

Isomorphic and almost-isometric problems in high-dimensional convex geometry

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Abstract. The classical theorems of high-dimensional convex geometry exhibit a surprising level of regularity and order in arbitrary high-dimensional convex sets. These theorems are mainly concerned with the rough geometric features of general convex sets; the so-called “isomorphic” features. Recent results indicate that, perhaps, high-dimensional convex sets are also very regular on the almost-isometric scale. We review some related research directions in high-dimensional convex geometry, focusing in particular on the problem of geometric symmetrization.

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1. Introduction

We will begin by quoting a sample of two fundamental theorems from the asymptotic theory of finite-dimensional normed spaces. The first will be Dvoretzky's theorem (for proofs, credits and history see, e.g., [47], [58] and references therein). We work in \mathbb{R}^n , endowed with the standard Euclidean norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$. A convex body in \mathbb{R}^n is a compact, convex set with a non-empty interior.

Theorem 1.1 (Dvoretzky's theorem). *Let $K \subset \mathbb{R}^n$ be a convex body that is centrally-symmetric (i.e. $K = -K$) and let $0 < \varepsilon < 1$. Then there exist $r > 0$ and a subspace $F \subset \mathbb{R}^n$ with $\dim(F) > c\varepsilon^2 \log n$ such that*

$$(1 - \varepsilon)rD_F \subset K \cap F \subset (1 + \varepsilon)rD_F,$$

where $D_F = \{x \in F; |x| \leq 1\}$ is the Euclidean unit ball in the subspace F and $c > 0$ is a universal constant.

Theorem 1.1 reveals a basic property of centrally-symmetric convex sets in high dimension: They all contain almost-spherical sections of logarithmic dimension. The second theorem we quote is Milman's quotient of subspace theorem [53]. It presents an almost full-dimensional approximate ellipsoid that is “hidden” in a certain way within any convex body in high dimension.

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Theorem 1.2 (Milman's quotient of subspace theorem). *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body, and let $0 < \delta < \frac{1}{2}$. Then there exist subspaces $E \subset F \subset \mathbb{R}^n$ with $\dim(E) > \lceil (1 - \delta)n \rceil$ and an ellipsoid $\mathcal{E} \subset E$ such that*

$$\frac{1}{c(\delta)}\mathcal{E} \subset \text{Proj}_E(K \cap F) \subset c(\delta)\mathcal{E}.$$

Here Proj_E stands for the orthogonal projection operator onto E in \mathbb{R}^n and $c(\delta) < c \frac{1}{\delta} \log \frac{1}{\delta}$, where $c > 0$ is a universal constant.

Asymptotic convex geometry is a discipline that emerged in the 1970s and 1980s from the geometric study of Banach spaces. It is also known by other names, such as the theory of high-dimensional normed spaces, asymptotic geometric analysis, etc. Theorem 1.1 and Theorem 1.2 are typical representatives of the achievements of asymptotic convex geometry. We refer the reader to, e.g., [31] for a more complete picture of this theory. Theorem 1.1, Theorem 1.2 and other results impose stringent regularity on the geometry of a general high-dimensional convex set (the central symmetry requirement is, in many cases, not entirely essential). An important feature of these results is their broad scope; a not-so-obvious fact that we learn from the asymptotic theory of convex geometry, is that there exist non-trivial, structural geometric statements that apply to all high-dimensional convex bodies.

The precise convexity is rarely used in this theory, and corresponding principles also hold under much weaker assumptions, such as quasi convexity. The focus is on the high dimension; the theory makes sense only when the dimension n is a very large number, tending to infinity. A protagonist in many proofs of high-dimensional results is the concentration of measure phenomenon, that is, the strong concentration inequalities that typical high-dimensional measures satisfy. This phenomenon and its applications were largely put forward by Milman, starting from his proof of Dvoretzky's theorem [52]. The concentration phenomenon forces regularity and simplicity on some a priori complicated objects such as a Lipschitz function on the sphere, and is one reason for the success of the high-dimensional theory (see, e.g., the review [54]).

A key characteristic of the theory is its "isomorphic" nature. That is, the scale in which convex bodies are viewed is such that two centrally-symmetric convex bodies $K, T \subset \mathbb{R}^n$ are considered to be "close enough" when

$$c_1 K \subset T \subset c_2 K \tag{1}$$

for $c_1, c_2 > 0$ being universal constants, independent of the dimension. In other words, the norms that have K and T as their unit balls, are uniformly isomorphic. This approach is most natural to functional analysis, the origin of the subject, and has led to an interesting and elegant theory. On the other hand, even some of the most basic questions of an "almost-isometric" nature in high dimension (as opposed to "isomorphic" nature) still remain unanswered. Let us present two such "almost-isometric" problems. The first is due to Bourgain [15], and will be discussed in more detail in Section 3. One of its many formulations reads as follows:

Question 1.1 (*The slicing problem*). Does there exist $c > 0$, such that for any dimension n and every convex body $K \subset \mathbb{R}^n$ of volume one, there exists at least one hyperplane section of K whose $(n - 1)$ -dimensional volume is larger than c ?

The second question we would like to present, is the almost-isometric version of Dvoretzky's theorem.

Question 1.2. Fix a positive integer k . Do there exist $c(k), c'(k) > 0$ such that for any $0 < \varepsilon < 1$, $N = \lfloor c'(k) \left(\frac{1}{\varepsilon}\right)^{c(k)} \rfloor$ and for any centrally-symmetric convex body $K \subset \mathbb{R}^N$, one may find $r > 0$ and a k -dimensional subspace $E \subset \mathbb{R}^N$ such that

$$(1 - \varepsilon)rD_E \subset K \cap E \subset (1 + \varepsilon)rD_E ?$$

In fact, it has been conjectured (see [55]) that the answer to Question 1.2 is affirmative, with $c(k) = \frac{k-1}{2}$. This was proven by Bourgain and Lindenstrauss [20], for $k \geq 4$ and up to a factor of $\log \frac{1}{\varepsilon}$, but only when the convex set K is assumed to have certain symmetries. Question 1.2 has not even been resolved for small values of k ; in particular $k = 3$. An exception is the case $k = 2$, where a proof due to Gromov appears in [55].

Question 1.1, Question 1.2 and problems of the same spirit are sensitive to the fine geometry of the convex body K . In this sense, these questions are more related to classical convexity theory. Moreover, the answers to the above two questions are both negative, if we relax the exact convexity requirement to quasi convexity, or even to isomorphic convexity (i.e., if we only assume that the convex hull of K is contained in $2K$).

We expect that in order to better understand the almost-isometric nature of high-dimensional convex bodies, new techniques should be employed, beyond the traditional concentration of measure phenomenon. Those techniques should take into account the precise convexity of the bodies, unlike in the isomorphic theory. Next, we demonstrate the transition from isomorphic to almost-isometric behavior in a specific test problem, that of geometric symmetrization.

2. Symmetrization of convex bodies

Let $K \subset \mathbb{R}^n$ be a convex body. For any hyperplane H that passes through the origin in \mathbb{R}^n we will consider two types of symmetrization procedures. Our first symmetrization technique was described by Steiner ([66], see also [13]) in his proof of the isoperimetric inequality in two and three dimensions. Let $h \in S^{n-1}$ be a unit vector such that $H = h^\perp$. The Steiner symmetral of K with respect to the hyperplane H is the unique set $\sigma_H(K)$ for which the following two conditions hold:

1. For any $y \in H$, the set $\sigma_H(K) \cap [y + \mathbb{R}h]$ is a translation of the (possibly empty) segment $K \cap [y + \mathbb{R}h]$.

2. For any $y \in H$, the segment $\sigma_H(K) \cap [y + \mathbb{R}h]$, whenever non-empty, is centered at H .

Here $y + \mathbb{R}h$ stands for the line through y that is orthogonal to H . The set $\sigma_H(K)$ is symmetric with respect to the hyperplane H , hence the term ‘‘symmetrization’’. In addition, $\sigma_H(K)$ is convex and has the same volume as that of K . We will examine processes of symmetrization, where one begins with a convex body $K \subset \mathbb{R}^n$, and consecutively applies Steiner symmetrizations with respect to varying hyperplanes. It is a classical fact (see [25]) that given an arbitrary convex body $K \subset \mathbb{R}^n$, one may select appropriate hyperplanes H_1, H_2, \dots in \mathbb{R}^n so that the sequence of bodies

$$\sigma_{H_m} \dots (\sigma_{H_2} (\sigma_{H_1}(K))) \quad \text{for } m = 1, 2, \dots$$

converges in the Hausdorff metric to a Euclidean ball. This Euclidean ball will clearly have the same volume as that of the body we started with. Moreover, suppose we symmetrize a given convex body $K \subset \mathbb{R}^n$ with respect to randomly chosen hyperplanes, that are selected independently and uniformly over the grassmannian. Then convergence to a Euclidean ball occurs with probability one [49].

The second symmetrization procedure we consider is Minkowski symmetrization (also known as Blaschke symmetrization [9]). As before, $K \subset \mathbb{R}^n$ is a convex body and $H \subset \mathbb{R}^n$ is a hyperplane through the origin. For $x \in \mathbb{R}^n$, let $\pi_H(x)$ stand for the reflection of x with respect to H . The Minkowski symmetrization of K with respect to H is the set

$$\tau_H(K) = \frac{K + \pi_H(K)}{2},$$

where $\frac{K + \pi_H(K)}{2} = \left\{ \frac{x + \pi_H(y)}{2}; x, y \in K \right\}$ is half of the Minkowski sum of K and $\pi_H(K)$. The set $\tau_H(K)$ is convex, yet its volume is usually different from that of K . Minkowski symmetrization preserves a different characteristic of the body, namely the mean width. The mean width of K is the quantity

$$w(K) = 2 \int_{S^{n-1}} \left[\sup_{x \in K} \langle x, \theta \rangle \right] d\mu(\theta),$$

where $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ is the unit sphere in \mathbb{R}^n and μ is the unique rotationally-invariant probability measure on S^{n-1} . Thus, $w(\tau_H(K)) = w(K)$. A simple relation between Steiner and Minkowski symmetrization is that

$$\sigma_H(K) \subset \tau_H(K). \quad (2)$$

As in the case of Steiner symmetrizations, by applying an appropriate series of consecutive Minkowski symmetrizations to a given convex body $K \subset \mathbb{R}^n$, we obtain a sequence of convex bodies that converges towards a Euclidean ball. This Euclidean ball has the same mean width as the original body K .

Many geometric inequalities in which the Euclidean ball is the extremal case, may be proven using symmetrization techniques. Once we know that a certain geometric

quantity is, say, decreasing under symmetrization, we deduce that this quantity is minimized for the Euclidean ball, among all convex bodies of a given volume or mean width. For instance, from (2) we conclude that the ratio $w(K)/\text{Vol}_n(K)^{\frac{1}{n}}$ is minimal for the Euclidean ball, among all convex bodies in \mathbb{R}^n . A sample of geometric inequalities proven via symmetrization includes the Brunn–Minkowski inequality (see, e.g., [13]), Santaló’s inequality [50], Sylvester’s problem [10], best approximation by polytopes [48] and a rearrangement inequality for integrals [23].

For a convex body $K \subset \mathbb{R}^n$ and $\varepsilon > 0$, we define $S(K, \varepsilon)$ (or $M(K, \varepsilon)$) to be the minimal number ℓ for which there exist ℓ Steiner symmetrizations (or Minkowski symmetrizations) that transform K into \tilde{K} such that

$$e^{-\varepsilon} r D \subset \tilde{K} \subset e^{\varepsilon} r D,$$

where $D = \{x \in \mathbb{R}^n; |x| \leq 1\}$ is the unit Euclidean ball and $r = \left(\frac{\text{Vol}_n(K)}{\text{Vol}_n(D)}\right)^{\frac{1}{n}}$ (or $r = \frac{w(K)}{2}$). An interpretation I learned from V. Milman (e.g., [57]), is that the functions $S(K, \varepsilon)$, $M(K, \varepsilon)$ measure the complexity of the body K in the following sense. We view the Euclidean ball as the simplest of all convex bodies. If few symmetrizations are sufficient in order to transform K to become only ε -far from a Euclidean ball, then we think of K as being geometrically “simple”. Convex bodies that require a large number of symmetrizations to attain this goal are viewed as more “complex”. Define

$$S(n, \varepsilon) = \sup_{K \subset \mathbb{R}^n} S(K, \varepsilon), \quad M(n, \varepsilon) = \sup_{K \subset \mathbb{R}^n} M(K, \varepsilon), \quad (3)$$

where the suprema run over all convex bodies in \mathbb{R}^n . Consider first the isomorphic problem, where we try to symmetrize a convex body to make it close to a Euclidean ball in the isomorphic sense, as in (1). That is, we take ε in (3) to be of the order of magnitude of 1.

Theorem 2.1 ([37], [44]). *There exists a universal constant $c > 0$ such that for any dimension $n \geq 1$,*

1. $S(n, c) \leq 3n$, and
2. $M(n, c) \leq 5n$.

In addition, the slightly better inequality $M\left(n, c \frac{\log \log(n+2)}{\sqrt{\log(n+1)}}\right) \leq 5n$ holds.

Previous estimates in the literature are $M(n, c) < c'n \log n$ [21] and $S(n, c) < \tilde{c}n \log n$ [22]. See also [33] and [67]. Here, and throughout this note, the letters c, C, c', \tilde{c} etc. denote positive universal constants. These constants need not be the same from one occurrence to the next. According to Theorem 2.1, all convex sets in \mathbb{R}^n are geometrically “simple” in the above sense, at least in the isomorphic scale. Indeed, the number of symmetrizations needed to transform an arbitrary convex body into an isomorphic Euclidean ball, the simplest body, is only linear in the dimension n .

The constants “3” and “5” in Theorem 2.1 are probably not optimal, and the best constants are yet to be found. Yet, the exact constant is essentially known for a variant of our problem: Suppose we apply consecutive Steiner symmetrizations to a given convex body, and we are already satisfied when we arrive at an isomorphic ellipsoid, rather than a Euclidean ball. It is not very difficult to see (e.g. [44]) that for some convex bodies, at least $(1 - o(1))n$ Steiner symmetrizations are required in order to arrive at an isomorphic ellipsoid. The following theorem expresses the fact that roughly n symmetrizations are also sufficient, for all n -dimensional convex sets.

Theorem 2.2 ([44]). *For any $\delta > 0$, there exists a number $c(\delta) > 0$ for which the following holds: For any dimension $n \geq 1$ and a convex body $K \subset \mathbb{R}^n$, there exist an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ and $\lfloor (1 + \delta)n \rfloor$ Steiner symmetrizations that transform K into a convex body \tilde{K} such that*

$$\frac{1}{c(\delta)}\mathcal{E} \subset \tilde{K} \subset c(\delta)\mathcal{E}.$$

Moreover, $c(\delta) < c' \frac{1}{\delta} \log \frac{1}{\delta}$, where c' is a universal constant.

The proofs of Theorem 2.1 and Theorem 2.2 utilize some of the cornerstones of the asymptotic theory of convex geometry, such as concentration of measure inequalities, Kashin's splitting ([35], and the precise estimates in [30]) and Milman's quotient of subspace theorem mentioned above. Let us discuss some details from the proof of Theorem 2.1. We will focus our attention on the case of Minkowski symmetrizations, which is easier to analyze.

Given a convex body $K \subset \mathbb{R}^n$, our task is to design a sequence of symmetrizations that transform K into an approximate Euclidean ball. A plausible solution is choosing the symmetrizations randomly, that is, the hyperplanes are selected independently and uniformly. This approach was manifested in [21], and leads to the bound $M(n, c) < c'n \log n$. In fact, the effect of random Minkowski symmetrizations may be described even more precisely: For any convex body $K \subset \mathbb{R}^n$, the minimal number of *random* Minkowski symmetrizations needed in order to transform K , with reasonable probability, into an isomorphic Euclidean ball, has the order of magnitude of

$$n \log \frac{\text{diam}(K)}{w(K)}. \quad (4)$$

Here $\text{diam}(K)$ is the diameter of K (See [36] for exact formulation of this statement, based on [21]. See also [36] for a related phase-transition of the diameter in the process). We would like to emphasize that (4) is not merely a bound; it is actually an asymptotic formula for the minimal number of random symmetrizations required, valid for each convex body in \mathbb{R}^n . Just two simple geometric parameters, the diameter and the mean width, suffice to completely characterize the performance of a complicated process such as random Minkowski symmetrizations. This is a typical situation in asymptotic convex geometry (compare with [56] and [59]). The ratio $\frac{\text{diam}(K)}{w(K)}$ is never larger than $c\sqrt{n}$, when $K \subset \mathbb{R}^n$. There are convex bodies in \mathbb{R}^n for which

$\frac{\text{diam}(K)}{w(K)} > c' \sqrt{n}$; a segment in \mathbb{R}^n is an example of such a body. We thus conclude from (4) that for some convex bodies $K \subset \mathbb{R}^n$, at least $cn \log n$ random Minkowski symmetrizations are necessary in order to transform, with reasonable probability, the body K into an isomorphic Euclidean ball.

Consequently, the proof of the estimate $M(n, c) \leq 5n$ must involve a different symmetrization process: It is not efficient to simply take random, independent, Minkowski symmetrizations. This is in contrast to some other results in the theory, where the random choice is essentially the best choice (see, e.g. [59]). The approach taken in [37] is to perform several iterations, each consisting of n symmetrizations, that are carried out with respect to n mutually orthogonal hyperplanes in \mathbb{R}^n . There is more than one way of selecting these n mutually orthogonal hyperplanes. For instance, one may choose them randomly; that is, the iterations are independent, and the choice of the hyperplanes corresponds to the uniform probability measure on the orthogonal group (this leads to a proof that $M(n, c) \leq 6n$). As in [21], the proof of Theorem 2.1 still involves randomness, but of a different type.

We have explained why high-dimensional convex bodies are “simple objects”, in some sense, in the isomorphic scale. One might be tempted to believe that the true complexity of high-dimensional convex sets resides in the almost-isometric scale. Perhaps the simplicity of convex bodies, as manifested in Theorem 2.1 and in the prominent theorems of asymptotic convex geometry (e.g., Theorem 1.1 and Theorem 1.2 above), is relevant only in the isomorphic scale? For the case of symmetrization, a negative answer is provided by the next theorem.

Theorem 2.3 ([39]). *There exists a universal constant $c > 0$ such that for any dimension $n \geq 1$ and $0 < \varepsilon < \frac{1}{2}$,*

1. $M(n, \varepsilon) \leq cn \log \frac{1}{\varepsilon}$, and
2. $S(n, \varepsilon) \leq cn^4 \log^2 \frac{1}{\varepsilon}$.

The proof of Theorem 2.3 involves harmonic analysis on the sphere S^{n-1} . The dependence on n and the dependence on ε in the bound for $M(n, \varepsilon)$ are each optimal, up to the exact value of the constant c . The exponents “4” and “2” in the bound for $S(n, \varepsilon)$ are probably not optimal. Yet, the dependence on ε in Theorem 2.3 is surprisingly good. Very few results with a logarithmic dependence on the distance ε are known in high-dimensional convex geometry. Another example is described in [7] (see S. Szarek’s contribution in these proceedings for an explanation). It would be interesting to also find such good dependencies in other problems in the theory.

3. Volume distribution in convex bodies

The next family of problems we consider is related to the distribution of mass in high-dimensional convex bodies. For a convex body $K \subset \mathbb{R}^n$, let $\mathcal{E} \subset \mathbb{R}^n$ be the

Legendre ellipsoid of inertia of K ; that is, \mathcal{E} is the unique ellipsoid that has the same barycenter as K and also

$$\int_K \langle x, \theta \rangle^2 dx = \int_{\mathcal{E}} \langle x, \theta \rangle^2 dx$$

for any $\theta \in \mathbb{R}^n$. A convex body $K \subset \mathbb{R}^n$ of volume one is called isotropic if its barycenter lies at the origin and its Legendre ellipsoid is a Euclidean ball. In that case,

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2$$

independently of θ in the unit sphere S^{n-1} . The quantity L_K is the isotropic constant of the convex body K . For any convex body $K \subset \mathbb{R}^n$ there exists a unique, up to orthogonal transformations, isotropic body \tilde{K} which is an affine image of K (see, e.g., [51]). The isotropic constant of a general convex body K is defined as $L_K := L_{\tilde{K}}$, where \tilde{K} is an isotropic affine image of K .

The isotropic constant of K encompasses many of the volumetric properties of the convex body K . See [51] for a list. For instance, if K is isotropic, then for any hyperplane H through the origin,

$$\frac{c_1}{L_K} \leq \text{Vol}_{n-1}(K \cap H) \leq \frac{c_2}{L_K} \quad (5)$$

where $c_1, c_2 > 0$ are universal constants (see [34], or the survey paper [51]). Note that the relation (5) is a non-trivial rigidity property of convex bodies, in the almost-isometric scale. It is well-known (e.g. [51]) that for any dimension n and a convex body $K \subset \mathbb{R}^n$, we have $L_K > c$ for some universal constant $c > 0$. Denote,

$$L_n = \sup_{K \subset \mathbb{R}^n} L_K$$

where the supremum runs over all convex bodies in \mathbb{R}^n . Question 1.1 is equivalent (see [51]) to the following question: Is it true that $\sup_{n \geq 1} L_n < \infty$? The best estimate for the isotropic constant known to date is

$$L_n < cn^{\frac{1}{4}} \quad (6)$$

for a universal constant $c > 0$. The estimate (6), proven in [41], is a slight improvement on a previous bound $L_n < cn^{1/4} \log n$, due to Bourgain (See [16], [17], [28]. See [60] or the last remark in [38] for the non-symmetric case of Bourgain's bound). Aside from the general bound (6), an affirmative answer to Question 1.1 was obtained for large classes of convex bodies, including unconditional convex sets, zonoids, duals to zonoids, convex bodies with a bounded outer volume ratio and unit balls of Schatten norms (see, e.g., references in [41]). A reduction of the slicing problem, from general convex bodies to the simpler class of finite-volume-ratio bodies, appears in [18], [19] (see [18], [19] for precise definitions and statements).

A possible relaxation of Question 1.1 is its isomorphic version. Rather than trying to bound the isotropic constant of a given convex body $K \subset \mathbb{R}^n$, the isomorphic version asks whether there exists another convex body K' , isomorphic to K in the sense of (1), for which the isotropic constant is bounded. A positive answer is provided in the following theorem.

Theorem 3.1 ([41]). *Let $K \subset \mathbb{R}^n$ be a convex body, and let $0 < \varepsilon < 1$. Then there exists another convex body $K' \subset \mathbb{R}^n$ such that*

1. $L_{K'} < \frac{c}{\sqrt{\varepsilon}}$, and
2. for some $x_0 \in \mathbb{R}^n$,

$$(1 - \varepsilon)K' \subset K + x_0 \subset (1 + \varepsilon)K'.$$

Here, $c > 0$ is a universal constant.

Theorem 3.1 reduces the slicing problem to a question regarding the stability of the isotropic constant under isomorphic change of the body. Theorem 3.1, together with Ball's observation (see [4] or [51, page 78]), provides another derivation of the existence of a Milman ellipsoid with a universal constant, for any convex body $K \subset \mathbb{R}^n$. A Milman ellipsoid for K with constant c is an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ with $\text{Vol}_n(K) = \text{Vol}_n(\mathcal{E})$ such that K may be covered by e^{cn} translations of \mathcal{E} (see, e.g., [56] for a detailed discussion).

Given a convex body $K \subset \mathbb{R}^n$, there are several ellipsoids or Euclidean structures associated with K , such as Milman ellipsoids, the maximal volume ellipsoid, the Legendre inertia ellipsoid, the minimal surface area ellipsoid, etc. It is customary to call these Euclidean structures various "positions" of K . The relations between different positions of a convex body are not clear in general. See [43] for a certain non-trivial relation, applicable only to 2-convex bodies. As is proven in [19], Question 1.1 is equivalent to the following question: Is it true that for any convex body $K \subset \mathbb{R}^n$, the Legendre ellipsoid of K is also a Milman ellipsoid for K , with a universal constant?

A very interesting development stems from the recent Paouris theorem. Suppose that $K \subset \mathbb{R}^n$ is an isotropic convex body. Let X be a random vector, that distributes uniformly over K . Then $\mathbb{E}X = 0$. What can be said about the distribution of $|X|$? Clearly $\sqrt{\mathbb{E}|X|^2} = \sqrt{n}L_K$. Moreover, a direct consequence of the Brunn–Minkowski inequality is that $\text{Prob}(|X| > \sqrt{n}L_K t)$ decays exponentially in t , i.e., at least as fast as e^{-ct} for a universal constant $c > 0$ (see [1] for a subgaussian decay). A surprisingly strong improvement is contained in the following theorem [63], [64].

Theorem 3.2 (Paouris theorem). *Let $K \subset \mathbb{R}^n$ be an isotropic convex body. Then, for any $t \geq 1$,*

$$\text{Vol}_n(\{x \in K; |x| \geq ct\sqrt{n}L_K\}) \leq \exp(-t\sqrt{n}) \quad (7)$$

where $c > 0$ is a universal constant.

The proof of Theorem 3.2 involves, among other ingredients, a clever use of Dvoretzky's theorem. In the case where the convex body K is also assumed to be unconditional, the conclusion of Theorem 3.2 was proven in [11], [12], and for K being the normalized ℓ_1^n -ball, the result was proven in [65]. The inequality (7) is actually tight for the normalized ℓ_1^n -ball, up to the value of the constant c .

According to Theorem 3.2, all the mass of an isotropic convex body, except for a mere $e^{-\sqrt{n}}$ -fraction, lies inside a ball of radius $c\sqrt{n}L_K$ around the origin. A conjecture put forward by Anttila, Ball and Perissinaki [2] suggests that there exists a sequence $\varepsilon_n \rightarrow 0$ with the following property: Whenever $K \subset \mathbb{R}^n$ is an isotropic convex body, then for some $\rho > 0$,

$$\text{Vol}_n(\{x \in K; (1 - \varepsilon_n)\rho \leq |x| \leq (1 + \varepsilon_n)\rho\}) \geq 1 - \varepsilon_n. \quad (8)$$

This ‘‘thin shell’’ conjecture (8) was verified in [2] for unit balls of l_p^n -spaces, and for a large family of uniformly convex bodies. A positive answer to this conjecture would imply, in particular, that all high-dimensional isotropic convex bodies have many near-gaussian one-dimensional marginal distributions. See [2] for the exact formulation and proof of this implication, and see [24] for a discussion pertaining to the question of existence of near-gaussian marginals, for all high-dimensional convex bodies.

Our next topic is related to large deviation estimates for marginal distributions of general convex sets. Suppose $K \subset \mathbb{R}^n$ is a convex body of volume one, and let $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ be a linear functional. Denote $\|\varphi\|_{L^1(K)} = \int_K |\varphi(x)| dx$. A well-known consequence of the Brunn–Minkowski inequality, observed by Borell [14], is that for all $t \geq 1$,

$$\text{Vol}_n(\{x \in K; |\varphi(x)| \geq t\|\varphi\|_{L^1(K)}\}) \leq \exp(-ct) \quad (9)$$

where $c > 0$ is a universal constant. Thus, a uniform sub-exponential estimate holds for the distribution of an arbitrary linear functional on an arbitrary convex set. A typical case in which (9) is sharp, is that of a cone over an $(n - 1)$ -dimensional base; the distribution of a linear functional that vanishes on the base of the cone, is very close to being an exact exponential.

When $K \subset \mathbb{R}^n$ is an ellipsoid of volume one, the sub-exponential bound (9) may be substantially improved. It is easy to see that in this case, any linear functional $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the sub-gaussian estimate

$$\text{Vol}_n(\{x \in K; |\varphi(x)| \geq t\|\varphi\|_{L^1(K)}\}) \leq \exp(-ct^2) \quad \text{for all } t \geq 1, \quad (10)$$

where $c > 0$ is a universal constant. Inequality (10) is rather sharp, since the distribution of a linear functional on an ellipsoid is very close to being gaussian. A question that is often attributed to Milman [6], [61], [62], asks whether for any convex body $K \subset \mathbb{R}^n$ of volume one, there exists a non-zero linear functional $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$, for which a sub-gaussian estimate holds as in (10). A positive answer to this question would have the interpretation that for any convex body $K \subset \mathbb{R}^n$, there exists a direction in which, in a sense, K does not look like an apex or a cone, but rather exhibits quite regular behavior, like that of an ellipsoid or a Euclidean ball.

An affirmative answer to Milman’s question was obtained for unconditional convex bodies [11], for zonoids [61] and for some other classes of convex sets [61], [62]. A recent, general principle provides an affirmative answer to Milman’s question, up to a logarithmic factor:

Theorem 3.3 ([42]). *Let $K \subset \mathbb{R}^n$ be a convex body of volume one. Then there exists a non-zero linear functional $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$, such that for any $t \geq 1$,*

$$\text{Vol}_n(\{x \in K; |\varphi(x)| \geq t \|\varphi\|_{L^1(K)}\}) \leq \exp\left(-c \frac{t^2}{\log^5(t+1)}\right),$$

where $c > 0$ is a universal constant.

The proofs of Theorem 3.1 and Theorem 3.3 make use of several properties of the logarithmic Laplace transform of log-concave functions. We would like to conclude this section with the “random cotype-2” result of Gluskin and Milman. Suppose $K \subset \mathbb{R}^n$ is a centrally-symmetric convex body, and X_1, \dots, X_n are independent, random vectors, distributed uniformly in K . In [32] it is proven that with probability larger than $1 - e^{-cn}$,

$$\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{i=1}^n \varepsilon_i \lambda_i X_i \right\|_K > c \sqrt{\sum_{i=1}^n \lambda_i^2} \quad \text{for all } (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n, \quad (11)$$

where $\|\cdot\|_K$ is the norm whose unit ball is K and $c > 0$ is a universal constant. Consequently, any finite-dimensional norm satisfies a cotype-2 condition as in (11), with high probability, when the vectors X_1, \dots, X_n are random vectors that are selected independently and uniformly in the unit ball of that norm. See, e.g., [58, Section 9], for definitions and basic properties of type and cotype of normed spaces.

4. Beyond Brunn–Minkowski and Santaló inequalities

Some of the recent developments regarding volume distribution in high-dimensional convex sets are connected with a better understanding of log-concave functions, which are functions $f: \mathbb{R}^n \rightarrow [0, \infty)$ whose logarithm is concave. The relation between the slicing problem and log-concave functions goes back at least to [5]. Recall that the Legendre transform of a function $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as

$$\mathcal{L}\varphi(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - \varphi(y)].$$

The following result follows from the Santaló and Bourgain–Milman inequalities [3], [45]: For any measurable function $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$, there exists $x_0 \in \mathbb{R}^n$ such that $\tilde{\varphi}(x) = \varphi(x - x_0)$ satisfies

$$c \leq \left(\int_{\mathbb{R}^n} e^{-\tilde{\varphi}} \int_{\mathbb{R}^n} e^{-\mathcal{L}\tilde{\varphi}} \right)^{\frac{1}{n}} \leq 2\pi, \quad (12)$$

where $c > 0$ is a universal constant (and we agree, for the purpose of (12), that $c \leq \infty \cdot 0 \leq 2\pi$). Equality on the right hand side of (12) holds if and only if φ is a.e. a positive definite quadratic form (see [3]). For the case where φ is an even function, we may select $x_0 = 0$ in (12). In that case, the right hand side of (12) was proven by K. Ball (see [4] and also [29], for related inequalities). When φ is an even function, the following generalization holds (see [40]):

$$\int_{\mathbb{R}^n} e^{-\varphi} d\mu \int_{\mathbb{R}^n} e^{-\mathcal{L}\varphi} d\mu \leq \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} d\mu \right)^2 \quad (13)$$

where μ is any log-concave measure on \mathbb{R}^n (for example, a measure on \mathbb{R}^n whose density is a log-concave function). This is closely related to an interesting theorem of Cordero-Erausquin [26]: Suppose $K, T \subset \mathbb{R}^{2n}$ are convex bodies. We endow \mathbb{R}^{2n} with a complex structure, and assume that K, T are unit balls of complex Banach norms, and $T = \overline{T}$ where \overline{T} is the conjugate of T . Then

$$\text{Vol}_{2n}(K \cap T) \text{Vol}_{2n}(K^\circ \cap T) \leq \text{Vol}_{2n}(D \cap T)^2, \quad (14)$$

where $K^\circ = \{x \in \mathbb{R}^{2n}; \text{ for all } y \in K, \langle x, y \rangle \leq 1\}$ is the dual body. The proof of (14) uses complex interpolation and a recent complex version of the Prékopa–Leindler inequality due to Berndtsson [8]. It is not clear at the moment whether (14) generalizes to arbitrary centrally-symmetric convex sets. Inequality (13) may be viewed as a functional version of this suspected generalization. See [40] for related inequalities.

Inequality (14) suggests that, perhaps, convex bodies obey some additional geometric inequalities, beyond the classical Santaló and Brunn–Minkowski inequalities. Further evidence for this stems from the result of Cordero-Erausquin, Fradelizi and Maurey in [27]. Solving a conjecture from [46], they show that for any centrally-symmetric convex body $K \subset \mathbb{R}^n$ and $s, t > 0$,

$$\gamma_n(\sqrt{st}K) \geq \sqrt{\gamma_n(sK)\gamma_n(tK)}, \quad (15)$$

where γ_n is the standard gaussian measure in \mathbb{R}^n , whose density is given by $d\gamma_n = (2\pi)^{-n/2} \exp(-|x|^2/2)dx$. The Brunn–Minkowski type arguments only yield (15) with \sqrt{st} replaced by $\frac{s+t}{2}$ (see also F. Barthe's article in this volume). In the case where n is even, and K is the unit ball of a complex Banach norm, it is possible to replace the gaussian measure in (15) with any log-concave measure that respects the complex structure in a natural way (this follows from [26, Theorem 3.2]). It would be desirable to understand whether (14) and (15) actually hold in the context of arbitrary centrally-symmetric convex sets and arbitrary even log-concave measures, without an underlying complex structure.

Note added in proof. We would like to report on two very recent developments: First, Giannopoulos, Pajor and Paouris have simplified and slightly improved the proof of

Theorem 3.3, see <http://arxiv.org/abs/math.FA/0604299>. Second, the “thin shell” conjecture (8) has been proved by the author for all isotropic, convex sets. Consequently, typical one-dimensional marginal distributions of high-dimensional, isotropic, convex sets are approximately gaussian. Similar principles also hold for multi-dimensional marginal distributions. See <http://arxiv.org/abs/math.MG/0605014>.

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