Volumes in High Dimension – Exercises

(0) Let $X = (X_1, \ldots, X_n)$ be a random vector, distributed uniformly in $Q^n = [-1/2, 1/2]^n$. Let $\theta = (\theta_1, \ldots, \theta) \in \mathbb{R}$ with $|\theta| = 1$. Denote by f_θ the density of the random variable $\sum_i \theta_i X_i$. Prove that for almost any $t \in \mathbb{R}$,

$$f_\theta(t) = Vol_{n-1}\left(Q^n \cap H_{\theta,t}\right)$$

where $H_{\theta,t} = \{x \in \mathbb{R}^n \,;\, x \cdot \theta = t\}$.

February 19, 2014: The high-dimensional cube

(1) Let X, Y be independent random vectors in $\mathbb{R}^n$, distributed uniformly in $Q^n = [-1/2, 1/2]^n$. Show that

$$\left(\mathbb{E}|X - Y|^4\right)^{1/4} = \alpha_n \sqrt{n}$$

where α_n tends to a finite, positive limit as $n \to \infty$. What is $\lim_{n\to\infty} \alpha_n$?

(2) Let $n \geq 100$. Let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}$ be a unit vector such that

$$\forall i, \qquad |\theta_i| \leq \frac{5}{\sqrt{n}}.$$

Let X be a random vector in $\mathbb{R}^n$, distributed uniformly in $[-\sqrt{3}, \sqrt{3}]^n$. Denote by $f_\theta(t)$ the continuous density of $\langle X, \theta \rangle$. Prove that

$$\left| f_\theta(t) \; - \; \frac{\exp(-t^2/2)}{\sqrt{2\pi}} \right| \leq \frac{C}{n} \qquad\qquad (t \in \mathbb{R})$$

for a universal constant $C > 0$. *(In class we did the case $\theta_i = 1/\sqrt{n}$, you need to explain how to modify the proof.)*

(3) Let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ be a unit vector, with $|\theta_j| \leq 1/10$ for all j. Consider the function

$$g_\theta(s) = \prod_{j=1}^{n} \varphi\left(s\theta_j\right)$$

where $\varphi(t) = \sin(\sqrt{12}\pi t)/(\sqrt{12}\pi t)$. Denote $\varepsilon = \sum_j \theta_j^4$. Prove that for any $|s| \leq \frac{1}{10\varepsilon^{1/4}}$,

$$\left| g_\theta(s) - e^{-2\pi^2 s^2} \right| \leq C\varepsilon s^4 e^{-2\pi^2 s^2}$$

where $C > 0$ is a universal constant. *(This exercise is a key step in the proof of the Central Limit Theorem for general θ_i. I hope that it helps understand where the term $\sum_i \theta_i^4$ comes from. For the full proof, you may consult Feller's book "Introduction to Probability, Vol. II")*

February 26, 2014: The high-dimensional sphere/ball

(4) For $\theta \in S^{n-1}, t \in \mathbb{R}$ we set $H_{\theta,t} = \{x \in \mathbb{R}^n \,;\, x \cdot \theta = t\}$. Set,

$$f_\theta(t) = \frac{\mathrm{Vol}_{n-1}\left\{\sqrt{n}B_2^n \cap H_{\theta,t}\right\}}{\mathrm{Vol}_n\left\{\sqrt{n}B_2^n\right\}} = \frac{\mathrm{Vol}_{n-1}\left(B_2^{n-1}\right)}{\mathrm{Vol}_n\left(\sqrt{n}B_2^n\right)} \cdot \left(n - t^2\right)^{(n-1)/2}.$$

Prove that

$$f_\theta(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}} + O\left(\frac{1}{n}\right).$$

(5) Let X and Y be independent random vectors supported in the unit sphere. Assume that Y is distributed uniformly in S^{n-1}. Then the random variables

$$X \cdot Y \quad \text{and} \quad Y_1$$

have exactly the same distribution.

(6) Let $X = (X_1, \ldots, X_n)$ be a random vector uniformly distributed in S^{n-1}. Then $(X_1, \ldots, X_{n-2})$ is uniformly distributed in B^{n-2}. Hint: The density of $(X_1, \ldots, X_{n-1})$ is proportional to $\dfrac{1}{\sqrt{1-|x|^2}}$ in B^{n-1}.

(7) For a Borel subset $A \subset S^{n-1}$ and $\varepsilon > 0$ we denote

$$A_\varepsilon = \{x \in S^{n-1} \,;\, \exists y \in A,\, |x - y| < \varepsilon\}.$$

Correct and fill in the details of the argument from the class, and prove that the set

$$\left\{\sigma(A_\varepsilon) \,;\, A \subseteq S^{n-1} \text{ is Borel and } \sigma(A) \geq 1/2\right\}$$

has a minimum.

March 5, 2014: Isoperimetry and concentration

(8) Fix $0 < t < 1$ and $\varepsilon > 0$. Among all Borel sets $A \subset S^{n-1}$ with $\sigma_{n-1}(A) = t$, can you guess a set for which

$$\sigma_{n-1}(A_\varepsilon)$$

is minimal? Prove your guess (in class we did the case $t = 1/2$).

(9) Suppose that X is a random vector in $\mathbb{R}^n$ with $\mathbb{E}|X|^2 < \infty$. Assume that X is not supported by a hyperplane. Prove that there exist a vector $b \in \mathbb{R}^n$ and a positive-definite matrix A such that $A(X) + b$ is isotropic.

(10)+ Let X be an isotropic random vector in $\mathbb{R}^n$ and let $0 < \varepsilon < 1/2$. Assume that there exists $R > 0$ with

$$\mathbb{E}\left(\frac{|X|}{R} - 1\right)^2 \leq \varepsilon^2.$$

Prove that

$$\mathbb{E}\left(\frac{|X|}{\sqrt{n}} - 1\right)^2 \leq C\varepsilon^2$$

for some universal constant $C > 0$.

March 12, 2014: The thin-shell theorem

(11) Let $Y = (Y_1, \ldots, Y_n)$ be a random vector, uniformly distributed on S^{n-1}. Set

$$\Phi_n(t) = \mathbb{P}\left(\sqrt{n}Y_1 \leq t\right).$$

Denoting $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2} ds$, prove that

$$\int_{-\infty}^{\infty} |\Phi(t) - \Phi_n(t)| dt \leq \frac{C}{n}$$

where $C > 0$ is a universal constant.

(12) Let $\mathcal{M}([0,1])$ be the class of all Borel probability measures on $[0,1]$. For $\mu, \nu \in \mathcal{M}([0,1])$ we set

$$d_W(\mu, \nu) = \sup_f \left[\int f d\mu - \int f d\nu\right]$$

where the supremum runs over all 1-Lipschitz functions $f : [0,1] \to \mathbb{R}$. Prove that d_W is a metric on $\mathcal{M}([0,1])$, which induces the weak*-topology on $\mathcal{M}([0,1])$.

March 19, 2014: Brunn-Minkowski and Prekopa-Leindler

(13) The modulus of convexity of a norm $\|\cdot\|_K$ is defined, for $0 < \varepsilon < 1$, as

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|_K \; ; \; \|x\|_K \leq 1, \; \|y\|_K \leq 1, \; \|x - y\|_K \geq \varepsilon\right\}.$$

Prove that when $K \subset \mathbb{R}^n$ is a ball or an ellipsoid centered at the origin,

$$\delta(\varepsilon) \geq \frac{\varepsilon^2}{8}.$$

(14) For $K = B(\ell_p^n)$ with $p \geq 2$, show that

$$\delta(\varepsilon) \geq c_p \varepsilon^p$$

where $c_p > 0$ depends only on p. [Hint: $\tilde{c}_p|a - b|^p + |(a + b)/2|^p \leq (|a|^p + |b|^p)/2$].

(15) In class we proved a concentration inequality for the uniform distribution on K with respect to the metric $d(x, y) = \|x - y\|_K$. State and prove an analogous statement for the cone measure on ∂K.

March 26, 2014: The Santalo inequality and the Legendre transform

(16) Prove that $f^{**} = f$ for any lower-semi continuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ which is not identically $+\infty$.

(17) For a function $u : \mathbb{R}^n \to \mathbb{R}$ denote

$$\mathcal{I}(u) = -\log \int_{\mathbb{R}^n} e^{-u^*}$$

Prove that $\mathcal{I}$ is a well-defined, finite convex function, i.e.,

$$\mathcal{I}\left(\lambda u_1 + (1-\lambda) u_2\right) \leq \lambda \mathcal{I}(u_1) + (1-\lambda)\mathcal{I}(u_2)$$

for any $0 < \lambda < 1$ and $u_1, u_2 : \mathbb{R}^n \to \mathbb{R}$.

(18) Let X be an n-dimensional linear space, X^* is the dual space and $f : X \times X^* \to \mathbb{R}$. Explain why the "integrability of f" is a well-defined concept, as well as the value of the integral

$$\int_{X \times X^*} f.$$

Similarly, for a compactly-supported, integrable function $f : X \to [0, \infty)$ with a positive integral, prove that the barycenter

$$bar(f) = \frac{1}{\int_X f} \int_X x f(x) \in X$$

is well-defined.

April 2, 2014: Log-concavity, reverse Hölder inequalities, Brascamp-Lieb

(19) Let $f : (0, \infty) \to [0, \infty)$ be an integrable, log-concave function. For $p > -1$ denote

$$M_f(p) = \frac{\int_0^\infty t^p f(t) dt}{\Gamma(p+1)}$$

and set $M_f(-1) = \lim_{t \to 0^+} f(t)$. Prove that M_f is log-concave on $[-1, \infty)$ (in class we proved log-concavity in $[0, \infty)$).

(20) Under the same assumptions of the previous exercise, show that

$$K_f(p) = \int_0^\infty \left(\frac{t}{p}\right)^p f(t) dt$$

is log-concave in $(0, \infty)$.

(21) Let μ be a probability measure on $\mathbb{R}^n$ that is absolutely-continuous (i.e., it has a density). Prove that μ has a log-concave density if and only if for any Borel sets $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

April 23, 2014: Poincaré inequalities, thin shell for unconditional convex sets

(22) Suppose that μ is a probability measure on $\mathbb{R}^n$. Assume that μ satisfies a Poincaré inequality with constant one, i.e., for any C^1-function $f \in L^2(\mu)$,

$$Var_\mu(f) \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

(i) Without using the Poincaré inequality, prove that for any $f \in L^p(\mu)$ and $p \geq 1$,

$$\|f - E_f\|_{L^p(\mu)} \leq C_1 \|f - M_f\|_{L^p(\mu)} \leq C_2 \|f - E_f\|_{L^p(\mu)}$$

where $C_1, C_2 > 0$ are universal constants, $E_f = \int f d\mu$ and M_f is a median of f.

(ii) Let $p \geq 1$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a 1-Lipschitz function with $M(f) = 0$. Use the Poincaré inequality for the function $sgn(f) f^p$ and conclude that

$$\|f\|_p \leq Cp,$$

where $C > 0$ is a universal constant.

(iii) Use the Markov-Chebyshev inequality, and prove that for any 1-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mu(\{x \in \mathbb{R}^n ; |f(x) - E_f| \geq t\}) \leq Ce^{-ct} \qquad (t > 0),$$

where $c, C > 0$ are universal constants.

(23) Let $K \subset \mathbb{R}^n_+$ be monotone and p-convex. Prove that K is also q-convex for any $q < p$.

(24) Let $X \in \mathbb{R}^n_+$ be a half-log-concave random vector. Prove that $(\sqrt{X_1}, \ldots, \sqrt{X_n})$ is log-concave.

April 30, 2014: Entropy, Covariance and the isotropic constant

(25) Among all random vector X in $\mathbb{R}^n$ with a fixed covariance matrix, prove that the Gaussian maximizes the entropy. [Recall the hint from class]

(26) Let X be a log-concave random vector in $\mathbb{R}^n$, with density f. Prove that

$$\left(\int_{\mathbb{R}^n} f^2 \right)^{1/n} \sim \int_{\mathbb{R}^n} f^{1+1/n} \sim \exp(-Ent(X)/n),$$

where $A \sim B$ means that $c_1 A \leq B \leq c_2 A$ for universal constants $c_1, c_2 > 0$. [Hint: Use the body $K(f)$ introduced in class]

(27) Let $K \subset \mathbb{R}^n$ be a convex set, $Vol(K) = 1$. Prove that there exists a hyperplane $H \subset \mathbb{R}^n$ with

$$Vol_{n-1}(K \cap H) \geq c/L_K$$

for a universal constant $c > 0$.

May 7, 2014: Volume Ratio, Kashin's splitting

(28) Let $k \leq n$ and let $X_1, \ldots, X_k$ be independent, identically-distributed random vectors, uniformly distributed on S^{n-1}.

(i) Prove that $\mathbb{P}(\dim sp\{X_1, \ldots, X_k\} = k) = 1$.

(ii) Denote $E = sp\{X_1, \ldots, X_k\}$, a random k-dimensional subspace in $\mathbb{R}^n$. Prove that for any fixed $U \in O(n)$, the random subspace E is equal in distribution to $U(E)$.

(iii) Consider the unit sphere $S_E = S^{n-1} \cap E$, and let Y be a random vector, uniformly distributed on the $k-1$-dimensional sphere S_E. Prove that Y is distributed uniformly over S^{n-1} [Hint: Use the uniqueness of the $O(n)$-invariant probability measure on S^{n-1}].

(29) Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body, $\|x\|_K = \inf\{\lambda > 0\,;\, x \in \lambda K\}$.

(i) Prove that $v.rad.(K) = \left(\int_{S^{n-1}} \|x\|_K^{-n} d\sigma_n(x)\right)^{1/n}$.

(ii) Denote $M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_n(x)$. Prove that $v.rad.(K) \cdot M(K) \geq 1$.

(iii) Assume that n is even, that $v.rad.(K) = 1$ and let $E \in G_{n,n/2}$ be a random subspace, uniformly distributed in $G_{n,n/2}$. Prove that with probability at least $1/2$,

$$Diam(K \cap E) \leq CM(K).$$

May 14, 2014: Log-Laplace Transform

(30) Let X be a log-concave random vector in $\mathbb{R}^n$ and set $\Lambda_X(y) = \log \mathbb{E} \exp(X \cdot y)$. Explain the notation and justify the differentiation under the integral sign: At any point $y \in \mathbb{R}^n$ with $\Lambda_X(y) < \infty$,

$$\left(\partial^\alpha e^{\Lambda_X}\right)(y) = \mathbb{E} X^\alpha e^{X \cdot y}$$

for any multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$.

May 21, 2014: Bourgain-Milman, reverse Brunn-Minkowski, Milman's ellipsoid

(31) Regarding the proof of the existence of M-ellipsoid we saw in class, show that the Milman ellipsoid $\mathcal{E}$ that we constructed satisfies the following property: For any $1 \leq \ell < n, \lambda = \ell/n$ and any subspace $F \in G_{n,\ell}$,

$$v.rad(K \cap F) \geq c_\lambda v.rad.(\mathcal{E} \cap F).$$

Hint: Use the fact that $\left(\int_{F+x_0} f\right)^{\frac{1}{codim(F)}} \geq \frac{f(x_0)^{1/n}}{L_X}$.

(32) Improve the bound obtained in class, and establish the Rogers-Shepherd inequality with best constant: For any centrally-symmetric, convex body $K \subset \mathbb{R}^n$ and a subspace $E \subset \mathbb{R}^n$,

$$|K \cap E| \cdot |Proj_{E^\perp}(K)| \leq \binom{n}{\ell} |K|$$

where $\ell = \dim(E)$.

(33+) Prove the Spingran's inequality: For any convex body $K \subset \mathbb{R}^n$ whose barycenter lies at the origin and any subspace

$$|K \cap E| \cdot |Proj_{E^\perp}(K)| \geq |K|$$

[Hint: Use Brunn-Minkowski, the case where $K = -K$ is easier.]

May 28, 2014: Quotient of Subspace,

(34) Suppose that $K \subset \mathbb{R}^n$ is a centrally-symmetric convex body and that $\mathcal{E} \subseteq \mathbb{R}^n$ is a centrally-symmetric ellipsoid with $|\mathcal{E}| = |K|$ and

$$|\mathcal{E} + K|^{1/n} \leq \alpha |K|^{1/n}.$$

Prove that $\mathcal{E}$ is a Milman ellipsoid of K with constant $c(\alpha)$ (i.e., that $|K \cap \mathcal{E}|^{1/n} \geq c(\alpha)|K|^{1/n}$).

(35) Same, but now instead of $|\mathcal{E} + K|^{1/n} \leq \alpha |K|^{1/n}$, assume that $N(K, \mathcal{E}) \leq \alpha^n$.