Symmetrization and isotropic constants of convex bodies

J. Bourgain
IAS, Princeton

B. Klartag
Tel Aviv

V. Milman
Tel Aviv

November 18, 2003

Abstract

We investigate the effect of a Steiner type symmetrization on the isotropic constant of a convex body. We reduce the problem of bounding the isotropic constant of an arbitrary convex body, to the problem of bounding the isotropic constant of a finite volume ratio body. We also add two observations concerning the slicing problem. The first is the equivalence of the problem to a reverse Brunn-Minkowski inequality in isotropic position. The second is the essential monotonicity in n of $L_n = \sup_{K \subset \mathbb{R}^n} L_K$ where the supremum is taken over all convex bodies in $\mathbb{R}^n$, and L_K is the isotropic constant of K.

1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex body whose barycenter is at the origin (i.e. $b(K) = \int_K \overrightarrow{x} dx = 0$). The inertia matrix of K is the matrix M_K whose entries are $M_{i,j} = \int_K x_i x_j dx$. The isotropic constant of K, denoted by L_K, is defined as

$$L_K^2 = \frac{det(M_K)^{\frac{1}{n}}}{Vol(K)^{1+\frac{2}{n}}}.$$

The isotropic constant is invariant under linear transformations of the body. If M_K is a scalar matrix and $Vol(K) = 1$, we say that K is isotropic, or that K is in isotropic position. In this case, for any $\theta \in \mathbb{R}^n$,

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2 |\theta|^2$$

where $| \cdot |$ is the standard Euclidean norm in $\mathbb{R}^n$. Any convex body K has a unique affine image of volume one which is in isotropic position. We refer the reader to [MP] for more information concerning the isotropic position and the isotropic constant.

A major unsolved problem asks whether there exists a numerical constant C such that $L_K < C$ for every convex body in any finite dimension. This problem is called the slicing problem or the hyperplane conjecture. A positive answer to this question has many interesting consequences, see [MP]. One of these is that every convex body of volume one, has an $n-1$ dimensional section whose $n-1$ dimensional volume is greater than some constant $c > 0$. The current best estimate is $L_K < cn^{1/4} \log n$, for an arbitrary convex body $K \subset \mathbb{R}^n$ (see [Bou], or the presentation in [D]. See [Pa] for the non-symmetric case). For certain classes of convex bodies the question is affirmatively answered, such as for unconditional bodies (as observed by Bourgain, see [MP]), zonoids, duals of zonoids (see [Ba2], also for the connection with the Gordon-Lewis constant), duals to bodies with finite volume ratio (see [MP]), and more (e.g. [J]). Here, we present a reduction of the general problem to the boundness of the isotropic constant of a certain class of convex bodies: those which have a finite volume ratio. For $K \subset \mathbb{R}^n$, the volume ratio of K is defined as,

$$v.r.(K) = \sup_{\mathcal{E} \subset K} \left(\frac{Vol(K)}{Vol(\mathcal{E})} \right)^{\frac{1}{n}}$$

where the supremum is over all ellipsoids contained in K. Here we prove the following conditional proposition:

Proposition 1.1 *There exists $v > 1$ such that the following holds:*

If there exists $c_1 > 0$ such that for any n and for any $K \subset \mathbb{R}^n$, the inequality $v.r.(K) < v$ implies that $L_K < c_1$,

then there exists $c_2 > 0$ such that for any n and for any $K \subset \mathbb{R}^n$ we have $L_K < c_2$.

Next, we shall state a qualitative version of Proposition 1.1. Denote $L_n = \sup_{K \subset \mathbb{R}^n} L_K$ where the supremum is over all convex sets in $\mathbb{R}^n$, and set

$$L_n(a) = \sup\{L_K \ ; \ K \subset \mathbb{R}^n \ , \ v.r.(K) \leq a\}.$$

Then we can bound L_n by a function of $L_n(a)$ for a suitable $a > 1$. As a matter of fact, this function is almost linear:

Proposition 1.2 *For any $\delta > 0$, there exist numbers $v(\delta) > 1$, $c(\delta) > 0$ such that for any n,*

$$L_n < c(\delta) \ L_n(v(\delta))^{1+\delta}.$$

A proof of these propositions, using a symmetrization technique, is presented in Section 4. The technique itself is presented in Section 2. We prove the following proposition in Section 3.

Proposition 1.3 *If $m < n$, then $L_m < cL_n$ where c is a numerical constant.*

As observed by K. Ball (see [MP]), the hyperplane conjecture implies that a reverse Brunn-Minkowski inequality holds in the isotropic position. Answering a question posed by K. Ball to one of the authors, we show that the slicing problem is actually equivalent to a reverse Brunn-Minkowski inequality in the isotropic position. The following conditional statement is proved in Section 5:

Proposition 1.4 *Assume that there exists a constant $C > 0$, such that for any n, and for any two isotropic convex bodies $K, T \subset \mathbb{R}^n$,*

$$Vol(K+T)^{1/n} \leq C\left(Vol(K)^{1/n} + Vol(T)^{1/n}\right). \tag{1}$$

Then it follows that for any convex body $K \subset \mathbb{R}^n$,

$$L_K < C'(C)$$

where $C'(C)$ is a number that depends solely on C.

Actually, Proposition 1.4 is correct even if we restrict T to be a Euclidean ball, as is evident from the proof. Note that as proved in [M1], inequality (1) which is a reverse Brunn-Minkowski inequality, holds when K and T are in a special position called M-position (see definition in Section 3). However, the connection of an M-ellipsoid with the isotropic position is not yet clear.

Throughout the paper we denote by $c, c', \tilde{c}, C$ etc. some positive universal constants whose value is not necessarily the same on different appearances. Whenever we write $A \approx B$, we mean that there exist universal constants $c, c' > 0$ such that $cA < B < c'A$. Also, $Vol(T)$ denotes the volume of a set $T \subset \mathbb{R}^n$, relative to its affine hull.

The paper [BKM] serves as an extended introduction to this paper.

2 Symmetrization

2.1 Definition

Let $K \subset \mathbb{R}^n$ be a convex body, let $E \subset \mathbb{R}^n$ be a subspace of dimension k, and let $T \subset E$ be a k-dimensional convex body, whose barycenter is at the origin. We define the "(T,E)-symmetrization" of K as the unique body K' such that:

(i) for any $x \in E^\perp$, $Vol(K \cap (x+E)) = Vol(K' \cap (x+E))$.

(ii) for any $x \in E^\perp$ the body $K' \cap (x+E)$ is homothetic to T, and its barycenter lies in $E^\perp$.

In other words, we replace any parallel section of K, with a homothetic copy of T of the appropriate volume. This procedure of symmetrization is known in convexity, see [BF], page 79. For completeness, we shall next prove that this symmetrization preserves convexity, as follows from Brunn-Minkowski inequality.

Lemma 2.1 *K' is a convex body.*

Proof: For any $z \in E^{\perp}$, the section $(z+E) \cap K'$ is convex, as a homothetic copy of T. Let $x, y \in Proj_{E^{\perp}}(K')$ be any points, where $Proj_{E^{\perp}}$ is the orthogonal projection onto $E^{\perp}$ in $\mathbb{R}^n$. We will show that

$$conv((x+E) \cap K', (y+E) \cap K')$$
$$= \bigcup_{0 \le \lambda \le 1} \lambda \left[(x+E) \cap K'\right] + (1-\lambda)\left[(y+E) \cap K'\right] \subset K'.$$

For $z \in E^{\perp}$, denote $v(z) = Vol((z+E) \cap K') = Vol((z+E) \cap K)$. Since K is convex, by Brunn-Minkowski,

$$v(\lambda x + (1-\lambda)y)^{1/k} \ge \lambda v(x)^{1/k} + (1-\lambda) v(y)^{1/k} \tag{2}$$

where $k = dim(E)$. Since $(z+E) \cap K' = z + \left(\frac{v(z)}{Vol(T)}\right)^{1/k} T$ for any point $z \in E^{\perp}$, inequality (2) entails that

$$(\lambda x + (1-\lambda)y + E) \cap K' \supset \lambda\left[(x+E) \cap K'\right] + (1-\lambda)\left[(y+E) \cap K'\right]$$

and the lemma is proved. $\square$

2.2 The effect of a symmetrization on the isotropic constant

Let us determine the eigenvectors of the inertia matrix $M_{K'}$. These eigenvectors are also called axes of inertia of the body K'. If K is an arbitrary body of volume one with its barycenter at zero, and $\{e_1, .., e_n\}$ are its axes of inertia, then since $L_K^2 = det(M_K)^{1/n}$,

$$L_K^2 = \left(\prod_{i=1}^{n} \int_K \langle x, e_i \rangle^2 dx\right)^{\frac{1}{n}}.$$

Lemma 2.2 *Assume that K is isotropic. Let $e_1, .., e_k$ be axes of inertia of the body $T \subset E$, and let $e_{k+1}, .., e_n$ be any orthonormal basis of $E^{\perp}$. Then the orthonormal basis $\{e_1, .., e_n\}$ is a basis of inertia axes of K'.*

Proof: By property (i) from the symmetrization definition, for any $v \in E^{\perp}$

$$\int_{K'} \langle x, v\rangle^2 dx = \int_K \langle x, v\rangle^2 dx = L_K^2 |v|^2 \tag{3}$$

since K is isotropic. By property (ii), for any $v \in E^{\perp}, u \in E$,

$$\int_{K'} \langle x, v\rangle \langle x, u\rangle dx = \int_{Proj_{E^{\perp}}(K')} \langle y, v\rangle \int_{K' \cap [y+E]} \langle z, u\rangle dz dy = 0$$

since the barycenter of T is at zero. Hence, E and $E^{\perp}$ are invariant subspaces of $M_{K'}$. According to (3), the operator $M_{K'}$ restricted to $E^{\perp}$ is simply a multiple of the identity. Therefore any orthogonal basis $e_{k+1}, .., e_n$ of $E^{\perp}$ is a basis of eigenvectors of $M_{K'}$. All that remains is to select k axes of inertia in E. Let $e_1, .., e_k$ be axes of inertia of the k-dimensional body T. It is straightforward to verify that for any $u_1, u_2 \in E$,

$$\int_{K'} \langle x, u_1\rangle \langle x, u_2\rangle dx = c(K, E, T) \int_T \langle x, u_1\rangle \langle x, u_2\rangle dx$$

where $c(K, E, T) = \frac{\int_{Proj_{E^{\perp}}(K)} Vol(K \cap (x+E))^{1+2/k} dx}{Vol(T)^{1+2/k}}$ depends only on K, E, T. Therefore $e_1, .., e_k$ are also axes of inertia of K'. $\square$

We postpone the proof of the following lemma to Section 6.

Lemma 2.3 *Let f be a compactly supported non-negative function on $\mathbb{R}^n$, such that $f^{1/k}$ is concave on its support, and $\int_{\mathbb{R}^n} f(x) dx = 1$. Denote $M = \max_{x \in \mathbb{R}^n} f(x)$. Then,*

$$\frac{(k+1)(k+2)}{(n+k+1)(n+k+2)} M^{2/k} \leq \int_{\mathbb{R}^n} f(x)^{1+\frac{2}{k}} dx \leq M^{2/k}.$$

Now we can estimate L_2 norms of some linear functionals over K'.

Lemma 2.4 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin. Let $E \subset \mathbb{R}^n$ be a subspace with $dim(E) = k$, and let $T \subset E$ be a k-dimensional convex body of volume one with zero as a barycenter. Denote by K' the "(T, E)-symmetrization" of K. Then for any $v \in E$,*

$$\int_{K'} \langle x, v\rangle^2 dx \geq \left(\frac{k+1}{n+1} \right)^2 Vol(K \cap E)^{2/k} \int_T \langle x, v\rangle^2 dx$$

and,

$$\int_{K'} \langle x, v\rangle^2 dx \leq \left(\frac{n+1}{k+1} \right)^2 Vol(K \cap E)^{2/k} \int_T \langle x, v\rangle^2 dx.$$

Proof:

$$\int_{K'} \langle x, v\rangle^2 dx = \int_{Proj_{E^\perp}(K')} \int_{K'\cap(E+x)} \langle y, v\rangle^2 dy dx$$

$$= \int_{Proj_{E^\perp}(K')} Vol(K' \cap (E+x))^{1+\frac{2}{k}} dx \int_T \langle y, v\rangle^2 dy.$$

Denote $g(x) = Vol(K' \cap (x+E)) = Vol(K \cap (x+E))$. Then by Brunn-Minkowski inequality, $g^{1/k}$ is concave on its support in $E^\perp$ and $\int g = vol(K) = 1$. By Lemma 2.3,

$$\frac{(k+1)(k+2)}{(n+1)(n+2)} M^{2/k} \int_T \langle y, v\rangle^2 dy \le \int_{K'} \langle x, v\rangle^2 dx \le M^{2/k} \int_T \langle y, v\rangle^2 dy$$

where $M = \max_{x\in E^\perp} g(x)$. Since the barycenter of K is at the origin, by Theorem 1 in [F],

$$g(0) \le M \le \left(\frac{n+1}{k+1}\right)^k g(0)$$

and since $g(0) = Vol(K \cap E)$, we get

$$\left(\frac{k+1}{n+1}\right)^2 \le \frac{(k+1)(k+2)}{(n+1)(n+2)} \le \frac{\int_{K'} \langle x, v\rangle^2 dx}{Vol(K\cap E)^{\frac{2}{k}} \int_T \langle x, v\rangle^2 dx} \le \left(\frac{n+1}{k+1}\right)^2.$$

□

The following theorem connects the isotropic constant of the symmetrized body with the isotropic constants of K, T.

Theorem 2.5 *Let K be an isotropic body of volume one, E a subspace of dimension k, T a k-dimensional convex body with its barycenter at the origin, and K' the "(T,E)-symmetrization" of K. Then*

$$L_{K'} \;\approx\; L_K^{1-\frac{k}{n}} \; L_T^{k/n} \; Vol(K\cap E)^{1/n}.$$

In fact, the ratio of these two quantities is always between $\left(\frac{k+1}{n+1}\right)^{k/n}$ and $\left(\frac{n+1}{k+1}\right)^{k/n}$.

Proof: We may assume that $Vol(T) = 1$. Let $\{e_1, .., e_n\}$ be selected according to Lemma 2.2. Then,

$$L_{K'} = \left(\prod_{i=1}^n \sqrt{\int_{K'} \langle x, e_i\rangle^2 dx}\right)^{1/n} = L_K^{1-\frac{k}{n}} \left(\prod_{i=1}^k \sqrt{\int_{K'} \langle x, e_i\rangle^2 dx}\right)^{1/n}$$

where the right-most equality follows from (3). By Lemma 2.4,

$$\begin{aligned} L_{K'} &\geq L_K^{1-\frac{k}{n}} \left(\prod_{i=1}^{k} \sqrt{\left(\frac{k+1}{n+1}\right)^2 Vol(K\cap E)^{\frac{2}{k}} \int_T \langle x, e_i\rangle^2 dx} \right)^{1/n} \\ &= L_K^{1-\frac{k}{n}} \left(\frac{k+1}{n+1}\right)^{\frac{k}{n}} Vol(K\cap E)^{\frac{1}{n}} L_T^{\frac{k}{n}} \end{aligned}$$

since the vectors $e_1, .., e_k$ are inertia axes of T. Therefore,

$$L_{K'} > cL_K^{1-\frac{k}{n}} L_T^{k/n} Vol(K\cap E)^{1/n}.$$

Regarding the inverse inequality, according to the opposite inequality in Lemma 2.4 we get,

$$\begin{aligned} L'_K &\leq \left(\frac{n+1}{k+1}\right)^{\frac{k}{n}} L_K^{1-\frac{k}{n}} Vol(K\cap E)^{\frac{1}{n}} L_T^{\frac{k}{n}} \\ &< cL_K^{1-\frac{k}{n}} L_T^{k/n} Vol(K\cap E)^{1/n} \end{aligned}$$

for a different constant c. □

3 Use of an M-ellipsoid

We will need to use a special ellipsoid associated with an arbitrary convex body, called an M-ellipsoid. An M-ellipsoid is defined by the following theorem (see [M1], or chapter 7 in the book [P]):

Theorem 3.1 *Let $K \subset \mathbb{R}^n$ be a convex body. Then there exists an ellipsoid $\mathcal{E}$ with $Vol(\mathcal{E}) = Vol(K)$ such that*

$$N(K, \mathcal{E}) = \min\{\sharp A; K \subset A + \mathcal{E}\} < e^{cn}$$

where $\sharp A$ is the number of elements in the set A, and c is a numerical constant. We say that $\mathcal{E}$ is an M-ellipsoid of K (with constant c).

An M-ellipsoid may replace K in various volume computations. For example, assume that $\mathcal{E}$ is an M-ellipsoid of K. If $E \subset \mathbb{R}^n$ is a subspace, and $Proj_E$ is the orthogonal projection onto E in $\mathbb{R}^n$, then by Theorem 3.1,

$$Vol(Proj_E(K))^{1/n} \leq (e^{cn} Vol(Proj_E(\mathcal{E})))^{1/n} = c' Vol(Proj_E(\mathcal{E}))^{1/n}.$$

Lemma 3.2 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin. Let $E \subset \mathbb{R}^n$ be a subspace of any dimension. Then,*

$$Vol(K\cap E)^{1/n} > c\frac{1}{Vol(Proj_{E^\perp}(K))^{1/n}}.$$

Proof: Denote $m = dim(E)$ and $M = \max_{x \in E^\perp} Vol(K \cap (E + x))$. By Fubini and by Theorem 1 in [F],

$$Vol(K) \leq M Vol(Proj_{E^\perp} K) \leq \left(\frac{n+1}{m+1} \right)^m Vol(Proj_{E^\perp} K) Vol(K \cap E).$$

Since $Vol(K) = 1$, we obtain

$$Vol(K \cap E)^{1/n} > \left(\frac{m+1}{n+1} \right)^{\frac{m}{n}} \frac{1}{Vol(Proj_{E^\perp}(K))^{1/n}}$$

and since $\left(\frac{m+1}{n+1} \right)^{\frac{m}{n}} > c$, the lemma follows. □

Proof of Proposition 1.3: First assume that $m \geq \frac{n}{2}$. Recall that $L_n = \sup_{C \subset \mathbb{R}^n} L_C$ where the supremum is taken over all isotropic convex bodies in $\mathbb{R}^n$. This supremum is attained by a compactness argument (the collection of all convex sets modulu affine transformations is compact). Define K to be one of the bodies where the supremum is attained; i.e.

$$L_K = L_n$$

and K is isotropic and of volume one. Let $\mathcal{E}$ be an M-ellipsoid of K. Since $\mathcal{E}$ is an ellipsoid of volume one, it has at least one projection onto a subspace $E^\perp$ of dimension $n - m$, such that

$$Vol(Proj_{E^\perp} K)^{1/n} < c Vol(Proj_{E^\perp} \mathcal{E})^{1/n} < C.$$

By Lemma 3.2,

$$Vol(K \cap E)^{1/n} > c'.$$

Let T be an m-dimensional body such that $L_T = L_m$, and T is of volume one and isotropic. Denote by K' the "(T, E)-symmetrization" of K. Then $L_K = L_n \geq L_{K'}$, and by Theorem 2.5,

$$L_K \geq L_{K'} > c L_K^{1 - \frac{m}{n}} L_T^{\frac{m}{n}} Vol(K \cap E)^{1/n} > \tilde{c} L_K^{1 - \frac{m}{n}} L_T^{\frac{m}{n}}$$

or equivalently,

$$L_n = L_K > \tilde{c}^{\frac{n}{m}} L_T = \tilde{c}^{\frac{n}{m}} L_m.$$

Since we assumed that $\frac{n}{m} \leq 2$, we get $L_m < c' L_n$. Regarding the case in which $m < \frac{n}{2}$: Note that $L_m \leq L_{2m}$, since the $2m$ dimensional body which is the cartesian product of T with itself, has the same isotropic constant as T. If s is the maximal integer such that $2^s m \leq n$, then clearly $2^s m > \frac{n}{2}$, and therefore

$$L_m \leq L_{2^s m} < c' L_n.$$

□

Remark 3.3: In the proof of Proposition 1.3 we showed that for every convex body $K \subset \mathbb{R}^n$ of volume one, and for any $1 \leq k \leq n$, there exists a k-dimensional subspace E such that $Vol(K \cap E)^{1/n} > c$. This fact is a direct consequence of the existence of an M-ellipsoid, but may not be very trivial to obtain directly.

We would like to mention an additional property attributed to a body $K \subset \mathbb{R}^n$, which has the largest possible isotropic constant. For this purpose, we will quote a useful result which appears in [Ba1] and in [MP]. Our formulation is closer to the one in [MP] (Lemma 3.10, and Proposition 3.11 there). Although results in that paper are stated only for centrally-symmetric bodies, the symmetry assumption is rarely used. The generalization to non-symmetric bodies is straightforward, and reads as follows:

Lemma 3.4 *Let $K \subset \mathbb{R}^n$ be an isotropic convex body of volume one. Let $1 \leq k \leq n$ and let E be a k-codimensional subspace. Define C as the unit ball of the (non-symmetric) norm defined on $E^\perp$ as*

$$\|\theta\| = |\theta|^{1+\frac{p}{p+1}} \Bigg/ \left(\int_{K \cap E(\theta)} |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p+1}}$$

for $p = k+1$, where $E(\theta) = \{x + t\theta; x \in E, t > 0\}$ is a half of a $k-1$-codimensional subspace. Then indeed C is convex, and

$$\frac{L_C}{L_K} \approx Vol(K \cap E)^{1/k}.$$

Corollary 3.5 *Let $K \subset \mathbb{R}^n$ be a convex isotropic body of volume one, such that $L_K = L_n$. Then for any subspace $E \subset \mathbb{R}^n$ of codimension k,*

$$Vol(K \cap E)^{1/k} < c$$

where c is a numerical constant.

Proof: By Lemma 3.4,

$$Vol(K \cap E)^{\frac{1}{k}} \approx \frac{L_C}{L_K} = \frac{L_C}{L_n} \leq \frac{L_{n-k}}{L_n} < c$$

where the last inequality follows from Proposition 1.3. $\square$

4 Proof of the reduction to bodies with finite volume ratio

In this section, assume that $K \subset \mathbb{R}^n$ is a convex isotropic body of volume one, such that $L_K = L_n$. Apriori, an M-ellipsoid of K may be very different from a Euclidean ball. We shall see that Corollary 3.5 imposes stringent conditions on the axes of an M-ellipsoid.

4.1 Controlling the axes of an M-ellipsoid

Denote by κ_m the volume of a unit Euclidean ball in $\mathbb{R}^m$. It is well known that $\kappa_m^{1/m} \approx \frac{1}{\sqrt{m}}$. Let $\mathcal{E} = \left\{x \in \mathbb{R}^n; \sum_i \frac{x_i^2}{n\lambda_i^2} \leq 1\right\}$ be an M-ellipsoid of K, whose existence is guaranteed in Theorem 3.1. The axes of this ellipsoid are of lengths $\sqrt{n}\lambda_1, .., \sqrt{n}\lambda_n$, and $\left(\prod_{i=1}^n \lambda_i\right)^{1/n} \approx 1$, since the volume of an M-ellipsoid is one. Assume that the λ_i's are ordered, i.e. $\lambda_1 \leq ... \leq \lambda_n$. For convenience, and without loss of generality, we assume that n is divisible by four.

Claim 4.1 *$\lambda_{n/2} < c$, for some numerical constant c.*

Proof: Let $E \subset \mathbb{R}^n$ be any subspace of any dimension. By Lemma 3.2 and Corollary 3.5,

$$Vol(Proj_E(K))^{1/n} > \frac{c}{Vol(K \cap E^\perp)^{1/n}} > c'.$$

Let $E = sp\{e_1, .., e_{n/2}\}$, the linear space spanned by $e_1, .., e_{n/2}$. Then,

$$c < Vol(Proj_E(K))^{1/n} \leq N(K, \mathcal{E})^{1/n} \left(\kappa_{n/2} \prod_{i=1}^{n/2} \sqrt{n}\lambda_i\right)^{1/n}$$

because $Vol(Proj_E(\mathcal{E})) = \kappa_{n/2} \prod_1^{n/2} \sqrt{n}\lambda_i$. Since $(\kappa_{n/2}\sqrt{n}^{n/2})^{1/n} \approx 1$, we get that

$$\left(\prod_{i=1}^{n/2} \lambda_i\right)^{2/n} > c.$$

Hence we obtain,

$$\lambda_{n/2} \leq \left(\prod_{i=\frac{n}{2}+1}^{n} \lambda_i\right)^{2/n} = \left(\prod_{i=1}^{n} \lambda_i\right)^{2/n} \left(\prod_{i=1}^{n/2} \lambda_i\right)^{-2/n} < \tilde{c}. \quad (4)$$

□

4.2 Finite volume ratio

The following lemma, whose proof involves the notion of an M-ellipsoid, originally appears in [M2]. It can also be deduced from the proof of Corollary 7.9 in [P].

Lemma 4.2 *Let $K \subset \mathbb{R}^n$ be a convex body. Let $0 < \lambda < 1$. Then there exists a subspace G of dimension $\lfloor \lambda n \rfloor$ such that if $P : \mathbb{R}^n \to \mathbb{R}^n$ is a projection (i.e. P is linear and $P^2 = P$) such that $ker(P) = G$, then $P(K)$ has a volume ratio smaller than $c(\lambda)$, where $c(\lambda)$ is some function which depends solely on λ.*

The central theme underlying the proof which follows, is the connection between an M-ellipsoid and the isotropy ellipsoid of a body with the largest possible isotropic constant. This connection arises when we project K onto the subspace $E = sp\{e_1, .., e_{n/2}\}$, together with its covering ellipsoid. According to (4) we get that $Proj_E(\mathcal{E}) \subset c\sqrt{n}D$, so in fact the normalized Euclidean ball is an M-ellipsoid for $Proj_E(K)$. In other words, the isotropy ellipsoid and the selected M-ellipsoid of K are equivalent in a large projection. Therefore, we may combine the properties of an M-ellipsoid with the properties of the isotropy ellipsoid, to create a finite volume ratio body.

Apply Lemma 4.2 to the body $Proj_E(K)$. There exists a subspace $F \subset E$ such that $dim(F) = n/4$ and

$$v.r.(Proj_F(K)) = v.r.(Proj_F(Proj_E(K))) < C.$$

Indeed, F is the orthogonal complement in E, to the subspace G from Lemma 4.2. Denote K' as the $(D_{F^\perp}, F^\perp)$-symmetrization of K, where $D_{F^\perp}$ is the standard Euclidean ball in $F^\perp$. Then,

$$K' \cap F = Proj_F(K') = Proj_F(K)$$

is a finite volume ratio body, i.e. there exists an ellipsoid $\mathcal{F} \subset K' \cap F$ such that $\left(\frac{Vol(K' \cap F)}{Vol(\mathcal{F})}\right)^{4/n} < C$. We claim that K' has a bounded volume ratio. Indeed, the ellipsoid

$$\mathcal{E}' = \left\{\lambda x + \mu y; \lambda^2 + \mu^2 \leq 1, x \in \mathcal{F}, y \in K' \cap F^\perp\right\}$$

satisfies

$$\frac{1}{\sqrt{2}}\mathcal{E}' \subset conv\{\mathcal{F}, K' \cap F^\perp\} \subset K',$$

$$Vol(\mathcal{E}')^{1/n} \geq \frac{1}{\sqrt{2}C} Vol(Proj_F(K'))^{1/n} Vol(K' \cap F^\perp)^{1/n} \geq \frac{1}{\sqrt{2}C},$$

by Lemma 3.2. Hence $\mathcal{E}'$ is evidence of the finite volume ratio property of K'. Note also that according to Claim 4.1,

$$Vol(Proj_F(K))^{1/n} \leq N(K, \mathcal{E})^{1/n} Vol(Proj_F(\sqrt{n}\lambda_{n/2}D))^{1/n} < c.$$

Hence by Lemma 3.2 and Theorem 2.5

$$L_{K'} \approx L_n^{1/4} Vol(K \cap F^\perp)^{1/n} > c\frac{L_n^{1/4}}{Vol(Proj_F(K))^{1/n}} > c' L_n^{1/4}$$

and therefore,

$$L_n < c(L_0)^4$$

where $L_0 = L_n(\tilde{c})$ is the largest possible L_K among all convex bodies in $\mathbb{R}^n$, having volume ratio not larger than $\tilde{c}$, and Proposition 1.1 is proved. $\square$

Remark 4.3: Regarding the connection between v, L_n and $L_n(v)$; Formally, we have proved for some $v > 1$ that $L_n \lesssim (L_n(v))^4$ for all n. However, by adjusting the dimensions of the subspaces E and F, we can reduce the power of $L_n(v)$, at the expense of increasing the volume ratio constant, v. The dependence obtained using this method is quite poor: For any $0 < \theta < 1$,

$$L_n \leq e^{\frac{c}{1-\theta}} L(e^{\frac{c}{1-\theta}})^{\frac{1}{\theta}}.$$

5 The isotropic position and an M-ellipsoid

Proof of Proposition 1.4: Denote $\mathcal{D}_m = \{x \in \mathbb{R}^m; |x| \leq \kappa_m^{-1/m}\}$, a Euclidean ball of volume one. Let $K \subset \mathbb{R}^n$ be a convex isotropic body of volume one. Denote,

$$K' = \left\{ (x_1, x_2); x_1 \in \sqrt{\frac{L_{\mathcal{D}_n}}{L_K}} K, x_2 \in \sqrt{\frac{L_K}{L_{\mathcal{D}_n}}} \mathcal{D}_n \right\} \subset \mathbb{R}^{2n}.$$

Let $E \subset \mathbb{R}^{2n}$ be the subspace spanned by the first n standard unit vectors, and let $F = E^{\perp}$. We claim that K' is an isotropic body. By a reasoning similar to that in Lemma 2.2, the subspaces E and F are invariant under the action of the matrix $M_{K'}$. In addition, $M_{K'}$ acts as a multiple of the identity in both subspaces. Let us show that it is the same multiple of the identity in both subspaces, and hence $M_{K'}$ is a scalar matrix. For any $v \in E$,

$$\int_{K'} \langle x, v \rangle^2 dx = \frac{L_{\mathcal{D}_n}}{L_K} \int_K \langle x, v \rangle^2 dx = L_{\mathcal{D}_n} L_K.$$

Also, for any $v \in F$,

$$\int_{K'} \langle x, v \rangle^2 dx = \frac{L_K}{L_{\mathcal{D}_n}} \int_{\mathcal{D}_n} \langle x, v \rangle^2 dx = L_K L_{\mathcal{D}_n}.$$

Therefore K' is isotropic. According to our assumption, a reverse Brunn-Minkowski inequality holds. Hence by (1),

$$Vol(K' + \mathcal{D}_{2n})^{1/2n} < C \left(Vol(K')^{1/2n} + Vol(\mathcal{D}_{2n})^{1/2n} \right) = 2C. \quad (5)$$

But $\sqrt{\frac{L_K}{L_{\mathcal{D}_n}}}\mathcal{D}_n + \mathcal{D}_{2n} \subset K' + \mathcal{D}_{2n}$. Hence,

$$Vol(K' + \mathcal{D}_{2n})^{1/2n} > Vol\left(\sqrt{\frac{L_K}{L_{\mathcal{D}_n}}}\mathcal{D}_n + \mathcal{D}_{2n}\right)^{1/2n} > c\left(\frac{L_K}{L_{\mathcal{D}_n}}\right)^{1/4}. \tag{6}$$

Combining (5) and (6), and using the fact that $L_{\mathcal{D}_n} < c'$ we get

$$L_K < (\tilde{c}C)^4$$

and since K is arbitrary, the isotropic constant of an arbitrary convex body K in $\mathbb{R}^n$ is universally bounded. □

Remark: The proof of Proposition 1.1 uses the close relation between an M-ellipsoid and the isotropy ellipsoid of the body whose isotropic constant is as large as possible. As follows from Proposition 1.4, if we could deduce such a relation between an M-ellipsoid and the isotropy ellipsoid of an arbitrary convex body $K \subset \mathbb{R}^n$, then a universal bound for the isotropic constant will follow.

6 Appendix: Concave Functions

This section proves Lemma 2.3 in a way similar to the proofs presented in [Ba1], [F]. The following lemma reflects the fact that among all concave functions on the line, the linear function is extremal.

Lemma 6.1 *Let $f : [0, \infty) \to [0, \infty)$ be a compactly supported function such that $f^{1/k}$ is concave on its support and $a = f(0) > 0$. Let $n > 0$ and choose b such that*

$$\int_0^\infty f(x)x^n dx = \int_0^\infty \left(a^{1/k} - bx\right)_+^k x^n dx$$

where $x_+ = \max\{x, 0\}$. Then for any $p > 1$

$$\int_0^\infty f(x)^p x^n dx \geq \int_0^\infty \left(a^{1/k} - bx\right)_+^{pk} x^n dx. \tag{7}$$

Proof: Since f has a compact support, $\int_0^\infty f(x)x^n dx < \infty$, so $b > 0$. Denote $h(x) = a^{1/k} - f(x)^{1/k}$. Then h is a convex function and $h(0) = 0$. Therefore $\tilde{h}(x) = \frac{h(x)}{x}$ is increasing. Since

$$\int_0^\infty (a^{1/k} - x\tilde{h})_+^k x^n dx = \int_0^\infty (a^{1/k} - bx)_+^k x^n dx$$

it is impossible that $\tilde{h}$ is always smaller or always larger than b. The function $\tilde{h}$ is increasing, so there exists $x_0 \in [0, \infty)$ such that $\tilde{h} \leq b$ on

$[0, x_0]$ and $\tilde{h} \geq b$ on $[x_0, \infty)$. Denote $g(x) = \left(a^{1/k} - bx\right)_+^k$. In order to obtain (7) we need to prove that

$$p \int_0^\infty \int_0^{f(x)} y^{p-1} dy x^n dx \geq p \int_0^\infty \int_0^{g(x)} y^{p-1} dy x^n dx.$$

Since $(g(x) - f(x))(x - x_0) \geq 0$, and g^{p-1} is a decreasing function,

$$\int_0^{x_0} \int_{g(x)}^{f(x)} y^{p-1} dy x^n dx \geq \int_0^{x_0} \int_{g(x)}^{f(x)} g(x_0)^{p-1} dy x^n dx, \tag{8}$$

$$\int_{x_0}^\infty \int_{f(x)}^{g(x)} y^{p-1} dy x^n dx \leq \int_{x_0}^\infty \int_{f(x)}^{g(x)} g(x_0)^{p-1} dy x^n dx. \tag{9}$$

Subtracting (9) from (8), we obtain

$$\int_0^\infty \int_{g(x)}^{f(x)} y^{p-1} dy x^n dx \geq g(x_0)^{p-1} \int_0^\infty (f(x) - g(x)) x^n dx = 0$$

and the lemma is proven. □

Proof of Lemma 2.3: The inequality on the right has nothing to do with log-concavity: Since $\int_{\mathbb{R}^n} f = 1$,

$$\int_{\mathbb{R}^n} f^{1+\frac{2}{k}} = \int_{\mathbb{R}^n} f \cdot f^{\frac{2}{k}} \leq \int_{\mathbb{R}^n} f \cdot M^{\frac{2}{k}} = M^{\frac{2}{k}}.$$

Let us prove the left-most inequality. By translating f if necessary, we may assume that $f(0) = M$. We shall begin by integrating in polar coordinates:

$$\int_{\mathbb{R}^n} f(x)^{1+\frac{2}{k}} dx = \int_{S^{n-1}} \int_0^\infty f(r\theta)^{1+\frac{2}{k}} r^{n-1} dr d\theta.$$

Fix $\theta \in S^{n-1}$, and denote $g(r) = f(r\theta)$. Then $g^{1/k}$ is concave, as a restriction of a concave function to a straight line. Now, by Lemma 6.1 for $p = 1 + \frac{2}{k}$,

$$\int_0^\infty g(x)^{1+\frac{2}{k}} x^{n-1} dx \geq \int_0^\infty \left(a^{1/k} - bx\right)_+^{k+2} x^{n-1} dx \tag{10}$$

where $a = g(0)$ and b is chosen as in Lemma 6.1, i.e. $\int_0^\infty g(x) x^{n-1} dx = \int_0^\infty \left(a^{1/k} - bx\right)_+^k x^{n-1} dx$. An elementary calculation yields that

$$\frac{\int_0^\infty \left(a^{1/k} - bx\right)_+^{k+2} x^{n-1} dx}{\int_0^\infty \left(a^{1/k} - bx\right)_+^k x^{n-1} dx} = a^{2/k} \frac{(k+1)(k+2)}{(n+k+1)(n+k+2)} \tag{11}$$

where we used the fact that $\int_0^1 x^a(1-x)^b dx = \frac{a!b!}{(a+b+1)!}$. Denote $c_{n,k} = \frac{(k+1)(k+2)}{(n+k+1)(n+k+2)}$. Combining (10) and (11) we obtain

$$\int_0^\infty g(x)^{1+\frac{2}{k}} x^{n-1} dx \geq a^{2/k} c_{n,k} \int_0^\infty \left(a^{1/k} - bx\right)_+^k x^{n-1} dx$$

$$= c_{n,k} g(0)^{2/k} \int_0^\infty g(x) x^{n-1} dx$$

or in other words, for every $\theta \in S^{n-1}$,

$$\int_0^\infty f(r\theta)^{1+\frac{2}{k}} r^{n-1} dr \geq c_{n,k} f(0)^{2/k} \int_0^\infty f(r\theta) r^{n-1} dr.$$

By integrating this inequality over the sphere S^{n-1},

$$\int_{\mathbb{R}^n} f(x)^{1+\frac{2}{k}} dx \geq c_{n,k} f(0)^{2/k} \int_{\mathbb{R}^n} f(x) dx = c_{n,k} f(0)^{2/k}.$$

□

References

[Ba1] K. M. Ball, Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$. Studia Math. **88** (1988) 69–84.

[Ba2] K. M. Ball, Normed spaces with a weak-Gordon-Lewis property, Proc. of Funct. Anal., University of Texas and Austin (1987–1989), Lecture Notes in Math., vol. **1470**, Springer (1991) 36–47.

[BF] T. Bonnesen, W. Fenchel, Theory of convex bodies. Translated from German and edited by L. Boron, C. Christenson and B. Smith. BCS Associates, Moscow, Idaho USA (1987).

[Bou] J. Bourgain, On the distribution of polynomials on high dimensional convex sets, Geometric aspects of functional analysis (1989–90), Lecture Notes in Math., vol. **1469**, Springer Berlin (1991) 127–137.

[BKM] J. Bourgain, B. Klartag, V.D. Milman, A reduction of the slicing problem to finite volume ratio bodies, C. R. Acad. Sci. Paris, Ser. I **336** (2003) 331–334.

[D] S. Dar, Remarks on Bourgain's problem on slicing of convex bodies, Geometric aspects of functional analysis, Operator Theory: Advances and Applications, vol. **77** (1995) 61–66.

[F] M. Fradelizi, Sections of convex bodies through their centroid, Arch. Math **69** (1997) 515–522.

[J] M. Junge, Hyperplane conjecture for quotient spaces of L_p. Forum Math. **6**, no. 5, (1994) 617–635.

[M1] V.D. Milman, Inégalité de Brunn-Minkowski inverse et applications à le théorie locale des espaces normés. C.R. Acad. Sci. Paris, Ser. I **302** (1986) 25–28.

[M2] V.D. Milman, Geometrical inequalities and mixed volumes in the local theory of Banach spaces. Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983). Astérisque No. **131** (1985), 373–400.

[MP] V.D. Milman, A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. Geometric aspects of functional analysis (1987–88), Lecture Notes in Math., vol. **1376**, Springer Berlin, (1989) 64–104.

[Pa] G. Paouris, On the isotropic constant of non-symmetric convex bodies. Geometric aspects of functional analysis, Lecture Notes in Math., **1745**, Springer, Berlin (2000) 239–243.

[P] G. Pisier, The volume of convex bodies and Banach space geometry, Cambridge Tracts in Mathematics, Cambridge univ. Press, vol. **94** (1997).