincide!

**Question 16.17.** The definition of the integral of a function clearly does not depend on the orientation of the curve, the definition of the integral of a form does depend on the orientation. How this could happen?

To finish this discussion, observe that ANY line integral can be rewritten as an ‘arc-length integral’: given a form  $\omega_x \in (T_x \mathbb{R}^n)^*$ , take the vector field  $F(x) \in T_x \mathbb{R}^n$  such that  $\omega = \omega_F$ . Then

$$\begin{aligned} \int_{\gamma} \omega &= \int_I \langle F(\gamma(t)), \dot{\gamma}(t) \rangle dt \\ &= \int_I \langle F(\gamma(t)), T(\gamma(t)) \rangle |\dot{\gamma}(t)| dt \\ &= \int_{\gamma} \langle F, T \rangle ds, \end{aligned}$$

that is

$$\int_{\gamma} \omega_F = \int_{\gamma} \langle F, T \rangle ds.$$

## 17 Green's theorem

**Theorem 17.1.** *Suppose  $G \subset \mathbb{R}^2$  is a domain whose boundary  $\Gamma = \partial G$  consists of finitely many piece-wise  $C^1$ -curves, and is positively oriented<sup>30</sup>. Suppose  $Pdx + Qdy$  is a  $C^1$ -differential form on  $\bar{G}$ . Then*

$$\int_{\Gamma} P dx + Q dy = \iint_G \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

As in the Divergence Theorem, the integral on the left-hand side is an additive set-function. It's easy to compute its 'density':

**Exercise 17.2.** Let  $S$  be a square centered at the point  $(\xi, \eta)$ , and  $\Gamma$  be its boundary with the positive orientation. Check that

$$\lim_{s \downarrow (\xi, \eta)} \frac{1}{\text{Area}(S)} \int_{\Gamma} P dx + Q dy = \frac{\partial Q}{\partial x}(\xi, \eta) - \frac{\partial P}{\partial y}(\xi, \eta).$$

Then using the properties of the line integrals we know already, it is not difficult to complete the proof Green's Theorem, approximating 'from inside' the domain  $G$  by the connected union of the squares.

However, we shall not do this. We show that Green's Theorem is a simply corollary to the Divergence Theorem. To this end, consider the unit normal field to a two-dimensional curve  $\gamma: I \rightarrow \mathbb{R}^2$ :

$$N(\gamma(t)) = \frac{1}{|\dot{\gamma}|} \begin{pmatrix} \dot{\gamma}_2 \\ -\dot{\gamma}_1 \end{pmatrix}$$

(This definition does not depend on parameterization of  $\gamma$ . Check!)

With this definition, the normal  $N$  lies to the right to the tangent  $T$ . Thus if  $\Gamma = \gamma(I)$  is the oriented boundary of  $G$ , then  $N$  is the outer normal to  $\Gamma$ .

Now, let  $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$  be a smooth vector field on  $\bar{G}$ . Consider the linear form

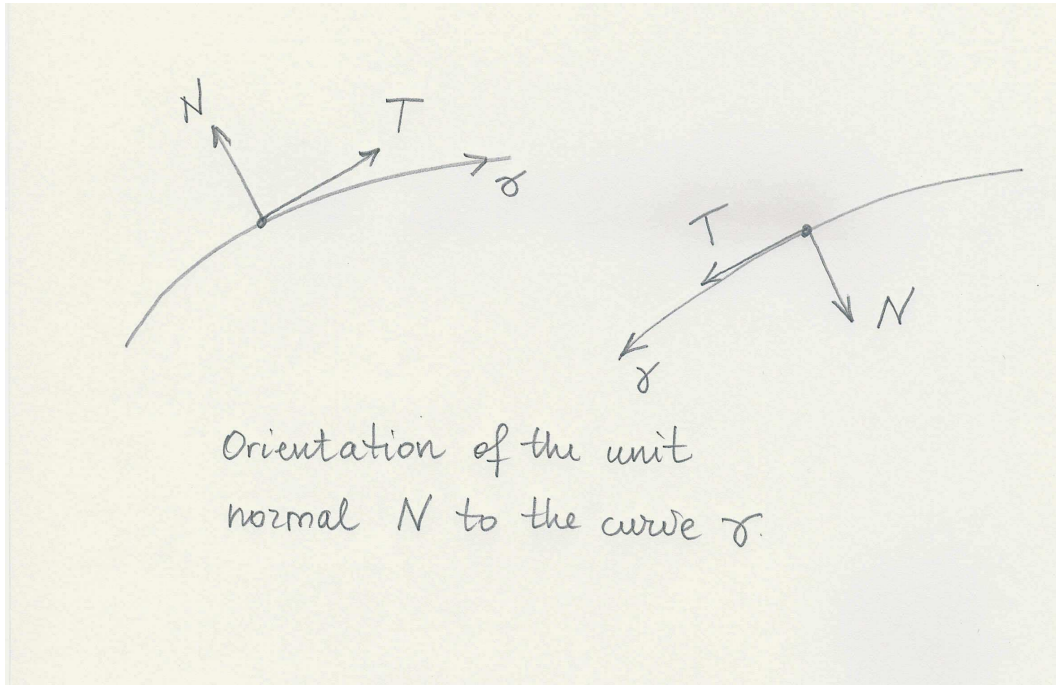
$$\omega = -F_2 dx + F_1 dy$$

(WARNING: this is NOT 'the work form' we defined earlier!) Then

$$\begin{aligned} \int_{\gamma} \omega &= \int_I [-F_2(\gamma(t))\dot{\gamma}_1(t) + F_1(\gamma(t))\dot{\gamma}_2(t)] dt \\ &= \int_I \langle F(\gamma(t)), N(\gamma(t)) \rangle |\dot{\gamma}(t)| dt = \int_{\gamma} \langle F, N \rangle ds. \end{aligned}$$

---

<sup>30</sup>This means that if one traverses the boundary in the positive direction, then his/her left foot is within the domain ('The Law of the Left Foot').



We can go in the opposite direction: if  $\omega = P dx + Q dy$  is a differential form in  $\mathbb{R}^2$ , then  $F = \begin{pmatrix} Q \\ -P \end{pmatrix}$  is the corresponding vector field. Thus

$$\int_{\Gamma} P dx + Q dy = \int_{\Gamma} \langle F, N \rangle ds$$

$$\stackrel{\text{Gauss}}{=} \iint_G \operatorname{div}(F) dx dy = \iint_G \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

We see that Green's Theorem is *equivalent* to the two-dimensional version of the Divergence Theorem<sup>31</sup>.

**Exercise 17.3.**  $C$  is the unit circle with positive orientation. Find

$$\int_C e^{x^2-y^2} (\cos 2xy dx + \sin 2xy dy)$$

**Exercise 17.4.**  $\Gamma$  is the boundary of the square  $[0, \pi] \times [0, \pi]$  with natural orientation. Find

$$\int_{\Gamma} \left( \cos x \cos y + 3x^2 \right) dx + \left( \sin x \sin y + (y^4 + 1)^{1/4} \right) dy.$$

<sup>31</sup>In fact, its proof for domains with piece-wise smooth boundaries is essentially simpler than the proof of the Divergence Theorem we gave.

**Exercise 17.5.** Suppose  $\gamma: [0, 1] \rightarrow \mathbb{R}^2$  be a closed non-constant regular  $C^1$  curve such that

$$(17.6) \quad \int_{\gamma} x^3 dy - y^3 dx = 0.$$

Show that the set  $\Gamma = \gamma([0, 1])$  cannot be a boundary of a domain in  $\mathbb{R}^2$ .

Give an example of a closed regular non-constant curve  $\gamma$  satisfying (17.6).

## 17.1 Application: Area computation

Suppose  $\omega$  is such a form that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

Then, by Green's Theorem,

$$\text{area}(G) = \int_{\partial G} \omega.$$

The most popular examples of such forms are

$$-y dx, \quad x dy, \quad \frac{1}{2}(-y dx + x dy).$$

**Example 17.7.** Consider a closed polygonal line  $\Gamma = [z_0, z_1, \dots, z_N = z_0]$ ,  $z_i = (x_i, y_i)$ . Suppose  $\Gamma = \partial G$ . Then

$$\text{area}(G) = \frac{1}{2} \sum_{j=0}^{N-1} (y_{j+1} - y_j)(x_{j+1} + x_j).$$

*Proof:* We have

$$\text{area}(G) = \sum_{j=0}^{N-1} \int_{\gamma_j} x dy,$$

where  $\gamma_j(t) = z_j(1-t) + z_{j+1}t$ ,  $0 \leq t \leq 1$ . Then  $\dot{\gamma}_j = z_{j+1} - z_j$ , and

$$\int_{\gamma_j} x dy = \int_0^1 (x_j(1-t) + x_{j+1}t)(y_{j+1} - y_j) dt = \frac{1}{2}(y_{j+1} - y_j)(x_{j+1} + x_j).$$

□

**Exercise 17.8.** Find the area of the domain bounded by the loop of the Cartesian leaf  $x^3 + y^3 = 3xy$ .

Hint: set  $y = tx$ , then the leaf is parameterized as follows:

$$x(t) = \frac{3t}{1+t^3}, \quad y(t) = \frac{3t^2}{1+t^3}, \quad 0 < t < \infty.$$

To compute the area, use the area form  $\omega = \frac{1}{2}(x dy - y dx) = \frac{1}{2}x^2 (y/x)' dt$ . In our case,  $y/x = t$ , and the form equals  $\frac{1}{2}x^2 dt$ .

**Exercise 17.9.** Find the length of the astroid  $\{(x, y): x^{2/3} + y^{2/3} = 1\}$ , and the area of the domain it bounds.

Answers: the length of the astroid equals 6, its area is  $3\pi/8$ .

**Exercise 17.10.** Suppose  $(r, \theta)$  are the polar coordinates in  $\mathbb{R}^2$ . Prove:

$$\text{area}(G) = \frac{1}{2} \int_{\Gamma} r^2 d\theta = - \int_{\Gamma} r\theta dr.$$

Here,  $\Gamma$  is the oriented boundary of  $\partial G$ . In the second formula we assume that  $0 \notin \Gamma$ .

## 17.2 Application: Cauchy integral theorem for smooth functions

Let  $f$  be a complex-valued  $C^1$ -function in a plane domain  $G$ . Set

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

The function  $f$  is analytic in  $G$  if and only if  $\partial f / \partial \bar{z} \equiv 0$  in  $G$ . Indeed, let  $f = u + iv$ , then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} (u_x + iv_x + iu_y - v_y).$$

We see that the equation  $f_{\bar{z}} = 0$  is equivalent to the Cauchy-Riemann system:

$$u_x = v_y, \quad u_y = -v_x.$$

**Exercise 17.11.** 1. If  $f$  is a  $C^2$ -function, then

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4} \Delta f.$$

2. If  $u$  is a real-valued  $C^2$ -function, then the complex derivative  $u_z$  is analytic if and only if  $u$  is harmonic.

**Exercise 17.12.** If  $f$  is a  $C^1$ -function, and  $g$  is analytic, then

$$(f \cdot g)_{\bar{z}} = g \cdot f_{\bar{z}}.$$

**Theorem 17.13** (Cauchy - Green). *Suppose  $G \subset \mathbb{C}$  is a bounded domain with a piece-wise  $C^1$ -boundary  $\Gamma$ , and suppose that  $f: G \rightarrow \mathbb{C}$  is a  $C^1$ -function in  $\bar{G}$ . Then*

$$f(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \iint_G \frac{\partial f / \partial \bar{z}}{z-w} dx dy, \quad w \in G.$$

**Corollary 17.14** (Cauchy's theorem). *If, in the assumptions of the theorem above,  $f$  is analytic in  $G$ , then*

$$f(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-w} dz, \quad w \in G.$$

**Corollary 17.15.** *If a complex-valued function  $f \in C^1(\mathbb{C})$  vanishes outside of a compact subset of  $\mathbb{C}$ , then*

$$f(w) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{z}}{z-w} dx dy.$$

*Proof of the Theorem:* First, observe that for the complex-valued function  $f$ , the Green's formula reads

$$(17.16) \quad \int_{\Gamma} f dz = 2i \iint_G \frac{\partial f}{\partial \bar{z}} dx dy$$

Indeed,

$$f dz = u dx - v dy + i(u dy + v dx),$$

$$2i \frac{\partial f}{\partial \bar{z}} = i(u_x + iv_x + iu_y - v_y) = -(u_y + v_x) + i(u_x - v_y),$$

and (17.16) follows from Green's theorem.

We apply (17.16) to the function  $z \mapsto f(z)/(z-w)$  in the domain  $G_{\epsilon} = \{z \in G: |z-w| > \epsilon\}$ ,  $\epsilon < \text{dist}(w, \partial G)$ . Since the function  $z \mapsto \frac{1}{z-w}$  is analytic in  $G_{\epsilon}$ ,

$$\frac{\partial}{\partial \bar{z}} \frac{f(z)}{z-w} = \frac{\partial f / \partial \bar{z}}{z-w}$$

(see the exercise above), and we get

$$2i \iint_{G_{\epsilon}} \frac{\partial f / \partial \bar{z}}{z-w} dx dy = \int_{\Gamma} \frac{f(z)}{z-w} dz - \int_0^{2\pi} f(w + \epsilon e^{i\theta}) i d\theta.$$

Letting  $\epsilon \rightarrow 0$  and using continuity of  $f$  at  $w$ , and integrability of  $\frac{\partial f / \partial \bar{z}}{z-w}$ , we get the result.  $\square$



## 18 Poincaré Lemma

### 18.1 Homotopies. Simply connected domains

**Definition 18.1** (homotopy). The curves  $\gamma_0, \gamma_1: [0, 1] \rightarrow \mathbb{R}^n$  with common starting and terminating points  $\gamma_0(0) = \gamma_1(0)$ ,  $\gamma_0(1) = \gamma_1(1)$  are *homotopic* to each other, if there exists a continuous map  $\gamma: [0, 1] \times [0, 1] \rightarrow \mathbb{R}^n$  such that

$$\gamma(\cdot, 0) = \gamma_0, \quad \gamma(\cdot, 1) = \gamma_1,$$

and

$$\gamma(0, \cdot) = \gamma_0(0) = \gamma_1(0), \quad \gamma(1, \cdot) = \gamma_0(1) = \gamma_1(1).$$

The mapping  $\gamma$  is called the *homotopy* of the curves  $\gamma_0$  and  $\gamma_1$ .

The notion of homotopy formalizes the intuitive idea of *continuous deformations* of curves that keep fixed the beginning and the end. Clearly, any two curves with common beginning and end are homotopic in  $\mathbb{R}^n$  (Prove this!). In particular, any closed curve in  $\mathbb{R}^n$  is homotopic to the point, i.e., the trivial ‘constant curve’.

Given domain  $\Omega \subset \mathbb{R}^n$  and two curves in  $\Omega$ , we can consider only those homotopies  $\gamma$  that do not leave  $\Omega$ . I.e., two curves in  $\Omega$  with common beginning and end, are *homotopic in  $\Omega$*  if it is possible to deform continuously one curve into another without leaving domain  $\Omega$ . Now, the property to be ‘homotopic in  $\Omega$ ’ depends both on the pair of curves and  $\Omega$ . For example, the unit circle is homotopic to the point in  $\mathbb{R}^2$ , but in  $\mathbb{R}^2 \setminus \{0\}$  such a homotopy does not exist. The circle  $(x - 2)^2 + y^2 = 1$  is homotopic to the point in  $\mathbb{R}^2 \setminus \{0\}$ .

**Theorem 18.2** (integrals over homotopic curves are equal). *Let  $\omega = \sum a_i(x) dx_i$  be a  $C^1$ -differential form in the domain  $\Omega \subset \mathbb{R}^n$  such that*

$$(18.3) \quad \frac{\partial a_i}{\partial x_k} = \frac{\partial a_k}{\partial x_i}, \quad 1 \leq i, k \leq n,$$

*and let the piece-wise  $C^1$ -curves  $\gamma_0, \gamma_1: [0, 1] \rightarrow \Omega$  be homotopic in  $\Omega$ . Then*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

**Definition 18.4.** The domain  $\Omega \subset \mathbb{R}^n$  is called *simply-connected*, if any closed curve  $\gamma_0: [0, 1] \rightarrow \Omega$ ,  $\gamma_0(0) = \gamma_0(1) = c$ , is homotopic in  $\Omega$  to the point.

For example, any ball in  $\mathbb{R}^n$  is simply connected, the half-space in  $\mathbb{R}^n$  is also simply connected. More generally, any convex domain in  $\mathbb{R}^n$  is simply connected. The punctured ball is not simply connected in  $\mathbb{R}^2$ , but is simply connected in  $\mathbb{R}^n$  for  $n \geq 3$ . Of course, if  $\Omega$  is simply-connected, and  $\Omega_1$  is its homeomorphic image (that is,  $\exists f: \Omega \rightarrow \Omega_1$ ,  $f$  is one-to-one,  $f$  and  $f^{-1}$  are continuous), then  $\Omega_1$  is also simply-connected. (Prove!)

**Corollary 18.5.** *Let  $\omega$  be a  $C^1$  form in a simply-connected domain  $\Omega$  satisfying (18.3). Then  $\omega$  has a primitive.*

## 18.2 Proof for smooth homotopies

First, we assume that the curves  $\gamma_0$  and  $\gamma_1$  are  $C^1$ , and that the homotopy  $\gamma: [0, 1]^2 \rightarrow \Omega$  is a  $C^1$  mapping, such that the mixed derivative  $\frac{\partial^2}{\partial t \partial s}$  is continuous and does not depend on the order of differentiation. Then

$$\left( \int_{\gamma_1} - \int_{\gamma_0} \right) \omega = \int_0^1 \left( \sum_{i=1}^n a_i(\gamma_1(t)) \dot{\gamma}_{1,i}(t) - \sum_{i=1}^n a_i(\gamma_0(t)) \dot{\gamma}_{0,i}(t) \right) dt.$$

The integrand on the RHS equals

$$\sum_{i=1}^n a_i(\gamma(t, s)) \dot{\gamma}_i(t, s) \Big|_{s=0}^{s=1} = \int_0^1 \frac{d}{ds} \left( \sum_{i=1}^n a_i(\gamma(t, s)) \dot{\gamma}_i(t, s) \right) ds$$

(here and henceforth,  $\gamma_i$  is the  $i$ -th coordinate component of the mapping  $\gamma$ ).  
Next,

$$\begin{aligned} \frac{d}{ds} \left( \sum_{i=1}^n a_i(\gamma(t, s)) \dot{\gamma}_i(t, s) \right) &= \sum_{i,k=1}^n \left( \frac{\partial a_i}{\partial x_k} \circ \gamma \right) \frac{\partial \gamma_i}{\partial t} \frac{\partial \gamma_k}{\partial s} + \sum_{i=1}^n (a_i \circ \gamma) \frac{\partial^2 \gamma_i}{\partial t \partial s} \\ &\stackrel{(18.3)}{=} \frac{d}{dt} \left( \sum_{k=1}^n a_k(\gamma(t, s)) \frac{\partial \gamma_k}{\partial s}(t, s) \right). \end{aligned}$$

Thereby,

$$\int_{\gamma_1} \omega - \int_{\gamma_0} \omega = \int_0^1 dt \int_0^1 ds \frac{d}{dt} \sum_{k=1}^n (a_k \circ \gamma) \frac{\partial \gamma_k}{\partial s}.$$

The integrand is uniformly continuous and hence we can switch the order of integration. We obtain

$$(18.6) \quad \int_0^1 ds \sum_{k=1}^n \left( a_k(\gamma(1, s)) \frac{\partial \gamma_k}{\partial s}(1, s) - a_k(\gamma(0, s)) \frac{\partial \gamma_k}{\partial s}(0, s) \right).$$

Since the end-points of the homotopy are fixed, the functions  $s \mapsto \frac{\partial \gamma_k}{\partial s}(0, s)$ ,  $s \mapsto \frac{\partial \gamma_k}{\partial s}(1, s)$  are constant, and

$$\frac{\partial \gamma_k}{\partial s}(0, s) = \frac{\partial \gamma_k}{\partial s}(1, s) = 0,$$

thus the RHS of (18.6) is zero.

It remains to explain how to smooth homotopies.

### 18.3 Smoothing

Suppose that  $\gamma_0$  and  $\gamma_1$  are homotopic  $C^1$ -curves in  $\Omega$ , and that  $\gamma: [0, 1]^2 \rightarrow \Omega$  is their homotopy. We'll smooth it using the idea similar to that one we used in the construction of the smooth cut-off function.

We extend by the mapping  $\gamma$  to a continuous mapping from a larger open square  $Q \supset [0, 1]^2$  to  $\Omega$ , and set

$$\gamma_\epsilon(t, s) \stackrel{\text{def}}{=} \frac{1}{4\epsilon^2} \int_{t-\epsilon}^{t+\epsilon} \int_{s-\epsilon}^{s+\epsilon} \gamma(\xi, \eta) d\xi d\eta$$

( $\epsilon$  is sufficiently small). Clearly,  $\gamma_\epsilon \in C^1$ , and the mixed derivative  $\frac{\partial^2 \gamma_\epsilon}{\partial t \partial s}$  is continuous and does not depend on the order of differentiation.

Since

$$\gamma_\epsilon(t, s) - \gamma(t, s) = \frac{1}{4\epsilon^2} \int_{t-\epsilon}^{t+\epsilon} \int_{s-\epsilon}^{s+\epsilon} [\gamma(\xi, \eta) - \gamma(t, s)] d\xi d\eta,$$

we see that  $\gamma_\epsilon$  approximate  $\gamma$  as  $\epsilon \rightarrow 0$  *uniformly* in  $t$  and  $s$ . In particular, we can choose  $\epsilon$  so small that  $\gamma_\epsilon \in \Omega$ .

Now, we change a bit the smooth curve  $\gamma_\epsilon$  to recover the 'boundary conditions' as  $t = 0$ ,  $t = 1$ ,  $s = 0$  and  $s = 1$ . First, we replace  $\gamma_\epsilon(t, s)$  by

$$\gamma_\epsilon(t, s) - (1-s) [\gamma_\epsilon(t, 0) - \gamma_0(t)] - s [\gamma_\epsilon(t, 1) - \gamma_1(t)].$$

We keep for the 'perturbed function' the same the notation  $\gamma_\epsilon$ . New function is still uniformly close to the original function  $\gamma$ . Now it 'connects' the curves  $\gamma - 0$  and  $\gamma_1$ :

$$\gamma_\epsilon(t, 0) = \gamma_0(t), \quad \gamma_\epsilon(t, 1) = \gamma_1(t).$$

However, it still does not stay constant when  $t = 0$  and  $t = 1$ . To mend this, we replace  $\gamma_\epsilon$  by

$$\gamma_\epsilon(t, s) - (1-t) [\gamma_\epsilon(0, s) - \gamma_0(0)] - t [\gamma_\epsilon(1, s) - \gamma_0(1)].$$

The new function  $\gamma_\epsilon$  already satisfies

$$\gamma_\epsilon(0, s) = \gamma_0(0) = \gamma_1(0), \quad \gamma_\epsilon(1, s) = \gamma_0(1) = \gamma_1(1).$$

(Note that the conditions at  $s = 0$  and  $s = 1$  have not been changed!)

This completes the proof of the theorem in the case when the curves  $\gamma_0$  and  $\gamma_1$  are  $C^1$ . If they are only *piece-wise*  $C^1$ , then we approximate them uniformly by  $C^1$ -curves  $\gamma_{0,\epsilon}$ ,  $\gamma_{1,\epsilon}$  with the same end-points. Clearly, the curves  $\gamma_{0,\epsilon}$  and  $\gamma_{1,\epsilon}$  are still homotopic. (Why?) Thus

$$\int_{\gamma_{0,\epsilon}} \omega = \int_{\gamma_{1,\epsilon}} \omega.$$

Letting  $\epsilon \rightarrow 0$ , we get the result. □