High-dimensional distributions with convexity properties

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Abstract

We review recent advances in the understanding of probability measures with geometric characteristics on $\mathbb{R}^n$, for large n. These advances include the central limit theorem for convex sets, according to which the uniform measure on a high-dimensional convex body has marginals that are approximately gaussian.

1 Introduction

This talk is concerned with probability measures in high dimension that satisfy certain geometric convexity assumptions. Probability distributions on high dimensional spaces appear in quite a few branches of mathematics and mathematical physics. From probability theory to quantum physics, from analysis and combinatorics to statistical mechanics, it is not uncommon to study a distribution, or a family of distributions, on a space of many "equally important" parameters. These high-dimensional measures are usually, but not always, quite concrete. A general study of probability distributions in high dimension is likely hopeless, as such distributions may exhibit a wide range of entirely unrelated phenomena (see [36] for a slight exception).

There seem to exist, nevertheless, some large classes of distributions which obey some interesting, non-trivial principles. One of the earliest such examples is provided by the classical *Central Limit Theorem.* Suppose we are given a probability density $f : \mathbb{R}^n \to [0, \infty)$ which is a product density, i.e.,

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i)$$

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for some functions $f_1, \dots, f_n$. Then f is the joint density of n independent random variables $X_1, \dots, X_n$. Assume that the dimension n is large. Then under mild integrability assumptions on the f_i's, it is guaranteed that

$$\mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i \leq t\right) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-(s-b)^2/2\right) ds \qquad \forall t \in \mathbb{R}, \tag{1}$$

for appropriate coefficients $b, \theta_1, \dots, \theta_n \in \mathbb{R}$. Stated differently, any product density f has marginals that are approximately gaussian. This fact demonstrates that product densities enjoy strong regularity properties in high dimension. Moreover, when the density f is properly normalized (such that $X_1, \dots, X_n$ have mean zero and variance one), the gaussian approximation (1) actually holds for "most" choices of $\theta_1, \dots, \theta_n \in \mathbb{R}$ with $\sum_i \theta_i^2 = 1$. By "most" we mean that the coefficients $\theta_1, \dots, \theta_n$ may be chosen randomly, uniformly on the unit sphere S^{n-1} in $\mathbb{R}^n$.

The case of independent random variables is perhaps the paradigmatic example for high-dimensional measures with a clear *structure*, distributions that are composed of basic building blocks. Another source for regularity in the study of high-dimensional measures is *symmetry*; Measures that possess symmetries, whether they be apparent or hidden, are usually easier to analyze.

In this talk, we revisit the central limit theorem and related principles from a more geometric point of view. Rather than exploiting the structure or symmetries of a given high-dimensional distribution, our plan is to investigate classes of densities with certain geometric characteristics. In particular, we shall see that *convexity* conditions fit very well with high dimensionality. The study of uniform measures on arbitrary high-dimensional convex sets turns out to be quite fruitful, as well as the study of probability densities of the form $?exp(-H)$, for a convex function $H : \mathbb{R}^n \to \mathbb{R}$. The spatial arrangement of volume due to the geometry of $\mathbb{R}^n$, for large n, imposes regularity and order on such convexity-related measures.

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2 An example: The sphere

Write $|x|$ for the standard Euclidean norm of $x \in \mathbb{R}^n$, and $x \cdot y$ for the usual scalar product of $x, y \in \mathbb{R}^n$. The unit sphere in $\mathbb{R}^n$ is $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$. For a set $A \subseteq S^{n-1}$ and $\varepsilon > 0$ denote

$$A_\varepsilon = \left\{x \in S^{n-1};\, \exists y \in A,\, d(x, y) \leq \varepsilon\right\},$$

the ε-neighborhood of A. Here, d is the geodesic distance on the sphere S^{n-1}, i.e., $\cos d(x,y) = x \cdot y$. As a first example of a truly high-dimensional measure of geometric origin, we will discuss the uniform probability measure on S^{n-1}, denoted by σ_{n-1}. The rotational symmetry of σ_{n-1} yields simple answers to many geometric questions. Consider for instance the isoperimetric inequality on S^{n-1}, going back to Lévy [38] and to Schmidt [51] (see the Appendix in Figiel, Lindenstrauss and Milman [22] or Benyamini [5] for simple proofs). This inequality states that for any Borel set $A \subset S^{n-1}$ and $\varepsilon > 0$,

$$\sigma_{n-1}(A) = 1/2 \qquad \Rightarrow \qquad \sigma_{n-1}(A_\varepsilon) \geq \sigma_{n-1}(H_\varepsilon), \tag{2}$$

where $H = \{x \in S^{n-1}; x_1 \leq 0\}$ is a hemisphere. There are only a handful of scenarios, in addition to the sphere, where the isoperimetric problem is completely solved (see the recent survey by Ros [50]). In order to appreciate the quantitative consequences of the isoperimetric inequality (2), we need to estimate $\sigma_{n-1}(H_\varepsilon) = \mathbb{P}(Y_1 \leq \sin\varepsilon)$, where $Y = (Y_1, \ldots, Y_n)$ is a random vector in S^{n-1}, distributed according to σ_{n-1}. When the dimension n is large, according to Maxwell's principle,

$$\mathbb{P}(Y_1 \leq t) = C_n^{-1} \int_{-1}^{t} \left(1 - \frac{s^2 n}{n}\right)^{\frac{n-3}{2}} ds \approx \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{t} e^{-\frac{s^2 n}{2}} ds, \tag{3}$$

for $C_n = \int_{-1}^{1} (1-s^2)^{(n-3)/2} ds \approx \sqrt{2\pi/n}$. Hence Y_1 is distributed approximately like a gaussian random variable of mean zero and variance $1/n$. The variance of Y_1 is very small; even though Y_1 attains values in the entire range $[-1,1]$, it is very rare for $|Y_1|$ to reach values as high as $1/10$. We thus arrive at the following surprising conclusion, which seems to contradict our low-dimensional intuition: Most of the mass of the sphere S^{n-1} in high dimension, is concentrated on a very narrow strip near the equator $\{x \in S^{n-1}; x_1 = 0\}$. The same is true, of course, for all other equators in S^{n-1}. This peculiar high-dimensional effect is called the "concentration of measure" phenomenon. See Milman [42, 43] for a thorough review of this phenomenon and its applications.

Returning to the isoperimetric inequality (2), standard computations (e.g., [44, Section 2]) show that

$$\sigma_{n-1}(H_\varepsilon) \geq 1 - \exp(-\varepsilon^2 n/2). \tag{4}$$

The strong quantitative information (4), when plugged into the isoperimetric inequality (2) shows that whenever $A \subset S^{n-1}$ has measure $1/2$, its ε-neighborhood covers almost the entire sphere, in sense of measure. Another useful consequence is the following corollary (see [44, Section 2 and Appendix V]).

Corollary 2.1 (Lévy's lemma). *Let $f : S^{n-1} \to \mathbb{R}$ be a 1-Lipschitz function (i.e., $f(x) - f(y) \leq d(x, y)$). Denote*

$$M = \int_{S^{n-1}} f(x) d\sigma_{n-1}(x).$$

Then for any $\varepsilon > 0$,

$$\sigma_{n-1}\left(\left\{x \in S^{n-1}; |f(x) - M| \geq \varepsilon\right\}\right) \leq C \exp(-c\varepsilon^2 n),$$

where $c, C > 0$ are universal constants.

Corollary 2.1 roughly states that Lipschitz functions on the high-dimensional sphere are effectively constant. When one evaluates such a function at, say, five randomly selected points, the typical answer will be five numbers that are very close to one another.

2.1 Sudakov's theorem

One of the conclusions we mentioned in passing was Maxwell's observation, that the marginals of σ_{n-1} are approximately gaussian when n is large. What other distributions in high dimension have approximately gaussian marginals? A fundamental result in this direction is a theorem going back to Sudakov [53] and to Diaconis and Freedman [20], to be described next. Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ with $\mathbb{E}|X|^2 < \infty$. We assume that X is normalized as follows:

$$\mathbb{E}X_i = 0, \quad \mathbb{E}X_i X_j = \delta_{i,j} \qquad \forall i, j = 1, \ldots, n. \tag{5}$$

Equivalently, all of the one-dimensional marginals of X have mean zero and variance one. A random vector that satisfies the normalization condition (5) will be called "isotropic". It turns out that the crucial property of X in the context of gaussian marginals is a certain *thin spherical shell* bound:

Theorem 2.2 *(Sudakov [53], Diaconis and Freedman [20], von Weizsäcker [54], Anttila, Ball and Perissinaki [1], Bobkov [6], ...). Let X be an isotropic random vector in $\mathbb{R}^n$ and let $\varepsilon > 0$. Assume that*

$$\mathbb{P}\left(\left|\frac{|X|}{\sqrt{n}} - 1\right| \geq \varepsilon\right) \leq \varepsilon. \tag{6}$$

Then, there exists a subset $\Theta \subseteq S^{n-1}$ with $\sigma_{n-1}(\Theta) \geq 1 - \exp(-c\sqrt{n})$, such that for any $\theta \in \Theta$ and $t \in \mathbb{R}$,

$$|\mathbb{P}(X \cdot \theta \leq t) - \Phi(t)| \leq C\left(\varepsilon + \frac{1}{n^\alpha}\right) \tag{7}$$

where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp\left(-s^2/2\right) ds$ and $C, c, \alpha > 0$ are universal constants.

The main assumption in Theorem 2.2, the inequality (6), states that most of the mass of the random vector X is contained in a thin spherical shell, whose width is only ε times its radius. This thin shell assumption is in fact necessary for the conclusion of the theorem to hold (the necessity follows from (8) below). The proof of Theorem 2.2 is a beautiful manifestation of the concentration of measure phenomenon. Let us briefly sketch the main ideas. Let Y be a random vector, distributed uniformly on the sphere S^{n-1}, which is independent of X. Fix $t \in \mathbb{R}$. Consider the function $F_t(\theta) = \mathbb{P}(X \cdot \theta \leq t)$, defined on the sphere S^{n-1}. Then,

$$\int_{S^{n-1}} F_t(\theta) d\sigma_{n-1}(\theta) = \mathbb{P}\left(|X|\, Y_1 \leq t\right).$$

Note that according to (6) and (3), the random variable $|X|$ is typically very close to $\sqrt{n}$, hence $|X| Y_1$ is approximately a standard normal random variable. Consequently,

$$\int_{S^{n-1}} F_t(\theta) d\sigma_{n-1}(\theta) = \mathbb{P}\left(|X|\, Y_1 \leq t\right) \approx \Phi(t). \tag{8}$$

In order to deduce Theorem 2.2, we would like to show that

$$F_t(\theta) \approx \Phi(t) \qquad \text{for most } \theta \in S^{n-1},$$

where "most" is interpreted in the sense of σ_{n-1}. We already know from (8) that the average of F_t on the unit sphere is close to $\Phi(t)$. We thus need to show that for most $\theta \in S^{n-1}$, the value $F_t(\theta)$ is close to the average of F. To that end, we will employ Corollary 2.1: Recall that Lipschitz functions deviate very little from their mean. The function F_t is not necessarily Lipschitz (nor continuous), yet it is possible to construct Lipschitz approximations for F_t: Take

$$\tilde{F}_t(\theta) = \mathbb{E} I_t(X \cdot \theta) \approx F_t(\theta)$$

where I_t is a Lipschitz approximation of the characteristic function of $(-\infty, t]$, see Bobkov [6] for details. This is roughly the sketch of the proof of (7) for a single, fixed value $t \in \mathbb{R}$. By considering simultaneously the values $t_i = \Phi^{-1}(i\varepsilon)$ for $i = 1, \ldots, \lfloor 1/\varepsilon \rfloor$, one concludes Theorem 2.2.

The above discussion demonstrates that the gaussian approximation property of the marginals is not necessarily associated with independent random variables. The geometry of the high-dimensional sphere is another protagonist related to gaussian approximation principles. As a matter of fact, in comparison with the case of independent random variables, the proof that the sphere's marginals are close to normal seems quite straightforward.

3 Convexity

It is easy to construct natural, isotropic probability distributions that strongly violate the *thin shell estimate* (6), and consequently do not have many approximately gaussian marginals. For instance, write σ_{n-1}^t for the uniform probability measure on the sphere of radius t, centered at the origin in $\mathbb{R}^n$, and consider the measure

$$\frac{1}{2}\left[\sigma_{n-1}^{r_1} + \sigma_{n-1}^{r_2}\right]$$

for $r_1 = \sqrt{n}/2$ and $r_2 = \sqrt{7n}/2$. All marginals of this probability measure are far from normal. Therefore, a geometric condition is needed in order to avoid this kind of examples and ensure the existence of approximately gaussian marginals. Here we follow the approach suggested by Anttila, Ball and Perissinaki [1] and by Brehm and Voigt [16], and consider the relationship between thin shell bounds and convexity assumptions.

3.1 Basic volumetric properties of convex sets

A convex body in $\mathbb{R}^n$ is a bounded, open convex set. The uniform measure on a convex body has several regularity features that are prominent mostly in high dimension. Some of these features will be reviewed next. For subsets $A, B \subset \mathbb{R}^n$ we write $A + B = \{a + b; a \in A, b \in B\}$ and also $\lambda A = \{\lambda a; a \in A\}$ for $\lambda \in \mathbb{R}$. The classical Brunn-Minkowski inequality states that for any Borel sets $A, B \subset \mathbb{R}^n$,

$$Vol_n\,(A+B)^{1/n} \geq Vol_n(A)^{1/n} + Vol_n(B)^{1/n},$$

where Vol_n is the Lebesgue measure. The Brunn-Minkowski inequality is a fundamental fact regarding the uniform measure on a convex domain (even though its formulation does not mention convexity), see, e.g., Schneider [52]. A function $f : E \to [0, \infty)$ is *log-concave* if for any $x, y \in E$ and $0 < \lambda < 1$,

$$f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

That is, a function f is log-concave if it takes the form $\exp(-H)$ for a convex function $H : E \to (-\infty, \infty]$. In particular, the characteristic function of a convex body is log-concave.

Let $K \subset \mathbb{R}^n$ be a convex body, and suppose that X is a random vector distributed uniformly on K. Let $E \subset \mathbb{R}^n$ be a subspace, and denote by $Proj_E$ the orthogonal projection operator onto E in $\mathbb{R}^n$. One of the consequences of the Brunn-Minkowski inequality is that the random vector $Proj_E(X)$ has a density in the subspace E, and this density is log-concave. Characteristic functions of convex bodies and their marginals are our main source of examples for log-concave densities. All marginals of all dimensions of a log-concave density are again log-concave, see Borell [13]. The latter fact also follows from the Prékopa-Leindler inequality which is a variant of the Brunn-Minkowski inequality, see, e.g., [31] and references therein, or the first pages of Pisier's book [49].

Certain questions regarding the uniform measure on a convex body may be reduced to one-dimensional calculus problems, by using the log-concavity of the marginals. For instance, suppose that $K \subset \mathbb{R}^n$ is a convex body of volume one whose barycenter lies at the origin, and let $\theta \in S^{n-1}$. Denote $H = \{x \in \mathbb{R}^n; x \cdot \theta = 0\}$. Then, as is proven in Ball [4], Fradelizi [24] and Hensley [29],

$$\frac{1}{\sqrt{12}} \leq Vol_{n-1}(K \cap H) \cdot \sqrt{\int_K (x \cdot \theta)^2 dx} \leq \frac{1}{\sqrt{2}}, \tag{9}$$

where Vol_{n-1} denotes $(n-1)$-dimensional volume. The inequalities in (9) are reduced, according to the Brunn-Minkowski inequality, to lower and upper bounds for $f^2(0) \int_{\mathbb{R}} t^2 f(t) dt$ where f is the log-concave density of a real-valued random variable of mean zero. It follows from (9) that when the uniform probability measure on a convex body K is isotropic, then all hyperplane sections of K through the origin have roughly the same volume.

An additional consequence of the Brunn-Minkowski inequality that may be proven in a similar way (see Borell [12]), goes as follows: For any random vector X that is distributed uniformly on a convex body in $\mathbb{R}^n$, and a linear functional φ,

$$\mathbb{P}(|\varphi(X)| \geq tM) \leq C \exp(-ct) \qquad \forall t \geq 0 \tag{10}$$

where $M = \mathbb{E}|\varphi(X)|$ and $c, C > 0$ are universal constants. In low dimension, the inequality (10) is trivial and probably useless (in two or three dimensions, the discussed probability is zero already for $t = 10$). It is the high-dimensional case in which (10) is meaningful. The large deviations estimate (10) may be generalized to the case where φ is a polynomial of degree d on $\mathbb{R}^n$, rather than a linear functional. In this case, the right-hand side of (10) has to be replaced by $C \exp(-ct^{1/d})$, see Bobkov [8], Bourgain [14], Carbery and Wright [17] and Nazarov, Sodin and Volberg [46].

3.2 Spectral gap

Let μ be an isotropic probability measure on $\mathbb{R}^n$ with a log-concave density. We are interested in approximately gaussian marginals of μ and consequently also in spherical thin shell bounds for μ. The thin shell estimate (6) would follow from a variance bound

$$\int_{\mathbb{R}^n} \left(\frac{|x|^2}{n} - 1 \right)^2 d\mu(x) \ll 1, \tag{11}$$

via Chebyshev's inequality. Here is a common line of attack on the thin-shell hypothesis (see, e.g., [9]): Rather than proving (11) directly, try to establish the inequality

$$\alpha \int_{\mathbb{R}^n} \varphi^2 d\mu \leq \int_{\mathbb{R}^n} |\nabla \varphi|^2 d\mu \tag{12}$$

for *all* smooth, μ-square-integrable functions φ with $\int \varphi d\mu = 0$. If (12) indeed holds with $\alpha \gg 1/n$, then (11) follows easily: It is simply the case $\varphi(x) = |x|^2/n - 1$. Note that (12) is actually a spectral gap problem: Write $\exp(-H)$ for the log-concave density of μ. For a smooth function φ satisfying certain growth conditions, denote

$$\triangle_\mu \varphi = \triangle \varphi - \nabla H \cdot \nabla \varphi$$

(for simplicity, assume that $H : \mathbb{R}^n \to \mathbb{R}$ is smooth). Integrating by parts, we see that

$$\int_{\mathbb{R}^n} |\nabla \varphi|^2 d\mu = - \int_{\mathbb{R}^n} \varphi \triangle_\mu \varphi d\mu.$$

The operator $-\triangle_\mu$ is thus a positive semi-definite, densely defined symmetric operator in $L^2(\mu)$, and hence it admits an extension to a self-adjoint operator (see, e.g., [19]). The minimal eigenvalue of $-\triangle_\mu$ is zero, with a constant eigenfunction. The inequality (12) is equivalent to a lower bound α for the second eigenvalue of $-\triangle_\mu$. A conjecture going back to Kannan, Lovász and Simonovits [30] is that (12) holds with $\alpha = c$, for all isotropic, log-concave probability measures, where $c > 0$ is a universal constant. Part of the appeal of this conjecture is its equivalent formulation in terms of an isoperimetric inequality for the measure μ, see Ledoux [37].

3.3 Strong uniform convexity assumptions

The spectral gap inequality (12) is known to hold, for reasonable values of α, under some strong uniform convexity assumptions. For example, denote by $\nabla^2 H$ the hessian of H. Then $\nabla^2 H \geq 0$, in the sense of symmetric matrices, as H is convex. Suppose that the following strong convexity assumption is fulfilled:

$$\nabla^2 H(x) \geq \delta I \qquad \text{for all } x \in \mathbb{R}^n, \tag{13}$$

for some $\delta > 0$, where I is the identity matrix. A Bochner-type integration by parts formula (see, e.g., [3, 18]) then states that

$$\int_{\mathbb{R}^n} (\triangle_\mu \varphi)^2 d\mu = \int_{\mathbb{R}^n} |\nabla^2 \varphi|_{HS}^2 d\mu + \int_{\mathbb{R}^n} (\nabla^2 H)(\nabla \varphi) \cdot \nabla \varphi d\mu \geq \delta \int_{\mathbb{R}^n} |\nabla \varphi|^2 d\mu \tag{14}$$

under some smoothness and growth conditions for φ, where $|\cdot|_{HS}$ is the Hilbert-Schmidt norm. Consequently,

$$(-\triangle_\mu)^2 \geq -\delta \triangle_\mu$$

in the sense of symmetric operators. Thus (12) holds with $\alpha = \delta$, as was observed by Brascamp and Lieb [15]. The assumption (13) implies, in fact, much stronger conclusions, see Bakry and Émery [3]. An additional strong convexity assumption that is known to imply a variance bound like (11) is related to the modulus of convexity. Suppose $K \subset \mathbb{R}^n$ is a convex body which is centrally symmetric (i.e., $K = -K$). Consider the norm $\| \cdot \|_K$ on $\mathbb{R}^n$ whose unit ball is K. The modulus of convexity of K is defined as

$$\delta_K(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|_K}{2} ; \|x\|_K \leq 1, \|y\|_K \leq 1, \|x-y\|_K \geq \varepsilon \right\}$$

for $\varepsilon > 0$. The modulus of convexity is always non-negative, and it is linearly invariant (unlike the condition (13)). The larger the modulus of convexity of K, the more "strictly-convex" is the boundary of K. Under certain assumptions on the modulus of convexity of K and its diameter, a thin shell bound (11) was proven by Anttila, Ball and Perissinaki [1], following the works of Arias-de-Reyna, Ball and Villa [2] and Gromov and Milman [28]. See also Milman and Sodin [40] for related isoperimetric inequalities.

4 A central limit theorem for convex bodies

Next we formulate a gaussian approximation result for marginals of general log-concave densities.

Theorem 4.1 *(see [33, 34]). Let X be an isotropic random vector in $\mathbb{R}^n$ with a log-concave density. Then there exists a subset $\Theta \subseteq S^{n-1}$, with $\sigma_{n-1}(\Theta) \geq 1 - \exp(-\sqrt{n})$, such that for any $\theta \in \Theta$ and any measurable set $A \subseteq \mathbb{R}$,*

$$\left| \mathbb{P}(X \cdot \theta \in A) - \frac{1}{\sqrt{2\pi}} \int_A \exp(-s^2/2) ds \right| \leq \frac{C}{n^\alpha},$$

where $C, \alpha > 0$ are universal constants.

The isotropic normalization in Theorem 4.1 is used only to infer that *most* marginals are approximately gaussian. Without assuming that X is isotropic, we can still assert the existence of at least one approximately gaussian marginal. In accordance with the discussion above, a central ingredient in the proof of Theorem 4.1 is the following thin shell estimate: For any isotropic random vector X with a log-concave density in $\mathbb{R}^n$,

$$\mathbb{E} \left(\frac{|X|^2}{n} - 1 \right)^2 \leq \frac{C}{n^\alpha}, \tag{15}$$

for universal constants $C, \alpha > 0$ (the proof in [34] yields $\alpha \approx 1/5$). We thus arrive at the following fundamental, non-intuitive conclusion, conjectured by Anttila, Ball and Perissinaki [1]: Most of the volume of a convex body in high dimension, with the isotropic normalization, is concentrated in a very thin spherical shell.

How can we prove the bound (15) for general log-concave densities, without making strong uniform convexity assumptions? Let us consider first the very particular case where the density of X is not only log-concave, but also *radially symmetric* in $\mathbb{R}^n$. Write $f(|x|)$ for the radial density of X, where f is a log-concave function on $[0, \infty)$. Integrating in polar coordinates, we see that the density of the random variable $|X|$ is

$$t \mapsto n\kappa_n t^{n-1} f(t) \qquad (t > 0),$$

where $\kappa_n = \pi^{n/2}/\Gamma(1 + n/2)$ is the volume of the n-dimensional unit ball. Such densities are necessarily very peaked: Denote by $t_0 > 0$ the point where the maximum of $t \mapsto t^{n-1} f(t)$ is attained. A standard application of Laplace's method (see [33]) yields the bound

$$t^{n-1} f(t) \leq t_0^{n-1} f(t_0) \exp\left(-c(t - t_0)^2\right) \qquad \text{for } |t - t_0| \leq c\sqrt{n}, \qquad (16)$$

where $c > 0$ is a universal constant. The log-concavity of f is crucial for the success of Laplace's method, since it implies upper bounds for the second derivative of $\log(t^{n-1} f(t))$ at the point t_0. The bound (16) entails that $|X|$ is very concentrated near its mean: Even though $\mathbb{E}|X|$ has the order of magnitude of $\sqrt{n}$, the standard deviation of $|X|$ is bounded by a universal constant. The inequality (15) follows with $\alpha = 1$, see [33] for details. An elegant argument leading to the same conclusion, using convexity properties of the moment function, is given by Bobkov [7].

We explained how to deduce (15) in the radial case. The general log-concave case may be reduced to the radial one by using concentration of measure techniques. This idea is very much related to a remark by Gromov [26, Section 1.2.F]. Denote by $G_{n,\ell}$ the grassmannian of all ℓ-dimensional subspaces in $\mathbb{R}^n$. The grassmannian $G_{n,\ell}$ is a metric space, and it carries a unique rotationally invariant probability measure, denoted by $\sigma_{n,\ell}$, which we refer to as the uniform measure on $G_{n,\ell}$. When the dimension n is large, the uniform measure on $G_{n,\ell}$ enjoys concentration properties similar to those described in Corollary 2.1 (see Gromov and Milman [27]).

Next, suppose that X is an isotropic random vector with a log-concave density in $\mathbb{R}^n$. For a subspace $E \subset \mathbb{R}^n$, denote by $f_E : E \to [0, \infty)$ the log-concave density of the random vector $Proj_E(X)$. Let ℓ be a parameter, which will have the order of magnitude of a small, positive power of n. Our main object of study is projections of X to different ℓ-dimensional subspaces of $\mathbb{R}^n$. Fix $r > 0$. Using the log-concavity of f, it is possible to show that the map

$$(E, \theta) \quad \mapsto \quad \log f_E(r\theta) \qquad (E \in G_{n,\ell}, \theta \in S^{n-1} \cap E)$$

may be approximated by a Lipschitz function. This is a rather technical part of the argument, see [34] for the details. Then, we use concentration of measure principles on the grassmannian $G_{n,\ell}$, to conclude that the function $f_E(r\theta)$, as a function of E and θ, is "effectively constant": For "most" subspaces $E \in G_{n,\ell}$ and for "most" $\theta \in S^{n-1} \cap E$, the value $\log f_E(r\theta)$ is approximately the same. With a bit of analysis, we deduce that for "most" subspaces $E \in G_{n,\ell}$ and for *all* $\theta \in S^{n-1} \cap E$, the value $f_E(r\theta)$ is roughly the same.

By considering several values of r simultaneously, we conclude that for most subspaces $E \in G_{n,\ell}$, the function f_E is approximately radial. Recall that f_E is the marginal of the log-concave density f, and consequently f_E is also log-concave. To summarize, for most ℓ-dimensional subspaces $E \subset \mathbb{R}^n$, the function f_E is the log-concave, approximately radial density of the isotropic random vector $Proj_E(X)$. According to the already established thin-shell bound for radial, log-concave densities, for most subspaces $E \in G_{n,\ell}$,

$$\mathbb{E}\left(\frac{|Proj_E(X)|}{\sqrt{\ell}} - 1\right)^2 \le \frac{C}{\ell}. \tag{17}$$

Introduce a random ℓ-dimensional subspace $E \subset \mathbb{R}^n$, uniformly distributed in $G_{n,\ell}$, independent of X. It is well-known that $|Proj_E(X)| \approx \sqrt{\ell/n}|X|$ with high probability of selecting E. The desired bound (15) thus follows from (17), modulo details we skipped, see [33, 34] or [23] for the complete proof.

Theorem 4.1 is concerned with one-dimensional marginals. There are also corresponding principles for multi-dimensional marginals:

Theorem 4.2 *(Eldan and Klartag [21], Klartag [33, 34]). Let X be an isotropic random vector in $\mathbb{R}^n$ with a log-concave density, and let $\ell \le cn^\alpha$ be an integer. Then there exists a subset $\mathcal{E} \subseteq G_{n,\ell}$, with $\sigma_{n,\ell}(\mathcal{E}) \ge 1 - \exp(-\sqrt{n})$, such that for all $E \in \mathcal{E}$,*

1. *For any measurable set $A \subseteq E$,*
$$\left|\mathbb{P}(Proj_E(X) \in A) - \int_A \varphi_E(x)dx\right| \le \frac{C}{n^\alpha},$$
where $\varphi_E(x) = (2\pi)^{-\ell/2}\exp(-|x|^2/2)$.
2. *Denote by f_E the density of the random vector $Proj_E(X)$. Then for any $x \in E$ with $|x| \le cn^\alpha$,*
$$\left|\frac{f_E(x)}{\varphi_E(x)} - 1\right| \le \frac{C}{n^\alpha}.$$

Here, $C, c, \alpha > 0$ are universal constants.

When X has a log-concave density but is not required to be isotropic, we can still assert that $Proj_E(X)$ is approximately gaussian for *some* ℓ-dimensional subspace $E \subset \mathbb{R}^n$. Theorem 4.2 should be compared with the classical Dvoretzky theorem. In Milman's form [41], Dvoretzky's theorem states that for any convex body $K \subset \mathbb{R}^n$, there exists a subspace $E \subset \mathbb{R}^n$ of dimension $\lfloor c \log n \rfloor$, such that the geometric projection

$$Proj_E(K) = \{Proj_E(x); x \in K\}$$

is approximately a Euclidean ball in the subspace E. Here, $c > 0$ is a universal constant. The logarithmic dependence on the dimension is tight. Theorem 4.2 is concerned with the full measure projection, or marginal, of the uniform measure on K. We learn that one can project the uniform measure of the convex body $K \subset \mathbb{R}^n$ to dimensions as large as $\lfloor n^c \rfloor$, and obtain an approximate gaussian. Here, again, $c > 0$ is a universal constant.

Both the geometric projection and the measure projection of a convex body bring regularity of the best kind, either in the form of a Euclidean ball or in the form of a gaussian distribution. One may argue, however, that on a quantitative level, the projection of the uniform measure on convex bodies behaves better, in a sense, than the geometric projection: We observe a power-law dependence on the dimension, rather than a logarithmic dependence.

5 Rate of convergence

We are still lacking optimal rate of convergence results for the central limit theorem for convex bodies. The exponents α that our proofs yield for Theorem 4.1 and Theorem 4.2 are probably inferior. The main problem is with the thin shell estimate (15); it is conceivable that the correct bound should be

$$\mathbb{E}\left(\frac{|X|^2}{n} - 1\right)^2 \leq \frac{C}{n}, \tag{18}$$

for all isotropic random vectors X with a log-concave density in $\mathbb{R}^n$, see Anttila, Ball and Perissinaki [1] and Bobkov and Koldobsky [9]. There are some cases where the sharp thin shell bound (18) is proven. For example, it is common to say that a log-concave density $f : \mathbb{R}^n \to [0, \infty)$ is "unconditional" if

$$f(x_1, \ldots, x_n) = f(\pm x_1, \ldots, \pm x_n) \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

for any choice of n signs. That is, f is unconditional if it is invariant under coordinate reflections. A convex body is called unconditional if its characteristic function is unconditional. Our general philosophy is that convexity is a great source of regularity in the study of high-dimensional distributions, which may substitute for structure and symmetry. The fact that sharp thin shell bounds were proven, at least so far, only under additional symmetry assumptions is certainly a weak point in our approach. Note that nevertheless, an unconditional log-concave density is only "mildly" symmetric, and that convexity plays a significant role in the analysis of these densities.

When X is an isotropic random vector in $\mathbb{R}^n$ with an unconditional, log-concave density, the bound (18) is proven and Theorem 4.1 holds with $\alpha = 1$. The proof in [35] for the unconditional case is based on the integration by parts formula (14) and some L^2-technique. An additional advantage of the unconditional case is that we may precisely describe the subset $\Theta \subseteq S^{n-1}$ from Theorem 4.1. Specifically, for any $\theta = (\theta_1, \dots, \theta_n) \in S^{n-1}$, it is proven in [35] that

$$\left| \mathbb{P}\left(\sum_{i=1}^n \theta_i X_j \leq t \right) - \Phi(t) \right| \leq C \sum_{i=1}^n \theta_i^4 \qquad \forall t \in \mathbb{R},$$

where $C > 0$ is a universal constant. We may thus take Θ in Theorem 4.1 to be the set of all $\theta \in S^{n-1}$ with $\sum_i \theta_i^4 \leq 50/n$. Note that for this choice, $\sigma_{n-1}(\Theta) \geq 1 - \exp(-\sqrt{n})$. Additionally, for $t \in [0, 1]$ let us define

$$Y_t = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lceil tn \rceil} X_j.$$

The analysis in [35] may be used to show that the stochastic process $(Y_t)_{0 \leq t \leq 1}$ converges, in an appropriate sense, to the standard Brownian motion.

In the unconditional case there are also available sharp large-deviations results, proven by Bobkov and Nazarov [10, 11]. For example, when X is an isotropic random vector in $\mathbb{R}^n$ with an unconditional, log-concave density, it is shown that

$$\mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \geq t \right) \leq C \exp\left(-ct^2\right) \qquad \forall 0 \leq t \leq \sqrt{n}, \tag{19}$$

where $c, C > 0$ are universal constants. When the random vector X is uniform in an unconditional convex body (a slightly stronger assumption than log-concavity), the inequality in (19) holds for all $t \geq 0$. The sub-gaussian behavior in (19) in the unconditional case should be compared with the general, sub-exponential bound of (10). One may also obtain an almost sub-gaussian bound in the general, non-unconditional case. The following was proven in [32] and in Giannopoulos, Pajor and Paouris [25]: For any random vector X distributed uniformly in a finite-dimensional convex body, there exists a non-zero linear functional φ for which the right-hand side of (10) may be improved upon to $C_\varepsilon \exp(-c_\varepsilon t^{2-\varepsilon})$, for arbitrarily small $\varepsilon > 0$. The positive coefficients $C_\varepsilon, c_\varepsilon$ depend solely on ε.

To prove (19), Bobkov and Nazarov use the Prékopa-Leindler inequality in order to reduce the problem from a general unconditional log-concave density to the "worst possible" unconditional one, which is $\exp(-\sum_i |x_i|)$ in this case. The latter density is then analyzed directly. A similar approach leads to the sharp large-deviations bound

$$\mathbb{P}\left(|X| \geq t\right) \leq C \exp\left(-ct\right) \qquad \text{for } t \geq C\sqrt{n}, \tag{20}$$

valid for all isotropic random vectors X with an unconditional, log-concave density (see [10, 11]). Here, $c, C > 0$ are universal constants.

One of the most significant and influential developments in recent years in the study of high-dimensional convex bodies is the Paouris theorem [47, 48]. It is one of the very few *sharp* quantitative results that are valid for all high-dimensional log-concave distributions. Paouris proved that the bound (20) actually holds for all isotropic random vectors X with a log-concave density, without the assumption that the density is unconditional. He observed that when E is a random ℓ-dimensional subspace in $\mathbb{R}^n$, for $\ell \leq c\sqrt{n}$, then the density f_E of $Proj_E(X)$ is typically approximately radial, in the following sense: The level set

$$\left\{x \in E; f_E(x) \geq e^{-2\ell} f_E(0)\right\} \tag{21}$$

is roughly a Euclidean ball. The dependence of ℓ on the dimension n is optimal. The proof that (21) is indeed approximately Euclidean is based on a clever use of the quantitative theory of Dvoretzky's theorem, developed mostly by Milman (see, e.g., [44]), with contributions by Litvak and Schechtman [45, 39]. Once it is known that the "effective support" of $Proj_E(X)$ (i.e., the set in (21)) is approximately a Euclidean ball, some analysis of log-concave densities – mostly one-dimensional analysis – leads to the bound (20), see [47, 48] for details.

There are currently no sharp inequalities that complement (20) for smaller values of t, in the general case. The best available thin shell bound is that for any isotropic random vector X with a log-concave density in $\mathbb{R}^n$,

$$\mathbb{P}\left(\left|\frac{|X|^2}{n} - 1\right| \geq t\right) \leq C \exp\left(-cn^{\alpha} t^{\beta}\right) \qquad \text{for } 0 < t < 1, \tag{22}$$

with, say, $\alpha = 0.33$ and $\beta = 3.33$, and $c, C > 0$ are universal constants (see [34]).

Some of the arguments we presented, especially those in Section 4, might be robust enough to permit possible generalizations to other notions of convexity. One may consider, for instance, probability measures on the surface of a convex body, rather than on the body itself, or probability densities of the form $V^{-\beta}$ for a convex function V and $\beta > 0$, as long as the tail is not "too heavy" (see Bobkov [8] for the terminology and for a review of such densities). We expect that convexity-related properties will play a role in the study of some high-dimensional distributions in the future.

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