

Centroid Bodies and the Logarithmic Laplace Transform - A Unified Approach

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Abstract

We unify and slightly improve several bounds on the isotropic constant of high-dimensional convex bodies; in particular, a linear dependence on the body's ψ_2 constant is obtained. Along the way, we present some new bounds on the volume of L_p -centroid bodies and yet another equivalent formulation of Bourgain's hyperplane conjecture. Our method is a combination of the L_p -centroid body technique of Paouris and the logarithmic Laplace transform technique of the first named author.

1 Introduction

This work combines two recent techniques in the study of volumes of high-dimensional convex bodies. The first technique is due to G. Paouris [25], and it relies on properties of the L_p -centroid bodies. The second technique was developed by the first named author [15], and it uses the logarithmic Laplace transform.

Suppose that μ is a Borel probability measure on \mathbb{R}^n endowed with a Euclidean structure $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$. We say that μ is a ψ_α -measure ($\alpha > 0$) with constant b_α if:

$$\left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p d\mu(x) \right)^{\frac{1}{p}} \leq b_\alpha p^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \quad \forall p \geq 2 \quad \forall \theta \in \mathbb{R}^n. \quad (1.1)$$

It is well-known that the uniform probability measure μ_K on any convex body $K \subset \mathbb{R}^n$ is a ψ_1 -measure with constant C , where $C > 0$ is a universal constant (this follows from Berwald's inequality [3], see also [21]). Here, as usual, a convex body in \mathbb{R}^n means a compact, convex set with a non-empty interior. The isotropic constant L_K of a convex body $K \subset \mathbb{R}^n$ is the following affine invariant parameter:

$$L_K := \text{Vol}_n(K)^{-\frac{1}{n}} (\det \text{Cov}(\mu_K))^{\frac{1}{2n}},$$

where Vol_n denotes Lebesgue measure and $\text{Cov}(\mu_k)$ denotes the covariance matrix of μ_K . The next theorem unifies and slightly improves several known bounds on the isotropic constant.

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Theorem 1.1. *Let $K \subset \mathbb{R}^n$ denote a convex body whose barycenter lies at the origin, and suppose that μ_K is a ψ_α -measure ($1 \leq \alpha \leq 2$) with constant b_α . Then:*

$$L_K \leq C \sqrt{b_\alpha^\alpha n^{1-\alpha/2}},$$

where $C > 0$ is a universal constant.

A central question raised by Bourgain [6, 7] is whether $L_K \leq C$ for some universal constant $C > 0$, for any convex body $K \subset \mathbb{R}^n$ (it is well-known that $L_K \geq c$ for a universal constant $c > 0$). This question is usually referred to as the *slicing problem* or *hyperplane conjecture*, see Milman and Pajor [21] for many of its equivalent formulations and for further background. Plugging $\alpha = 1$ in Theorem 1.1, we match the best known bound on the isotropic constant, which is $L_K \leq Cn^{1/4}$ for any convex body $K \subset \mathbb{R}^n$ (see Bourgain [8] and Klartag [15]). In the case $\alpha = 2$, Theorem 1.1 yields $L_K \leq Cb_2$. This slightly improves upon the previously known bound, which is:

$$L_K \leq Cb_2 \sqrt{\log b_2}, \quad (1.2)$$

due to Dafnis and Paouris [11] in the precise form (1.2) and to Bourgain [9] (with a different power of the logarithmic factor). Here, as elsewhere in this text, we use the letters $c, \tilde{c}, C, \tilde{C}, \bar{C}$ etc. to denote positive universal constants, whose value may not necessarily be the same in different occurrences.

We proceed by recalling the definition of the L_p -centroid bodies $Z_p(\mu)$, originally introduced by E. Lutwak and G. Zhang in [19] (under different normalization), which lie at the heart of Paouris' remarkable work [25]. Given a Borel probability measure μ on \mathbb{R}^n and $p \geq 1$, denote:

$$h_{Z_p(\mu)}(\theta) = \left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p d\mu(x) \right)^{\frac{1}{p}}, \quad \theta \in \mathbb{R}^n.$$

The function $h_{Z_p(\mu)}$ is a norm on \mathbb{R}^n , and it is the supporting functional of a convex body $Z_p(\mu) \subseteq \mathbb{R}^n$ (see e.g. Schneider [29] for information on supporting functionals). Clearly $Z_p(\mu) \subseteq Z_q(\mu)$ for $p \leq q$.

Now suppose that $K \subset \mathbb{R}^n$ is a convex body whose barycenter lies at the origin, and denote $Z_p(K) = Z_p(\mu_K)$, where μ_K is as before the uniform probability measure on K . As realized by Paouris, obtaining volumetric and other information on $Z_p(K)$ is very useful for understanding the volumetric properties of K itself. For instance, note that:

$$\text{V.Rad.}(Z_2(K)) = \det \text{Cov}(\mu_K)^{\frac{1}{2n}}, \quad (1.3)$$

where the volume-radius of a compact set $T \subset \mathbb{R}^n$ is defined as:

$$\text{V.Rad.}(T) = \left(\frac{\text{Vol}_n(T)}{\text{Vol}_n(B_n)} \right)^{\frac{1}{n}},$$

measuring the radius of the Euclidean ball whose volume equals the volume of T . Here, $B_n = \{x \in \mathbb{R}^n; |x| \leq 1\}$; note that $cn^{-\frac{1}{2}} \leq \text{Vol}_n(B_n)^{\frac{1}{n}} \leq Cn^{-\frac{1}{2}}$, as verified by direct calculation. Furthermore, it is known (e.g. [26, Lemma 3.6]) that:

$$c \cdot Z_\infty(K) \subseteq Z_n(K) \subseteq Z_\infty(K) := \text{conv}(K, -K), \quad (1.4)$$

where $\text{conv}(K, -K)$ denotes the convex hull of K and $-K$.

A sharp *lower* bound on the volume of $Z_p(K)$ due to Lutwak, Yang and Zhang [13] states that ellipsoids minimize $\text{V.Rad.}(Z_p(K))/\text{V.Rad.}(K)$ among all convex bodies K , for all $p \geq 1$. An elementary calculation yields:

$$\text{V.Rad.}(Z_p(K)) \geq c \sqrt{\frac{p}{n}} \text{V.Rad.}(K) \quad \text{for } 1 \leq p \leq n, \quad (1.5)$$

which is best possible (up to the value of the constant $c > 0$) in terms of $\text{Vol}_n(K)$. However, in view of the slicing problem and (1.3), one may try to strengthen (1.5) by replacing its right-hand side by $c\sqrt{p}\text{V.Rad.}(Z_2(K))$. The next two theorems are a step in this direction. Before formulating the results, we first broaden our scope.

It was realized by K. Ball [1, 2] that many questions regarding the volume of convex bodies are better formulated in the broader class of logarithmically-concave measures. A function $\rho : \mathbb{R}^n \rightarrow [0, \infty)$ is called log-concave if $-\log \rho : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is a convex function. A probability measure on \mathbb{R}^n is log-concave if its density is log-concave. For example, the uniform probability measure on a convex body and its marginals are all log-concave measures (see Borell [5] for a characterization).

Theorem 1.2. *Let μ be a log-concave probability measure on \mathbb{R}^n with barycenter at the origin. Let $1 \leq \alpha \leq 2$, and assume that μ is a ψ_α -measure with constant b_α . Then:*

$$\text{V.Rad.}(Z_p(\mu)) \geq c\sqrt{p}\text{V.Rad.}(Z_2(\mu)),$$

for all $2 \leq p \leq Cn^{\alpha/2}/b_\alpha^\alpha$. Here $c, C > 0$ denote universal constants.

Theorem 1.1 follows immediately from Theorem 1.2. Indeed, simply observe that for p in the specified range:

$$c\sqrt{p} \leq \frac{\text{V.Rad.}(Z_p(K))}{\text{V.Rad.}(Z_2(K))} \leq \frac{\text{V.Rad.}(\text{conv}(K, -K))}{\text{V.Rad.}(Z_2(K))} \leq C\sqrt{n} \frac{\text{Vol}_n(K)^{1/n}}{\text{V.Rad.}(Z_2(K))} = \frac{C\sqrt{n}}{L_K},$$

where the last inequality follows from the Rogers-Shephard inequality [28]. This completes the proof of Theorem 1.1, reducing it to that of Theorem 1.2. We remark here that the proof (of both theorems) only requires that the ψ_α condition (1.1) hold for $p \geq 2$ so that $\text{diam}(Z_p(\mu)) \leq c\sqrt{n}$, and only in an average sense (see Subsection 5.3).

Our next theorem contains an additional lower bound on the volume of $Z_p(\mu)$ which complements that of Theorem 1.2 in some sense. A Borel probability measure μ on $(\mathbb{R}^n, |\cdot|)$ is called isotropic when its barycenter lies at the origin, and its covariance matrix equals the identity matrix (i.e. $Z_2(\mu) = B_n$). Any measure with finite second moments and full-dimensional support may be brought into isotropic ‘‘position’’ by means of an affine transformation.

Theorem 1.3. *Let μ be an isotropic log-concave probability measure on \mathbb{R}^n . Then:*

$$\text{V.Rad.}(Z_p(\mu)) \geq c\sqrt{p},$$

for all $p \geq 2$ for which:

$$\text{diam}(Z_p(\mu))\sqrt{\log p} \leq C\sqrt{n}. \quad (1.6)$$

Here, $\text{diam}(T) = \sup_{x, y \in T} |x - y|$ stands for the diameter of $T \subset \mathbb{R}^n$, and $c, C > 0$ are universal constants.

Note that the ψ_α -condition (1.1) is precisely the requirement that $Z_p(\mu) \subseteq b_\alpha p^{\frac{1}{\alpha}} Z_2(\mu)$ for all $p \geq 2$, and so the conclusion of Theorem 1.3 agrees with that of Theorem 1.2, up to the logarithmic factor in (1.6). This discrepancy is explained by the fact that in Theorem 1.2, we actually make full use of the growth of $\text{diam}(Z_p(\mu))$ for all $p \geq 2$, whereas in Theorem 1.3 we only assumed this control for the end value of p . We emphasize that this constitutes a genuine difference in assumptions, and that the logarithmic factor in (1.6) is not just a mere technicality: we show in Section 6 that removing this factor is actually *equivalent* to Bourgain’s original hyperplane conjecture.

We find condition (1.6) quite interesting from other respects as well. It is very much related to Paouris’ parameter $q^*(\mu)$, to be discussed in Section 4. In fact, we show there that the parameter:

$$q^\#(\mu) := \sup \left\{ q \geq 1; \text{diam}(Z_q(\mu)) \leq c^\# \sqrt{n} \det \text{Cov}(\mu)^{\frac{1}{2n}} \right\} ,$$

for a small-enough universal constant $c^\# > 0$, is essentially equivalent to and has the same functionality as Paouris’ $q^*(\mu)$ parameter, in addition to being rather convenient to work with.

The lower bounds in Theorem 1.2 and Theorem 1.3 compare with the matching *upper* bounds on $\text{V.Rad.}(Z_p(\mu))$, obtained by Paouris [25, Theorem 6.2], which are valid for *all* $2 \leq p \leq n$:

$$\text{V.Rad.}(Z_p(\mu)) \leq C \sqrt{p} \text{V.Rad.}(Z_2(\mu)) . \quad (1.7)$$

This implies that the lower bounds in both theorems above are sharp, up to constants, and so the only pertinent question is the optimality of the range of p ’s for which their conclusion is valid. In this direction, Paouris obtained a partial converse to (1.7) in the following range of p ’s:

$$W(Z_p(\mu)) \geq c \sqrt{p} \text{V.Rad.}(Z_2(\mu)) \quad \forall 2 \leq p \leq q^\#(\mu) . \quad (1.8)$$

Here $W(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta)$ denotes half the mean width of K , σ is the Haar probability measure on the Euclidean unit sphere S^{n-1} , and $h_K(\theta) = \sup_{x \in K} \langle x, \theta \rangle$ is the supporting functional of K . Note that according to the Urysohn inequality, $W(K) \geq \text{V.Rad.}(K)$ (see e.g. [22]), and so Theorem 1.3 should be thought of as a formal strengthening of (1.8), if it were not for the logarithmic factor in (1.6).

The rest of this work is organized as follows. We begin with some more or less known preliminaries in Section 2. In Section 3, we deduce a new formula for $\text{V.Rad.}(Z_p(\mu))$ involving the “tilts” of the measure μ from [15, 16], and we relate between the Z_p -bodies of the original measure and its tilts. In Section 4, we deviate from our discussion to review Paouris’ q^* -parameter, and compare it with $q^\#$; this section may be read independently of this work. In Section 5, we use projections and the $q^\#$ -parameter to relate between the determinant of the covariance matrix of μ and its tilts, and conclude the proofs of Theorems 1.2 (in fact, a more general version) and 1.3. In Section 6, we show that removing the log-factor from Theorem 1.3 is equivalent to the slicing problem.

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2 Preliminaries

Given $1 \leq k \leq n$, the Grassmann manifold of all k -dimensional linear subspaces of \mathbb{R}^n is denoted by $G_{n,k}$. Given $E \in G_{n,k}$, the orthogonal projection onto E is denoted by $Proj_E$, and given a Borel probability measure μ on \mathbb{R}^n , we denote by $\pi_E \mu := (Proj_E)_*(\mu)$ the push-forward of μ via $Proj_E$. For a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, its polar body is denoted by:

$$K^\circ = \{x \in \mathbb{R}^n ; \langle x, y \rangle \leq 1 \quad \forall y \in K\} .$$

Finally, we denote by ∇ and Hess the gradient and Hessian of a sufficiently differentiable function, respectively.

Throughout this text, $x \simeq y$ is an abbreviation for $cx \leq y \leq Cx$ for universal constants $c, C > 0$. Similarly, we write $x \lesssim y$ ($x \gtrsim y$) when $x \leq Cy$ ($x \geq cy$). Additionally, for two convex sets $K, T \subset \mathbb{R}^n$ we write $K \simeq T$ when:

$$cK \subseteq T \subseteq CK$$

for universal constants $c, C > 0$.

2.1 Extension of the Slicing Problem to log-concave measures

We first recall the well-known extension of the slicing problem from the class of convex bodies to the class of all log-concave measures, due to Ball [1, 2]. Given a log-concave probability measure μ on \mathbb{R}^n , define its isotropic constant L_μ by:

$$L_\mu := \|\mu\|_{L_\infty}^{\frac{1}{n}} \det \text{Cov}(\mu)^{\frac{1}{2n}} , \quad (2.1)$$

where $\|\mu\|_{L_\infty} := \sup_{x \in \mathbb{R}^n} \rho(x)$ and ρ is the log-concave density of μ . It was shown by Ball [1, 2] that given $n \geq 1$:

$$\sup_{\mu} L_\mu \leq C \sup_K L_K ,$$

where the suprema are taken over all log-concave probability measures μ and convex bodies K in \mathbb{R}^n , respectively (see e.g. [15] for the non-even case). Similarly, the following theorem slightly generalizes Theorem 1.1:

Theorem 2.1. *Let μ denote a log-concave probability measure on \mathbb{R}^n with barycenter at the origin. Suppose that μ is in addition a ψ_α -measure ($1 \leq \alpha \leq 2$) with constant b_α . Then:*

$$L_\mu \leq C \sqrt{b_\alpha^\alpha n^{1-\alpha/2}} .$$

As was the case with Theorem 1.1, deducing Theorem 2.1 from Theorem 1.2 is equally elementary. We only require the following additional well-known lemma, which will come in handy in other instances in this work as well. This lemma serves as an extension of (1.4) to the class of log-concave measures.

Lemma 2.2. *Let μ denote a log-concave probability measure on \mathbb{R}^n with barycenter at the origin. Then:*

$$V.Rad.(Z_n(\mu)) \simeq \frac{\sqrt{n}}{\|\mu\|_{L_\infty}^{\frac{1}{n}}} .$$

Given Lemma 2.2, the reduction of Theorem 2.1 to Theorem 1.2 is indeed immediate, since for $p \leq n$ in the range specified in the latter:

$$c\sqrt{p} \leq \frac{\text{V.Rad.}(Z_p(\mu))}{\text{V.Rad.}(Z_2(\mu))} \leq \frac{\text{V.Rad.}(Z_n(\mu))}{\det \text{Cov}(\mu)^{\frac{1}{2n}}} \simeq \frac{\sqrt{n}}{\|\mu\|_{L^\infty}^{\frac{1}{n}} \det \text{Cov}(\mu)^{\frac{1}{2n}}} = \frac{\sqrt{n}}{L_\mu} .$$

Proof of Lemma 2.2. Denote by ρ the log-concave density of μ . According to [26, Proposition 3.7] (compare with [16, Lemma 2.8] and Lemma 2.3 below):

$$\text{V.Rad.}(Z_n(\mu)) \simeq \frac{\sqrt{n}}{\rho(0)^{\frac{1}{n}}} .$$

However, according to Fradelizi [12]:

$$e^{-n}M \leq \rho(0) \leq M \quad , \quad M := \|\mu\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} \rho(x) ,$$

and so the assertion immediately follows. \square

2.2 Λ_p -bodies

Now suppose that μ is an arbitrary Borel probability measure on \mathbb{R}^n . Its logarithmic Laplace transform is defined as:

$$\Lambda_\mu(\xi) := \log \int_{\mathbb{R}^n} \exp(\langle \xi, x \rangle) d\mu(x) \quad , \quad \xi \in \mathbb{R}^n .$$

The function Λ_μ is always convex (e.g. by Hölder's inequality), and clearly $\Lambda_\mu(0) = 0$. If in addition the barycenter of μ lies at the origin, then Λ_μ is non-negative (by Jensen's inequality). In this case, for any $t \geq 0$ and $\alpha \geq 1$:

$$\frac{1}{\alpha} \{\Lambda_\mu \leq \alpha t\} \subseteq \{\Lambda_\mu \leq t\} \subseteq \{\Lambda_\mu \leq \alpha t\} \quad , \quad (2.2)$$

where we abbreviate $\{\Lambda_\mu \leq t\} = \{\xi \in \mathbb{R}^n; \Lambda_\mu(\xi) \leq t\}$. When μ is log-concave, the convex function Λ_μ possesses several additional regularity properties. For instance $\{\Lambda_\mu < \infty\}$ is an open set, and Λ_μ is C^∞ -smooth and strictly-convex in this open set (see, e.g., [16, Section 2]).

The following lemma describes a certain equivalence, known to specialists, between the L_p -centroid bodies and the level-sets of the logarithmic Laplace Transform Λ_μ . See Latała and Wojtaszczyk [17, Section 3] for a proof of a dual version in the symmetric case (i.e., when $\mu(A) = \mu(-A)$ for all Borel subsets $A \subset \mathbb{R}^n$).

Definition. *The Λ_p -body associated to μ , for $p \geq 0$, is defined as:*

$$\Lambda_p(\mu) := \{\Lambda_\mu \leq p\} \cap -\{\Lambda_\mu \leq p\} .$$

Lemma 2.3. *Suppose μ is a log-concave probability measure on \mathbb{R}^n whose barycenter lies at the origin. Then for any $p \geq 1$:*

$$\Lambda_p(\mu) \simeq pZ_p(\mu)^\circ .$$

These two equivalent points of view turn out to complement each other well, and play a synergetic role in this work. Before providing a proof, we illustrate this in the following naive example. Given a log-concave probability measure μ , a well known consequence of Berwald's inequality (see e.g. [21]) is that:

$$q \geq p \geq 1 \quad \Rightarrow \quad Z_p(\mu) \subset Z_q(\mu) \subset C \frac{q}{p} Z_p(\mu) . \quad (2.3)$$

In view of Lemma 2.3, note that this is nothing else but a reformulation (up to constants) of the trivial set of inclusions in (2.2).

Proof of Lemma 2.3. First, suppose that $\xi \in \Lambda_p(\mu)$. Then:

$$\int_{\mathbb{R}^n} \exp(|\langle \xi, x \rangle|) d\mu(x) \leq \int_{\mathbb{R}^n} \exp(\langle \xi, x \rangle) d\mu(x) + \int_{\mathbb{R}^n} \exp(-\langle \xi, x \rangle) d\mu(x) \leq 2e^p .$$

Using the inequality $t^p/p! \leq e^t$, valid for any $t \geq 0$, we see that:

$$h_{Z_p(\mu)}(\xi) = \left(\int_{\mathbb{R}^n} |\langle \xi, x \rangle|^p d\mu(x) \right)^{\frac{1}{p}} \leq (2e^p p!)^{\frac{1}{p}} \leq Cp .$$

Since $\xi \in \Lambda_p(\mu)$ was arbitrary, this amounts to $\Lambda_p(\mu) \subseteq CpZ_p(\mu)^\circ$, the first desired inclusion.

For the other inclusion, suppose $\xi \in \mathbb{R}^n$ is such that $h_{Z_p(\mu)}(\xi) \leq p$, that is:

$$\left(\int_{\mathbb{R}^n} |\langle \xi, x \rangle|^p d\mu(x) \right)^{1/p} \leq p . \quad (2.4)$$

Write X for the random vector in \mathbb{R}^n that is distributed according to μ . Then the function:

$$\varphi(t) = \mathbb{P}(\langle X, \xi \rangle \geq t) \quad , \quad t \in \mathbb{R} ,$$

is log-concave, according to the Prékopa-Leindler inequality (see, e.g., the first pages of [27]). Furthermore, since the barycenter of μ lies at the origin, we have $1/e \leq \varphi(0) \leq 1 - 1/e$ by the Grünbaum–Hammer inequality (see e.g. [4, Lemma 3.3]). Using Markov's inequality, (2.4) entails that:

$$\varphi(3ep) \leq (3e)^{-p} .$$

Since φ is log-concave, then:

$$\mathbb{P}(\langle X, \xi \rangle \geq t) = \varphi(t) \leq \varphi(0) \left(\frac{\varphi(3ep)}{\varphi(0)} \right)^{\frac{t}{3ep}} \leq C \exp(-t/(3e)) \quad , \quad \forall t \geq 3ep .$$

An identical bound holds for $\mathbb{P}(\langle X, \xi \rangle \leq -t)$, and combining the two, we obtain:

$$\mathbb{P}(|\langle X, \xi \rangle| \geq t) \leq C \exp(-t/(3e)) \quad , \quad \forall t \geq 3ep .$$

Therefore:

$$\begin{aligned} \mathbb{E} \exp \left(\frac{|\langle \xi, X \rangle|}{6e} \right) &= \frac{1}{6e} \int_0^\infty \exp \left(\frac{t}{6e} \right) \mathbb{P}(|\langle X, \xi \rangle| \geq t) dt \\ &\leq \frac{1}{6e} \int_0^{3ep} \exp \left(\frac{t}{6e} \right) dt + C \int_{3ep}^\infty \exp(-t/(6e)) dt \leq \exp(\tilde{C}p) . \end{aligned}$$

Consequently:

$$\max \left\{ \Lambda_\mu \left(\frac{1}{6e} \xi \right), \Lambda_\mu \left(-\frac{1}{6e} \xi \right) \right\} \leq \log \mathbb{E} \exp \left(\frac{|\langle \xi, X \rangle|}{6e} \right) \leq Cp ,$$

for some $C \geq 1$, and using (2.2), this implies:

$$\max \left\{ \Lambda_\mu \left(\frac{1}{6eC} \xi \right), \Lambda_\mu \left(-\frac{1}{6eC} \xi \right) \right\} \leq p ,$$

for any $\xi \in \mathbb{R}^n$ with $h_{Z_p(\mu)}(\xi) \leq p$. This is precisely the second desired inclusion $pZ_p(\mu)^\circ \subseteq C'\Lambda_p(\mu)$, and the assertion follows. \square

2.3 Level Sets of Convex Functions Under Gradient Maps

The last topic we would like to review pertains to some properties of level sets of convex functions and their gradient images. The possibility to use the gradient image of Λ_μ as in [15] is one of the main reasons for additionally employing the logarithmic Laplace transform, rather than working exclusively with the L_p -centroid bodies.

Lemma 2.4. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a non-negative convex function, which is C^1 -smooth in $\{F < \infty\}$. Let $q, r \geq 0$. Then:*

$$\langle z, \nabla F(x) \rangle \leq q + r \quad \text{for any } z \in \{F \leq r\}, x \in \frac{1}{2}\{F \leq q\}.$$

In other words:

$$\nabla F \left(\frac{1}{2}\{F \leq q\} \right) \subset (q + r)\{F \leq r\}^\circ .$$

Proof. Since F is non-negative and its graph lies above any tangent hyperplane, then:

$$\left\langle \nabla F(x), \frac{z}{2} \right\rangle \leq F(x) + \left\langle \nabla F(x), \frac{z}{2} \right\rangle \leq F(x + z/2) \leq \frac{F(2x) + F(z)}{2} \leq \frac{q + r}{2} .$$

\square

The following lemma was proved in [16, Lemma 2.3] for an even function F .

Lemma 2.5. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a non-negative convex function, C^2 -smooth and strictly-convex in $\{F < \infty\}$, with $F(0) = 0$. Let $p > 0$, and set:*

$$F_p := \{F \leq p\} \cap -\{F \leq p\} .$$

Assume that:

$$\Psi_p := \left(\frac{1}{\text{Vol}_n(\frac{1}{2}F_p)} \int_{\frac{1}{2}F_p} \det \text{Hess} F(x) dx \right)^{\frac{1}{n}} > 0 .$$

Then:

$$V.\text{Rad.}(F_p) \leq 2 \frac{\sqrt{p}}{\sqrt{\Psi_p}} .$$

Proof. Applying Lemma 2.4 with $q = r = p$, and using the change of variables $x = \nabla F(y)$, we obtain:

$$\text{Vol}_n(2p(F_p)^\circ) \geq \text{Vol}_n\left(\nabla F\left(\frac{1}{2}F_p\right)\right) = \int_{\frac{1}{2}F_p} \det \text{Hess}F(y) dy = \text{Vol}_n\left(\frac{1}{2}F_p\right) \Psi_p^n .$$

Equivalently, we obtain:

$$\text{Vol}_n((F_p)^\circ) \geq \left(\frac{\Psi_p}{4p}\right)^n \text{Vol}_n(F_p) .$$

Note that F_p is a centrally-symmetric convex body, i.e., $F_p = -F_p$. The Blaschke–Santaló inequality (see, e.g., [29]) for a centrally-symmetric convex body K asserts that:

$$\text{V.Rad.}(K^\circ)\text{V.Rad.}(K) \leq 1 .$$

Combining the last two estimates with $K = F_p$, the result immediately follows. \square

3 A formula for $\text{V.Rad.}(Z_p(\mu))$ involving tilted measures

Let μ denote a log-concave probability measure on \mathbb{R}^n with density ρ , and let $\xi \in \{\Lambda_\mu < \infty\}$. We denote by μ_ξ the “tilt” of μ by ξ , defined via the following procedure. First, define the probability density:

$$\rho_\xi(x) := \frac{1}{Z_\xi} \rho(x) \exp(\langle \xi, x \rangle) \quad \text{for } x \in \mathbb{R}^n ,$$

where $Z_\xi > 0$ is a normalizing factor. Denoting by $b_\xi \in \mathbb{R}^n$ the barycenter of ρ_ξ , we set μ_ξ to be the probability measure with density $\rho_\xi(\cdot - b_\xi)$. Note that μ_ξ is a log-concave probability measure, having the origin as its barycenter. Furthermore, as verified in [16, Section 2], we have:

$$b_\xi = \nabla \Lambda_\mu(\xi) \quad , \quad \text{Cov}(\mu_\xi) = \text{Hess} \Lambda_\mu(\xi) . \quad (3.1)$$

The following proposition is one of the main results in this section:

Proposition 3.1. *Let μ denote a log-concave probability measure on \mathbb{R}^n whose barycenter lies at the origin. Then, for all $1 \leq p \leq n$:*

$$\text{V.Rad.}(Z_p(\mu)) \simeq \sqrt{p} \inf_{x \in \frac{1}{2}\Lambda_p(\mu)} \det \text{Cov}(\mu_x)^{\frac{1}{2n}} . \quad (3.2)$$

In the proofs of the theorems stated in the Introduction, we will not use the full force of Proposition 3.1, but rather only the lower bound for $\text{V.Rad.}(Z_p(\mu))$. This lower bound has a short proof, as the reader will see below. However, the observation that we actually obtain an equivalence seems interesting, hence we provide the arguments for both directions. Before going into the proof, as a testament of its usefulness, we state the following immediate corollary of Proposition 3.1:

Corollary 3.2. *Let μ be a log-concave probability measure on \mathbb{R}^n whose barycenter lies at the origin. Then:*

$$1 \leq p \leq q \leq n \quad \Rightarrow \quad \frac{V.\text{Rad.}(Z_p(\mu))}{\sqrt{p}} \geq c \frac{V.\text{Rad.}(Z_q(\mu))}{\sqrt{q}} .$$

Remark 3.3. Using $q = n$ above and the fact that $V.\text{Rad.}(Z_n(K)) \simeq V.\text{Rad.}(K)$ for a convex body K whose barycenter lies at the origin, which follows from (1.4) as in the Introduction, we immediately verify that:

$$\forall 1 \leq p \leq n \quad V.\text{Rad.}(Z_p(K)) \geq c \sqrt{\frac{p}{n}} V.\text{Rad.}(K) . \quad (3.3)$$

This recovers up to a constant the lower bound of Lutwak, Yang and Zhang (1.5). Moreover, recalling that $V.\text{Rad.}(Z_n(\mu)) \simeq \sqrt{n} / \|\mu\|_{L^\infty}^{\frac{1}{n}}$ by Lemma 2.2 and the definition (2.1) of L_μ , the same argument yields the following analog of (3.3):

$$\forall 1 \leq p \leq n \quad V.\text{Rad.}(Z_p(\mu)) \geq c \frac{\sqrt{p}}{L_\mu} \det \text{Cov}(\mu)^{\frac{1}{2n}} = c \frac{\sqrt{p}}{L_\mu} V.\text{Rad.}(Z_2(\mu)) .$$

This may also be deduced by only employing the lower-bound in (3.2), as in Remark 5.1.

We now turn to the proof of Proposition 3.1, and begin with the lower bound for $V.\text{Rad.}(Z_p(\mu))$. In fact, we show a formally stronger statement:

Lemma 3.4. *Let μ denote a log-concave probability measure on \mathbb{R}^n whose barycenter lies at the origin. Then, for all $1 \leq p \leq n$,*

$$V.\text{Rad.}(Z_p(\mu)) \geq c \sqrt{p} \sqrt{\Psi_p} ,$$

where $c > 0$ is a universal constant and:

$$\Psi_p := \left(\frac{1}{\text{Vol}_n(\frac{1}{2}\Lambda_p(\mu))} \int_{\frac{1}{2}\Lambda_p(\mu)} \det \text{Cov}(\mu_x) dx \right)^{\frac{1}{n}} .$$

Proof. Apply Lemma 2.5 with $F = \Lambda_\mu$. Since $\det \text{Hess} \Lambda_\mu(x) = \det \text{Cov}(\mu_x)$ according to (3.1), we deduce that:

$$V.\text{Rad.}(\Lambda_p(\mu)) \leq 2 \frac{\sqrt{p}}{\sqrt{\Psi_p}} . \quad (3.4)$$

Applying Lemma 2.3 in order to pass from $\Lambda_p(\mu)$ to $Z_p(\mu)$, and the Bourgain–Milman inequality (see, e.g., [27]) for a centrally-symmetric convex set $K \subset \mathbb{R}^n$:

$$V.\text{Rad.}(K^\circ) V.\text{Rad.}(K) \geq c ,$$

we deduce from (3.4) that:

$$V.\text{Rad.}(Z_p(\mu)) \simeq p V.\text{Rad.}(\Lambda_p(\mu)^\circ) \gtrsim p V.\text{Rad.}(\Lambda_p(\mu))^{-1} \gtrsim \sqrt{p} \sqrt{\Psi_p} .$$

□

In order to deduce the upper bound of Proposition 3.1, and of crucial importance to the main results of this work, is the following elementary observation:

Proposition 3.5. *Let μ denote a log-concave probability measure in \mathbb{R}^n with barycenter at the origin. Then:*

$$\forall x \in \frac{1}{2}\Lambda_p(\mu) \quad , \quad \Lambda_p(\mu_x) \simeq \Lambda_p(\mu) .$$

Indeed, it is clear that the logarithmic Laplace transform should commute nicely with the tilt operation, and the following identity is verified by direct calculation:

$$\Lambda_{\mu_x}(z) = \Lambda_\mu(z+x) - \Lambda_\mu(x) - \langle z, b_x \rangle \quad , \quad b_x = \nabla \Lambda_\mu(x) . \quad (3.5)$$

Geometrically, this means that the graph of Λ_{μ_x} is obtained from that of Λ_μ by subtracting the tangent plane at x (given by the linear function $z \mapsto \Lambda_\mu(x) + \langle z-x, \nabla \Lambda_\mu(x) \rangle$), and translating everything by $-x$ (so that x gets mapped to the origin). In particular, we verify that $\Lambda_{\mu_x}(0) = 0$ and that $\Lambda_{\mu_x} \geq 0$, as required from the logarithmic Laplace transform of a probability measure with barycenter at the origin.

It remains to manipulate level sets of convex functions, once again. We require the following:

Lemma 3.6. *Let F be as in Lemma 2.4, and let $y \in \mathbb{R}^n$ and $D, p > 0$. Define a function G by:*

$$G(z) := F(z+y) - F(y) - \langle z, \nabla F(y) \rangle .$$

Then:

$$y \in \frac{1}{2}\{F \leq Dp\} \quad , \quad z \in \{F \leq p\} \cap -\{F \leq p\} \quad \implies \quad z \in 2\{G \leq (D+1)p\} .$$

Proof. We apply Lemma 2.4 with $q = Dp$ and $r = p$. Since $-z \in \{F \leq p\}$ and $y \in \frac{1}{2}\{F \leq Dp\}$, then by the conclusion of that lemma, $\langle -z, \nabla F(y) \rangle \leq (D+1)p$. Since F is non-negative and convex, we deduce that:

$$G(z/2) \leq F(z/2+y) + \frac{D+1}{2}p \leq \frac{F(z)+F(2y)}{2} + \frac{D+1}{2}p \leq (D+1)p .$$

□

Proof of Proposition 3.5.

- (1) If $z \in \Lambda_p(\mu)$, we apply Lemma 3.6 with $D = 1$ and $y = x$ to $F = \Lambda_\mu$. By (3.5), we deduce that $\Lambda_{\mu_x}(z/2) = G(z/2) \leq 2p$. Using (2.2), we conclude that $\Lambda_{\mu_x}(z/4) \leq p$. The same argument applies to $-z$ by the symmetry of our assumptions, and so we conclude that $z \in 4\Lambda_p(\mu_x)$.
- (2) If $z \in \Lambda_p(\mu_x)$, we would like to apply Lemma 3.6 with $y = -x$ to $F = \Lambda_{\mu_x}$, since tilting μ_x by $-x$ gives back μ . To this end, we must verify that $\Lambda_{\mu_x}(-2x) \leq Dp$ for some $D > 0$. According to (3.5):

$$\Lambda_{\mu_x}(-2x) = \Lambda_\mu(-x) - \Lambda_\mu(x) + 2\langle x, \nabla \Lambda_\mu(x) \rangle .$$

By Lemma 2.4, we know that $\langle x, \nabla \Lambda_\mu(x) \rangle \leq 2p$, and using that Λ_μ is non-negative, convex and vanishes at the origin, we obtain:

$$\Lambda_{\mu_x}(-2x) \leq \frac{1}{2} \Lambda_\mu(-2x) + 4p \leq 4.5p .$$

We conclude that we may use $D = 4.5$ above, and so Lemma 3.6 finally implies that $\Lambda_\mu(z/2) = G(z/2) \leq 5.5p$. As in the first part of the proof, we deduce that $\Lambda_\mu(z/11) \leq p$. The same argument applies to $-z$ by the symmetry of our assumptions, and so we conclude that $z \in 11\Lambda_p(\mu)$. □

Using Lemma 2.3, we equivalently reformulate Proposition 3.5 as:

Proposition 3.7. *Let μ denote a log-concave probability measure in \mathbb{R}^n with barycenter at the origin. Then:*

$$\forall x \in \frac{1}{2}\Lambda_p(\mu) \quad , \quad Z_p(\mu_x) \simeq Z_p(\mu) .$$

To complete the proof of Proposition 3.1, we state again Paouris' upper bound (1.7) on $V.\text{Rad.}(Z_p(\nu))$:

Proposition 3.8 (Paouris). *For any log-concave probability measure ν with barycenter at the origin, and $2 \leq p \leq n$:*

$$V.\text{Rad.}(Z_p(\nu)) \leq C\sqrt{p}V.\text{Rad.}(Z_2(\nu)) .$$

Proof. The statement is invariant under linear transformations, so we may assume that ν is isotropic. The claim is then the content of [25, Theorem 6.2]. □

Proof of Proposition 3.1. Lemma 3.4 implies the lower bound:

$$V.\text{Rad.}(Z_p(\mu)) \geq c\sqrt{p} \inf_{x \in \frac{1}{2}\Lambda_p(\mu)} \det \text{Cov}(\mu_x)^{\frac{1}{2n}} .$$

Since $\det \text{Cov}(\mu_x)^{\frac{1}{2n}} = V.\text{Rad.}(Z_2(\mu_x))$, then applying Proposition 3.8, we obtain:

$$\inf_{x \in \frac{1}{2}\Lambda_p(\mu)} V.\text{Rad.}(Z_p(\mu_x)) \leq C\sqrt{p} \inf_{x \in \frac{1}{2}\Lambda_p(\mu)} \det \text{Cov}(\mu_x)^{\frac{1}{2n}} . \quad (3.6)$$

But by Proposition 3.7, $Z_p(\mu_x) \simeq Z_p(\mu)$ for all $x \in \frac{1}{2}\Lambda_p(\mu)$, and hence the left-hand side in (3.6) is equivalent to $V.\text{Rad.}(Z_p(\mu))$, completing the proof. □

Remark 3.9. It follows that all of the inequalities which we used in the proof of Proposition 3.1 above, are actually equivalences up to numeric constants. This fact has some interesting consequences; we omit a detailed account of these here, and only remark on the following point. Given $1 \leq p \leq n$, denote:

$$x_p := \operatorname{argmin}_{x \in \frac{1}{2}\Lambda_p(\mu)} \det \text{Cov}(\mu_x)^{\frac{1}{2n}} ,$$

so that μ_{x_p} is the “worst” tilt we need to account for when evaluating $V.\text{Rad.}(Z_p(\mu))$. It follows that for this tilt:

$$V.\text{Rad.}(Z_p(\mu_{x_p})) \simeq \sqrt{p}V.\text{Rad.}(Z_2(\mu_{x_p})) ,$$

and in particular, the argument described in Subsection 2.1 implies that $L_{\mu_{x_p}} \leq C\sqrt{n/p}$. It is interesting to compare this with the approach from [15] for resolving the isomorphic slicing problem. The latter approach is in some sense dual to our current one, since in this work our goal will be to bound $\det \text{Cov}(\mu_{x_p})^{\frac{1}{2n}}$ from below, whereas the goal in [15] was to bound this expression from above. Compare also with Remark 5.1.

4 On Paouris' definition of q^*

Given a centrally-symmetric convex body $K \subset \mathbb{R}^n$, its “(dual) Dvoretzky-dimension” $k^*(K)$ was defined by V. Milman and G. Schechtman [23] as the largest positive integer $k \leq n$ so that:

$$\sigma_{n,k} \left\{ E \in G_{n,k}; \frac{1}{2}W(K)B_E \subset \text{Proj}_E K \subset 2W(K)B_E \right\} \geq \frac{n}{n+k} ,$$

where $\sigma_{n,k}$ denotes the Haar probability measure on $G_{n,k}$ and B_E denotes the Euclidean unit ball in the subspace E . It was shown in [23], following Milman's seminal work [20], that:

$$k^*(K) \simeq n \left(\frac{W(K)}{\text{diam}(K)} \right)^2 . \quad (4.1)$$

Define $W_q(K) = \left(\int_{S^{n-1}} h_K(\theta)^q d\sigma(\theta) \right)^{\frac{1}{q}}$, the q -th moment of the supporting functional of K . According to Litvak, Milman and Schechtman [18]:

$$c_1 W_q(K) \leq \max \left\{ W(K), \sqrt{q/n} \text{diam}(K) \right\} \leq c_2 W_q(K) . \quad (4.2)$$

The quantity $W_q(Z_q(\mu))$ has a simple equivalent description: a direct calculation as in [24] confirms that for any Borel probability measure μ on \mathbb{R}^n and $q \geq 1$:

$$W_q(Z_q(\mu)) \simeq \frac{\sqrt{q}}{\sqrt{n+q}} I_q(\mu) , \quad I_q(\mu) := \left(\int_{\mathbb{R}^n} |x|^q d\mu(x) \right)^{\frac{1}{q}} . \quad (4.3)$$

Finally, observe that when the barycenter of μ is at the origin, then $I_2(\mu)^2 = \text{trace Cov}(\mu)$.

In [25], Paouris defines $q^*(\mu)$ as follows:

$$q^*(\mu) := \sup \{ q \in \mathbb{N}; k^*(Z_q(\mu)) \geq q \} .$$

It is straightforward to check that all of Paouris' results involving $q^*(\mu)$ from [25, 26] remain valid when replacing it with $q_c^*(\mu)$ when $c > 0$ is a fixed universal constant, where q_c^* is defined as follows:

$$q_c^*(\mu) := \sup \{ q \geq 1; k^*(Z_q(\mu)) \geq \delta^{-2} q \} .$$

Although the particular value of $c > 0$ seems insignificant for the results of [25, 26], the definition we require in this work is essentially that of q_c^* for some *small enough* universal constant $c > 0$. Our preference to work with a variant of q_c^* is motivated by Lemma 4.1 below and the subsequent remarks.

We proceed as follows. Given a log-concave probability measure μ on \mathbb{R}^n , $q \geq 1$ and $\delta > 0$, consider the following four related properties:

- (1) $P_1(\delta)$ is the property that $k^*(Z_q(\mu)) \geq \delta^{-2}q$.
- (2) $P'_1(\delta)$ is the property that $\text{diam}(Z_q(\mu)) \leq \delta\sqrt{n}\frac{W(Z_q(\mu))}{\sqrt{q}}$.
- (3) $P_2(\delta)$ is the property that $\text{diam}(Z_q(\mu)) \leq \delta\sqrt{n}\det \text{Cov}(\mu)^{\frac{1}{2n}}$.
- (4) P_W is the property that $W(Z_q(\mu)) \geq c\sqrt{q}\det \text{Cov}(\mu)^{\frac{1}{2n}}$, for some specific, appropriately small universal constant $c > 0$, as in the proof of Lemma 4.1(2) below.

According to (4.1), we have:

$$P_1(\delta) \Rightarrow P'_1(C_1\delta) \Rightarrow P_1(C_2\delta) , \quad (4.4)$$

for all $\delta > 0$, where $C_1, C_2 > 1$ are universal constants. The next lemma relates between the other properties above:

Lemma 4.1. *Suppose μ is a log-concave probability measure in \mathbb{R}^n whose barycenter lies at the origin. Let $q \in [1, n]$ and $\delta \in (0, 1]$. Then:*

- (1) *If μ is isotropic and $P_1(\delta)$ holds, then $P_2(C_3\delta)$ holds.*
- (2) (a) *If $P'_1(\delta)$ holds, then so does P_W .*
 (b) *Suppose $\delta < \delta_0$ for a certain appropriately small universal constant $\delta_0 > 0$. If $P_2(\delta)$ holds, then so does P_W .*
- (3) *If $P_2(\delta)$ and P_W hold, then so does $P'_1(C_4\delta)$.*

Proof.

- (1) Clearly $P_1(\delta)$ implies $P_1(1)$. Using (4.3), Paouris's main result [25, Theorem 8.1] and the isotropicity of μ , we know that:

$$W_q(Z_q(\mu)) \simeq \frac{\sqrt{q}}{\sqrt{n}}I_q(\mu) \simeq \frac{\sqrt{q}}{\sqrt{n}}I_2(\mu) = \frac{\sqrt{q}}{\sqrt{n}}(\text{trace Cov}(\mu))^{\frac{1}{2}} = \sqrt{q} .$$

In particular, $W(Z_q(\mu)) \leq W_q(Z_q(\mu)) \leq C\sqrt{q}$. Since $P_1(\delta)$ implies $P'_1(C_1\delta)$, then:

$$\text{diam}(Z_q(\mu)) \leq C_1\delta\sqrt{n}\frac{W(Z_q(\mu))}{\sqrt{q}} \leq CC_1\delta\sqrt{n} = C_3\delta\sqrt{n}\det \text{Cov}(\mu)^{\frac{1}{2n}} ,$$

and $P_2(C_3\delta)$ holds true.

- (2) Since all properties are invariant under scaling, we may assume that $\det \text{Cov}(\mu) = 1$. Using (4.3) and the arithmetic-geometric mean inequality:

$$\frac{1}{n}I_2(\mu)^2 = \frac{1}{n}\text{trace Cov}(\mu) \geq \det \text{Cov}(\mu)^{\frac{1}{n}} ,$$

we see that:

$$W_q(Z_q(\mu)) \geq c_0\frac{\sqrt{q}}{\sqrt{n}}I_q(\mu) \geq c_0\frac{\sqrt{q}}{\sqrt{n}}I_2(\mu) \geq c_0\sqrt{q} . \quad (4.5)$$

- (a) Assuming $P'_1(\delta)$, (4.2) implies that $W(Z_q(\mu)) \geq c_1W_q(Z_q(\mu))$, and together with (4.5), P_W follows.

- (b) Set $\delta_0 = c_0 c_1$, where c_0 is the constant from (4.5) and c_1 is the constant from (4.2). Using (4.5), the property $P_2(\delta)$ with $0 < \delta < \delta_0$ implies:

$$\frac{\sqrt{q}}{\sqrt{n}} \text{diam}(Z_q(\mu)) \leq \delta \sqrt{q} < c_0 c_1 \sqrt{q} \leq c_1 W_q(Z_q(\mu)) .$$

Therefore by (4.2), $W(Z_q(\mu)) \geq c_1 W_q(Z_q(\mu)) \geq c_0 c_1 \sqrt{q}$, and P_W follows.

- (3) This is immediate by plugging the estimates on $\text{diam}(Z_q(\mu))$ and $W(Z_q(\mu))$ into the definition of $P'_1(\delta)$. □

Remark 4.2. Inspecting the proof, one may check that the assumption that $\delta \leq 1$ is not essential for the proof of parts (1), (2a) and (3), if one allows different dependence on δ in the conclusion of the assertions. However, the assumption that $\delta < \delta_0$ was *crucially* used in the proof of part (2b).

We conclude from Lemma 4.1 and (4.4) that $P_1(\delta)$ implies all the other properties if μ is isotropic, and that $P_2(\delta)$ implies all the other properties if δ is small enough. Neither of these restrictions are essential for the purposes of this work, but nevertheless we prefer to proceed with the more accessible $P_2(\delta)$ property, since in addition and in contrast to the $P_1(\delta)$ one, it is more stable in the following sense:

- (1) For any μ , if $P_2(\delta)$ holds for q , then it also holds for all p with $1 \leq p < q$.
- (2) If μ is isotropic and $P_2(\delta)$ holds for μ with q , then $P_2(\delta \sqrt{n/k})$ holds for $\pi_E \mu$ with q , simply because $Z_q(\pi_E \mu) = \text{Proj}_E Z_q(\mu)$ for all $E \in G_{n,k}$.

Consequently, we make the following:

Definition.

$$\begin{aligned} q^\sharp(\mu) &:= \sup \left\{ q \geq 1 ; \text{diam}(Z_q(\mu)) \leq c^\sharp \sqrt{n} \det \text{Cov}(\mu)^{\frac{1}{2n}} \right\} \\ &= \Delta_\mu^{-1} (c^\sharp \sqrt{n} \det \text{Cov}(\mu)^{\frac{1}{2n}}) , \end{aligned}$$

where $[1, \infty) \ni q \mapsto \Delta_\mu(q) := \text{diam}(Z_q(\mu))$ and $c^\sharp > 0$ is a small enough constant, to be prescribed in Lemma 4.3 below.

As a convention, if $\text{diam}(Z_1(\mu)) \geq c^\sharp \sqrt{n} \det \text{Cov}(\mu)^{\frac{1}{2n}}$, we set $q^\sharp(\mu) = 1$.

Lemma 4.3. *We may choose the numeric constant $c^\sharp > 0$ small enough so that:*

- (1) $q^\sharp(\mu) \leq n$.
- (2) $1 \leq q \leq q^\sharp(\mu)$ implies $k^*(Z_q(\mu)) \geq q$ and $W(Z_q(\mu)) \geq c \sqrt{q} \det \text{Cov}(\mu)^{\frac{1}{2n}}$.

Proof. Assume first that $q^\sharp(\mu) > 1$. The second point follows immediately from Lemma 4.1 and (4.4). The first point follows from (4.3), since:

$$n \cdot \det \text{Cov}(\mu)^{\frac{1}{n}} \leq \text{trace Cov}(\mu) = I_2(\mu)^2 \leq I_n(\mu)^2 \simeq W_n(Z_n(\mu))^2 \leq \text{diam}(Z_n(\mu))^2 .$$

It remains to deal with the degenerate case $q^\sharp(\mu) = 1$. By definition, $k^*(Z_1(\mu)) \geq 1$, and e.g. by (4.1):

$$W(Z_1(\mu)) \geq c \frac{\text{diam}(Z_1(\mu))}{\sqrt{n}} \geq c c^\sharp \det \text{Cov}(\mu)^{\frac{1}{2n}} ,$$

as required. □

Consequently $[q^\sharp(\mu)] \leq q^*(\mu)$, and all of Paouris' results for $q \leq q^*(\mu)$ continue to hold for $q \leq q^\sharp(\mu)$. Similarly, by Lemma 4.1, if μ is isotropic then $q_c^*(\mu) \leq q^\sharp(\mu)$ for some small constant $c > 0$. To conclude this section, we reiterate the stability of $q^\sharp(\mu)$ under projections in the following corollary, which is one of the key ingredients in the proof of Theorem 1.3:

Corollary 4.4. *Let μ denote an isotropic log-concave probability measure in \mathbb{R}^n , let $1 \leq k \leq n, q \geq 1$. Then for all $E \in G_{n,k}$ with $k \geq (c^\sharp)^{-2} \text{diam}^2(Z_q(\mu))$, we have $q^\sharp(\pi_E \mu) \geq q$. In particular $k^*(\text{Proj}_E Z_q(\mu)) \geq q$ and $W(\text{Proj}_E Z_q(\mu)) \geq c\sqrt{q}$.*

Proof. Since $\pi_E \mu$ remains isotropic, $Z_q(\pi_E \mu) = \text{Proj}_E Z_q(\mu)$ and $\text{diam}(\text{Proj}_E Z_q(\mu)) \leq \text{diam}(Z_q(\mu)) \leq c^\sharp \sqrt{k}$, the assertion follows by definition of $q^\sharp(\pi_E \mu)$ and Lemma 4.3. \square

5 Controlling $\det \text{Cov}(\mu_x)$ via projections

In view of Proposition 3.1, our goal now is to bound from below $\det \text{Cov}(\mu_x)^{\frac{1}{2n}}$ for the tilted measures μ_x , where $x \in \frac{1}{2}\Lambda_p(\mu)$. Our only available information is given by Proposition 3.7, stating that $Z_p(\mu_x) \simeq Z_p(\mu)$, where μ itself is assumed isotropic.

5.1 Finding a single good direction

Suppose ν is a log-concave probability measure on \mathbb{R}^n whose barycenter lies at the origin. Recall that its isotropic constant is defined as:

$$L_\nu := \|\nu\|_{L_\infty}^{\frac{1}{n}} \det \text{Cov}(\nu)^{\frac{1}{2n}}. \quad (5.1)$$

Since the isotropic constant L_ν satisfies $L_\nu \geq c > 0$ (see e.g. [21, 16]), then according to Lemma 2.2:

$$\det \text{Cov}(\nu)^{\frac{1}{2n}} \gtrsim \frac{1}{\|\nu\|_{L_\infty}^{\frac{1}{n}}} \simeq \frac{\text{V.Rad.}(Z_n(\nu))}{\sqrt{n}}. \quad (5.2)$$

Remark 5.1. Since $Z_n(\mu_x) \simeq Z_n(\mu)$ whenever $x \in \frac{1}{2}\Lambda_n(\mu)$, we immediately see by (5.2) and (5.1) that in this case:

$$\det \text{Cov}(\mu_x)^{\frac{1}{2n}} \gtrsim \frac{\text{V.Rad.}(Z_n(\mu_x))}{\sqrt{n}} \simeq \frac{\text{V.Rad.}(Z_n(\mu))}{\sqrt{n}} \simeq \frac{1}{\|\mu\|_{L_\infty}^{\frac{1}{n}}} \simeq \frac{\det \text{Cov}(\mu)^{\frac{1}{2n}}}{L_\mu},$$

as already noted in [16, Formula (50)]. Using the lower bound on $\text{V.Rad.}(Z_p(\mu))$ given by Lemma 3.4, it follows that:

$$\text{V.Rad.}(Z_p(\mu)) \gtrsim \frac{\sqrt{p}}{L_\mu} \text{V.Rad.}(Z_2(\mu)) \quad , \quad \forall 1 \leq p \leq n,$$

recovering the extended Lutwak–Yang–Zhang lower-bound from Remark 3.3. This however misses our goal in this section by a factor of L_μ .

We next generalize the basic estimate (5.2) to handle other (say integer) values of k between 1 and n , by projecting onto a lower dimensional subspace:

Lemma 5.2. *Let ν denote a log-concave probability measure in \mathbb{R}^n with barycenter at the origin, and let k denote an integer between 1 and n . Then:*

$$\exists \theta \in S^{n-1} \quad \sqrt{\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\nu(x)} \geq \frac{c}{\sqrt{k}} \sup_{E \in G_{n,k}} \text{V.Rad.}(Proj_E Z_k(\nu)) . \quad (5.3)$$

Proof. Given $E \in G_{n,k}$, apply (5.2) to $\pi_E \nu$ and note that $Z_k(\pi_E \nu) = Proj_E Z_k(\nu)$. \square

The idea now is to compare $\text{V.Rad.}(Proj_E Z_k(\mu_x))$ with $\text{V.Rad.}(Proj_E Z_k(\mu))$. Note that if $Z_p(\nu) \simeq Z_p(\mu)$, then by (2.3):

$$1 \leq q \leq p \quad \Rightarrow \quad c \frac{q}{p} Z_q(\mu) \subset Z_q(\nu) \subset C \frac{p}{q} Z_q(\mu) ,$$

and so $\text{V.Rad.}(Proj_E Z_k(\nu)) \geq c \frac{k}{p} \text{V.Rad.}(Proj_E Z_k(\mu))$ for all $E \in G_{n,k}$, whenever $k \leq p$. To control $\text{V.Rad.}(Proj_E Z_k(\mu))$, we have:

Lemma 5.3. *Let μ denote a log-concave probability measure in \mathbb{R}^n with barycenter at the origin, and let $1 \leq k \leq q^\sharp(\mu)$. Then:*

$$\exists E \in G_{n,k} \quad \text{V.Rad.}(Proj_E Z_k(\mu)) \geq c \sqrt{k} \det \text{Cov}(\mu)^{\frac{1}{2n}} .$$

Proof. Lemma 4.3 asserts that $1 \leq k \leq q^\sharp(\mu)$ implies that $k^*(Z_k(\mu)) \geq k$. Consequently, there exists at least one (in fact, many) $E \in G_{n,k}$ so that:

$$\frac{1}{2} W(Z_k(\mu)) B_E \subset Proj_E Z_k(\mu) \subset 2W(Z_k(\mu)) B_E ,$$

and hence $\text{V.Rad.}(Proj_E Z_k(\mu)) \geq \frac{1}{2} W(Z_k(\mu))$. It remains to appeal to Lemma 4.3 again and deduce from $1 \leq k \leq q^\sharp(\mu)$ that $W(Z_k(\mu)) \geq c \sqrt{k} \det \text{Cov}(\mu)^{\frac{1}{2n}}$. \square

Combining all of the preceding discussion, we obtain the following fundamental:

Proposition 5.4. *Let ν, μ denote two log-concave probability measures in \mathbb{R}^n with barycenters at the origin, and let $1 \leq p \leq n$. Assume that $Z_p(\nu) \simeq Z_p(\mu)$. Then:*

$$\exists \theta \in S^{n-1} \quad \sqrt{\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\nu(x)} \geq c \min \left\{ 1, \frac{q^\sharp(\mu)}{p} \right\} \det \text{Cov}(\mu)^{\frac{1}{2n}} .$$

Remark 5.5. To avoid ambiguity of our notation, we explicitly remark that throughout this section, all statements which *assume* that $Z_p(\nu) \simeq Z_p(\mu)$, in fact apply whenever $\frac{1}{B} Z_p(\mu) \subseteq Z_p(\nu) \subseteq B Z_p(\mu)$ for *any* parameter $B \geq 1$, with the resulting constants in the conclusion of those statements depending in addition on B .

5.2 Controlling the entire $\det \text{Cov}(\nu)$

We can now proceed to control the entire $\det \text{Cov}(\nu)$ by projecting onto the flag of subspaces spanned by the eigenvectors of $\text{Cov}(\nu)$. To apply Proposition 5.4, we require good control over $q^\sharp(\pi_E \mu)$. One way to obtain such control is to make a definition:

Definition. The Hereditary- q^\sharp constant of a log-concave probability measure μ on \mathbb{R}^n , denoted $q_H^\sharp(\mu)$, is defined as:

$$q_H^\sharp(\mu) := n \inf_k \inf_{E \in G_{n,k}} \frac{q^\sharp(\pi_E \mu)}{k} .$$

Remark 5.6. It is useful to note the following alternative formula for $q_H^\sharp(\mu)$, valid only for an *isotropic*, log-concave probability measure μ on \mathbb{R}^n . Recalling the definitions of $q^\sharp(\nu)$, $\Delta_\nu(q) = \text{diam}(Z_q(\nu))$, and using $\sup_{E \in G_{n,k}} \text{diam}(Proj_E Z_q(\mu)) = \text{diam}(Z_q(\mu))$, we obtain:

$$q_H^\sharp(\mu) = n \inf_{1 \leq k \leq n} \frac{\Delta_\mu^{-1}(c^\sharp \sqrt{k})}{k} \simeq n \inf_{1 \leq q \leq q^\sharp(\mu)} \frac{q}{\text{diam}(Z_q(\mu))^2} , \quad (5.4)$$

where we use (2.3) and our convention for when $q^\sharp(\nu) = 1$ to justify the last equivalence.

Proposition 5.7. Let ν, μ denote two log-concave probability measures in \mathbb{R}^n with barycenters at the origin, and assume that μ is isotropic. Let $1 \leq p \leq Aq_H^\sharp(\mu)$ with $A \geq 1$, and assume that $Z_p(\nu) \simeq Z_p(\mu)$. Then:

$$\det \text{Cov}(\nu)^{\frac{1}{2n}} \geq \frac{c}{A} ,$$

where $c > 0$ denotes a universal constant.

Proof. Let $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ denote the eigenvalues of $\text{Cov}(\nu)$, and let $E_k \in G_{n,k}$ denote the subspace spanned by the eigenvectors corresponding to $\lambda_1, \dots, \lambda_k$. Since $Proj_{E_k} Z_p(\nu) \simeq Proj_{E_k} Z_p(\mu)$, Proposition 5.4 applied to $\pi_{E_k} \nu$ and $\pi_{E_k} \mu$ implies that:

$$\sqrt{\lambda_k} \geq c \min \left(1, \frac{q^\sharp(\pi_{E_k} \mu)}{p} \right) \geq c \min \left(1, \frac{q_H^\sharp(\mu) k}{p n} \right) \geq \frac{c}{A} \frac{k}{n} .$$

Taking geometric average over the λ_k 's, the assertion immediately follows. \square

Remark 5.8. It is clear from the proof that we may actually replace in the definition of $q_H^\sharp(\mu)$ the infimum over k with a geometric-average over the terms. For future reference, we denote this variant by $q_{GH}^\sharp(\mu)$, and as in Remark 5.6, obtain the following expression for it when μ is in addition *isotropic*:

$$q_{GH}^\sharp(\mu) = n \left(\prod_{k=1}^n \frac{\Delta_\mu^{-1}(c^\sharp \sqrt{k})}{k} \right)^{\frac{1}{n}} \simeq \left(\prod_{k=1}^n \Delta_\mu^{-1}(c^\sharp \sqrt{k}) \right)^{\frac{1}{n}} . \quad (5.5)$$

Another way to obtain some (partial) control over $q^\sharp(\pi_E \mu)$ is to invoke Corollary 4.4:

Proposition 5.9. Let ν, μ denote two log-concave probability measures in \mathbb{R}^n with barycenters at the origin, and assume that μ is isotropic. Let $1 \leq p \leq n$ and $A \geq 1$. Assume that $Z_p(\nu) \simeq Z_p(\mu)$ and that:

$$\text{diam}(Z_p(\mu)) \sqrt{\log(p)} \leq A \sqrt{n} . \quad (5.6)$$

Then:

$$\det \text{Cov}(\nu)^{\frac{1}{2n}} \geq \exp(-CA^2) .$$

Proof. We employ the same notation as in the previous proof. Setting:

$$k_0 := \lceil (c^\sharp)^{-2} \text{diam}^2(Z_p(\mu)) \rceil ,$$

Corollary 4.4 states that $q^\sharp(\pi_{E_{k_0}} \mu) \geq p$. Consequently, applying Proposition 5.4 to $\pi_{E_{k_0}} \nu$ and $\pi_{E_{k_0}} \mu$, we obtain that $\lambda_{k_0} \geq c > 0$, and hence the largest $n - k_0 + 1$ eigenvalues of $\text{Cov}(\nu)$ are bounded below by the same $c > 0$. To bound the contribution of the other eigenvalues, we use (2.3) to obtain the following trivial bound (which may be improved, but ultimately only results in better numeric constants):

$$\begin{aligned} \det \text{Cov}(\pi_{E_{k_0}} \nu)^{\frac{1}{2k_0}} &= \text{V.Rad.}(Z_2(\pi_{E_{k_0}} \nu)) \gtrsim \frac{1}{p} \text{V.Rad.}(Z_p(\pi_{E_{k_0}} \nu)) \\ &\simeq \frac{1}{p} \text{V.Rad.}(Z_p(\pi_{E_{k_0}} \mu)) \geq \frac{1}{p} \text{V.Rad.}(Z_2(\pi_{E_{k_0}} \mu)) = \frac{1}{p} . \end{aligned}$$

Using our estimates separately on E_{k_0} and $E_{k_0}^\perp$, we obtain:

$$\det \text{Cov}(\nu)^{\frac{1}{2n}} = \left(\det \text{Cov}(\pi_{E_{k_0}} \nu) \det \text{Cov}(\pi_{E_{k_0}^\perp} \nu) \right)^{\frac{1}{2n}} \geq c \left(\frac{1}{p} \right)^{\frac{k_0}{n}} .$$

Our assumption (5.6) precisely ensures that $k_0 \log(p) \leq C \cdot A^2 n$, and the assertion follows. \square

Remark 5.10. Our choice of working in this section with $q^\sharp(\mu)$ instead of $q_c^*(\mu)$ is only a matter of convenience and is not of essence, as justified in Section 4.

5.3 Proofs of Main Theorems

Theorem 1.3 now follows immediately from Proposition 5.9, combined with Propositions 3.1 and 3.7. Similarly, Proposition 5.7 and Remark 5.8, combined with Propositions 3.1 and 3.7, yield:

Theorem 5.11. *Let μ denote an isotropic log-concave probability measure in \mathbb{R}^n . Then:*

$$\text{V.Rad.}(Z_p(\mu)) \geq c\sqrt{p} \quad , \quad \forall 2 \leq p \leq Cq_H^\sharp(\mu) .$$

Moreover, the same bound remains valid for $2 \leq p \leq Cq_{GH}^\sharp(\mu)$.

Now if μ is a log-concave isotropic measure on \mathbb{R}^n which is in addition a ψ_α -measure with constant b_α (for $\alpha \in [1, 2]$), by definition:

$$\text{diam}(Z_p(\mu)) \leq 2b_\alpha p^{\frac{1}{\alpha}} .$$

It therefore follows immediately from (5.4) that:

$$q_H^\sharp(\mu) \geq \frac{c}{b_\alpha^\alpha} n^{\alpha/2} ,$$

and thus Theorem 1.2 follows from Theorem 5.11.

Lastly, it may be worthwhile to record the following generalization of Theorems 1.1 and 2.1, which follows immediately, as in Subsection 2.1, from Theorem 5.11 and (5.5):

Theorem 5.12. *Let μ denote a log-concave probability measure in \mathbb{R}^n with barycenter at the origin. Then:*

$$L_\mu \leq C \left(\prod_{k=1}^n \frac{k}{\Delta_\mu^{-1}(c^\sharp \sqrt{k})} \right)^{\frac{1}{2n}} .$$

Observe that in this formulation, we only require an *on-average* control over the growth of $\Delta_\mu(p) = \text{diam}(Z_p(\mu))$, as opposed to all previously mentioned bounds on L_μ .

6 Equivalence to the Slicing Problem

Denote:

$$L_n := \sup_{K \subseteq \mathbb{R}^n} L_K , \quad (6.1)$$

where the supremum runs over all convex bodies $K \subset \mathbb{R}^n$. Recall that K is called isotropic if μ_K , the uniform measure on K , is isotropic. Recall also that $Z_p(K) = Z_p(\mu_K)$. In this section, we observe that removing the logarithmic factor in Theorem 1.3 is in fact equivalent to Bourgain's hyperplane conjecture.

Theorem 6.1. *Given $n \geq 1$, the following statements are equivalent:*

- (1) *There exists $A > 0$ so that $L_n \leq A$.*
- (2) *There exists $B > 0$ so that for any isotropic convex body $K \subset \mathbb{R}^n$, we have:*

$$\text{V.Rad.}(Z_p(K)) \geq \sqrt{p}/B \quad \forall 1 \leq p \leq q^\sharp(\mu_K)/B . \quad (6.2)$$

The equivalence is in the sense that the parameters A, B above depend solely one on the other, and not on the dimension n .

The proof is based on the following construction from Bourgain, Klartag and Milman [10]. Given $m \geq 1$, let K_m denote an isotropic convex body with $L_{K_m} \geq cL_m$. Choosing $c > 0$ appropriately, it is well-known (see, e.g., the last remark in [14]) that we may assume that K_m is centrally-symmetric and satisfies $K_m \subset 10\sqrt{m}B_m$. We also set $D_m := \sqrt{m+2}B_m$, and note that D_m is isotropic. Given $1/n \leq \lambda < 1$, consider the cartesian product:

$$T_\lambda = K_{\lfloor \lambda n \rfloor} \times D_{\lceil (1-\lambda)n \rceil} \subseteq \mathbb{R}^n .$$

Clearly, T_λ is a centrally-symmetric isotropic convex body, and since $L_{D_m} \simeq 1$, it follows that:

$$L_{T_\lambda} \simeq L_{\lfloor \lambda n \rfloor}^{\lfloor \lambda n \rfloor/n} \simeq L_{\lfloor \lambda n \rfloor}^\lambda . \quad (6.3)$$

Lemma 6.2. *For any pair of centrally-symmetric convex bodies $K_1 \subset \mathbb{R}^{n_1}, K_2 \subset \mathbb{R}^{n_2}$ and $p \geq 1$, we have:*

$$\frac{1}{2}(Z_p(K_1) \times Z_p(K_2)) \subset Z_p(K_1 \times K_2) \subset Z_p(K_1) \times Z_p(K_2) .$$

Proof. Denote $E_1 := \mathbb{R}^{n_1} \times \{0\}$ and $E_2 := \{0\} \times \mathbb{R}^{n_2}$. By definition, $Z_p(K_1 \times K_2) \cap E_1 = Z_p(K_1) \times \{0\}$ and $Z_p(K_1 \times K_2) \cap E_2 = \{0\} \times Z_p(K_2)$. By the symmetries of K_1, K_2 and convexity of $Z_p(K_1 \times K_2)$, it follows that:

$$Z_p(K_1 \times K_2) \subseteq Z_p(K_1) \times Z_p(K_2) .$$

On the other hand, an elementary argument ensures that:

$$Z_p(K_1 \times K_2) \supseteq \text{conv}(Z_p(K_1) \times \{0\}, \{0\} \times Z_p(K_2)) \supseteq \frac{1}{2} (Z_p(K_1) \times Z_p(K_2)) .$$

□

Corollary 6.3. *For any $1/n \leq \lambda \leq 1/2$:*

$$\text{diam}(Z_{\lambda n}(T_\lambda)) \leq C\sqrt{\lambda n} .$$

Proof. By Lemma 6.2 we see that:

$$\text{diam}(Z_{\lambda n}(T_\lambda)) \leq \text{diam}(Z_{\lambda n}(K_{\lfloor \lambda n \rfloor})) + \text{diam}(Z_{\lambda n}(D_{\lceil (1-\lambda)n \rceil})) .$$

Observe that $\text{diam}(Z_{\lambda n}(K_{\lfloor \lambda n \rfloor})) \leq \text{diam}(K_{\lfloor \lambda n \rfloor}) \leq 20\sqrt{\lambda n}$. As for the other summand, a straightforward computation reveals that when $1/n \leq \lambda \leq 1/2$:

$$Z_{\lambda n}(D_{\lceil (1-\lambda)n \rceil}) \simeq \sqrt{\lambda}\sqrt{n}B_{\lceil (1-\lambda)n \rceil} .$$

The assertion now follows. □

Recall that for any isotropic convex body $K \subset \mathbb{R}^n$:

$$q^\sharp(K) = q^\sharp(\mu_K) := \sup \{q \geq 1; \text{diam}(Z_q(K)) \leq c^\sharp \sqrt{n}\} , \quad (6.4)$$

where $c^\sharp > 0$ is an appropriate universal constant (as in Section 4).

Corollary 6.4. *For any $n \geq 1$, there exists a centrally-symmetric isotropic convex body $K \subset \mathbb{R}^n$, such that:*

- (a) $q^\sharp(K) \geq cn$; and
- (b) $\log L_K \geq c \log L_n$,

where $c > 0$ is a universal constant.

Proof. Take $\lambda_0 := \min\{(c^\sharp/C)^2, 1/2\}$, where C is the constant from Corollary 6.3. Then $K = T_{\lambda_0}$ satisfies the first assertion in view of the choice of λ_0 , and by (6.3):

$$L_K \simeq L_{\lfloor \lambda_0 n \rfloor}^{\lambda_0} \gtrsim L_n^{\lambda_0} ,$$

where the inequality $L_{\lfloor \lambda n \rfloor} \gtrsim L_n$ for any $0 < c \leq \lambda \leq 1$ follows from the techniques in [10, Section 3]. Since $L_K \geq c > 0$, the second assertion follows. □

Proof of Theorem 6.1. If $L_n \leq A$, then $\text{Vol}_n(K)^{\frac{1}{n}} \geq 1/A$ for any isotropic convex body $K \subset \mathbb{R}^n$. Consequently, by the Lutwak–Yang–Zhang lower-bound (1.5), we even have:

$$\text{V.Rad.}(Z_p(K)) \geq \frac{c}{A} \sqrt{p} \quad \forall 1 \leq p \leq n .$$

For the other direction, apply our assumption (6.2) to the isotropic convex body $K \subset \mathbb{R}^n$ from Corollary 6.4, and obtain:

$$\frac{\sqrt{p}}{B} \leq \text{V.Rad.}(Z_p(K)) \leq \text{V.Rad.}(K) \simeq \frac{\sqrt{n}}{L_K} \quad \forall 1 \leq p \leq q^\sharp(K)/B .$$

Corollary 6.4 then implies that:

$$L_n \leq (L_K)^C \leq \left(C' B^{\frac{3}{2}} \sqrt{\frac{n}{q^\sharp(K)}} \right)^C \leq C_1 B^{C_2} ,$$

as required. □

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