Inner Regularization of Log-Concave Measures and Small-Ball Estimates

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Abstract

In the study of concentration properties of isotropic log-concave measures, it is often useful to first ensure that the measure has super-Gaussian marginals. To this end, a standard preprocessing step is to convolve with a Gaussian measure, but this has the disadvantage of destroying small-ball information. We propose an alternative preprocessing step for making the measure seem super-Gaussian, at least up to reasonably high moments, which does not suffer from this caveat: namely, convolving the measure with a random orthogonal image of itself. As an application of this "inner-thickening", we recover Paouris' small-ball estimates.

1 Introduction

Fix a Euclidean norm $|\cdot|$ on $\mathbb{R}^n$, and let X denote an isotropic random vector in $\mathbb{R}^n$ with log-concave density g. Recall that a random vector X in $\mathbb{R}^n$ (and its density) is called isotropic if $\mathbb{E}X = 0$ and $\mathbb{E}X \otimes X = Id$, i.e. its barycenter is at the origin and its covariance matrix is equal to the identity one. Taking traces, we observe that $\mathbb{E}|X|^2 = n$. Here and throughout we use $\mathbb{E}$ to denote expectation and $\mathbb{P}$ to denote probability. A function $g : \mathbb{R}^n \to \mathbb{R}_+$ is called log-concave if $-\log g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex. Throughout this work, C,c,c_2,C', etc. denote universal positive numeric constants, independent of any other parameter and in particular the dimension n, whose value may change from one occurrence to the next.

Any high-dimensional probability distribution which is absolutely continuous has at least one super-Gaussian marginal (e.g. [13]). Still, in the study of concentration properties of X as above, it is many times advantageous to know that *all* of the one-dimensional marginals of X are super-Gaussian, at least up to some level (see e.g. [24, 9, 14]). By this we mean that for some $p_0 \geq 2$:

$$\forall 2 \leq p \leq p_0 \quad \forall \theta \in S^{n-1} \qquad (E\,|\langle X, \theta \rangle|^p)^{\frac{1}{p}} \geq c(E\,|G_1|^p)^{\frac{1}{p}} \;, \tag{1.1}$$

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where G_1 denotes a one-dimensional standard Gaussian random variable and S^{n-1} is the Euclidean unit sphere in $\mathbb{R}^n$. It is convenient to reformulate this using the language of L_p-centroid bodies, which were introduced by E. Lutwak and G. Zhang in [16] (under a different normalization). Given a random vector X with density g on $\mathbb{R}^n$ and $p \geq 1$, the L_p-centroid body $Z_p(X) = Z_p(g) \subset \mathbb{R}^n$ is the convex set defined via its support functional $h_{Z_p(X)}$ by:

$$h_{Z_p(X)}(y) = \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^p \, g(x) dx \right)^{1/p} \ , \ y \in \mathbb{R}^n \ .$$

More generally, the *one-sided* L_p-centroid body, denoted $Z_p^+(X)$, was defined in [9] (cf. [10]) by:

$$h_{Z_p^+(X)}(y) = \left(2 \int_{\mathbb{R}^n} \langle x, y \rangle_+^p \, g(x) dx \right)^{1/p} \ , \ y \in \mathbb{R}^n \ ,$$

where as usual $a_+ := \max(a, 0)$. Note that when g is even then both definitions above coincide, and that when the barycenter of X is at the origin, $Z_2(X)$ is the Euclidean ball B_2^n if and only X is isotropic. Observing that the right-hand side of (1.1) is of the order of $\sqrt{p}$, we would like to have:

$$\forall 2 \leq p \leq p_0 \quad Z_p^+(X) \supset c\sqrt{p} B_2^n \ , \tag{1.2}$$

where $B_2^n = \{x \in \mathbb{R}^n ; |x| \leq 1\}$ is the unit Euclidean ball.

Unfortunately, we cannot in general expect to satisfy (1.2) for p_0 which grows with the dimension n. This is witnessed by X which is uniformly distributed on the n-dimensional cube $[-\sqrt{3}, \sqrt{3}]^n$ (the normalization ensures that X is isotropic), whose marginals in the directions of the axes are uniform on a constant-sized interval. Consequently, some preprocessing on X is required, which on one hand transforms it into another random variable Y whose density g satisfies (1.2), and on the other enables deducing back the desired concentration properties of X from those of Y.

A very common such construction is to convolve with a Gaussian, i.e. define $Y := (X+G_n)/\sqrt{2}$, where G_n denotes an independent standard Gaussian random vector in $\mathbb{R}^n$. In [11] (and in subsequent works like [12, 5]), the Gaussian played more of a regularizing role, but in [9], its purpose was to "thicken from inside" the distribution of X, ensuring that (1.2) is satisfied for all $p \geq 2$ (see [9, Lemma 2.3]). Regarding the transference of concentration properties, it follows from the argument in the proof of [11, Proposition 4.1] that:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq C \mathbb{P}\left(|Y| \geq \sqrt{\frac{(1+t)^2+1}{2}} \sqrt{n} \right) \quad \forall t \geq 0 \ , \tag{1.3}$$

and:

$$\mathbb{P}(|X| \leq (1-t)\sqrt{n}) \leq C \mathbb{P}\left(|Y| \leq \sqrt{\frac{(1-t)^2+1}{2}} \sqrt{n} \right) \quad \forall t \in [0,1] \ , \tag{1.4}$$

for some universal constant $C > 1$. The estimate (1.3) is perfectly satisfactory for transferring (after an adjustment of constants) deviation estimates above the expectation from $|Y|$ to $|X|$. However, note that the right-hand side of (1.4) is bounded below by $P(|Y| \leq \sqrt{n/2})$ (and in particular does not decay to 0 when $t \to 1$), and so (1.4) is meaningless for transferring *small-ball* estimates from $|Y|$ to $|X|$. Consequently, the strategies employed in [11, 12, 5, 9] did not and could not deduce the concentration properties of $|X|$ in the small-ball regime. This seems an inherent problem of adding an independent Gaussian: small-ball information is lost due to the "Gaussian-thickening".

The purpose of this note is to introduce a different inner-thickening step, which does not have the above mentioned drawback. Before formulating it, recall that X (or its density) is said to be "ψ_α with constant $D > 0$" if:

$$Z_p(X) \subset D p^{1/\alpha} Z_2(X) \quad \forall p \geq 2 \; . \tag{1.5}$$

We will simply say that "X is ψ_α", if it is ψ_α with constant $D \leq C$, and not specify explicitly the dependence of the estimates on the parameter D. By a result of Berwald [1] (or applying Borell's Lemma [3] as in [21, Appendix III]), it is well known that any X with log-concave density satisfies:

$$1 \leq p \leq q \quad \Rightarrow \quad Z_p(X) \subset Z_q(X) \subset C \frac{q}{p} Z_p(X) \; . \tag{1.6}$$

In particular, such an X is always ψ_1 with some universal constant, and so we only gain additional information when $\alpha > 1$.

Theorem 1.1. *Let X denote an isotropic random vector in $\mathbb{R}^n$ with a log-concave density, which is in addition ψ_α ($\alpha \in [1,2]$), and let X' denote an independent copy of X. Given $U \in O(n)$, the group of orthogonal linear maps in $\mathbb{R}^n$, denote:*

$$Y^U_\pm := \frac{X \pm U(X')}{\sqrt{2}} \; .$$

Then:

1. *For any $U \in O(n)$, the concentration properties of $|Y^U_\pm|$ are transferred to $|X|$ as follows:*

   $$P(|X| \geq (1+t)\sqrt{n}) \leq \left(2 \max\left(P(|Y^U_+| \geq (1+t)\sqrt{n}), P(|Y^U_-| \geq (1+t)\sqrt{n})\right)\right)^{1/2} \quad \forall t \geq 0 \; ,$$

   and:

   $$P(|X| \leq (1-t)\sqrt{n}) \leq \left(2 \max\left(P(|Y^U_+| \leq (1-t)\sqrt{n}), P(|Y^U_-| \leq (1-t)\sqrt{n})\right)\right)^{1/2} \quad \forall t \in [0,1] \; .$$

2. *For any $U \in O(n)$:*

   $$Z^+_p(Y^U_\pm) \subset C p^{1/\alpha} B^n_2 \quad \forall p \geq 2 \; . \tag{1.7}$$

3. *There exists a subset $A \subset O(n)$ with:*

$$\mu_{O(n)}(A) \geq 1 - \exp(-cn) \ ,$$

where $\mu_{O(n)}$ denotes the Haar measure on $O(n)$ normalized to have total mass 1, so that if $U \in A$ then:

$$Z_p^+(Y_\pm^U) \supset c_1 \sqrt{p} B_2^n \quad \forall p \in [2, c_2 n^{\frac{\alpha}{2}}] \ . \tag{1.8}$$

Remark 1.2. Note that when the density of X is even, then Y_+^U and Y_-^U in Theorem 1.1 are identically distributed, which renders the formulation of the conclusion more natural. However, we do not know how to make the formulation simpler in the non-even case.

Remark 1.3. Also note that $Y_\pm^U$ are isotropic random vectors, and that by the Prékopa–Leindler Theorem (e.g. [7]), they have log-concave densities.

As our main application, we manage to extend the strategy in the second named author's previous work with O. Guédon [9] to the small-ball regime, and obtain:

Corollary 1.4. *Let X denote an isotropic random vector in $\mathbb{R}^n$ with log-concave density, which is in addition ψ_α ($\alpha \in [1,2]$). Then:*

$$\mathbb{P}(\big||X| - \sqrt{n}\big| \geq t\sqrt{n}) \leq C \exp(-cn^{\frac{\alpha}{2}} \min(t^{2+\alpha}, t)) \quad \forall t \geq 0 \ , \tag{1.9}$$

and:

$$\mathbb{P}(|X| \leq \varepsilon \sqrt{n}) \leq (C\varepsilon)^{cn^{\frac{\alpha}{2}}} \quad \forall \varepsilon \in [0, 1/C] \ . \tag{1.10}$$

Corollary 1.4 is an immediate consequence of Theorem 1.1 and the following result, which is the content of [9, Theorem 4.1] (our formulation below is slightly more general, but this is what the proof gives):

Theorem (Guédon–Milman)**.** *Let Y denote an isotropic random vector in $\mathbb{R}^n$ with a log-concave density, so that in addition:*

$$c_1 \sqrt{p} B_2^n \subset Z_p^+(Y) \subset c_2 p^{1/\alpha} B_2^n \quad \forall p \in [2, c_3 n^{\frac{\alpha}{2}}] \ , \tag{1.11}$$

for some $\alpha \in [1,2]$. Then (1.9) and (1.10) hold with $X = Y$ (and perhaps different constants $C, c > 0$).

We thus obtain a preprocessing step which fuses perfectly with the approach in [9], allowing us to treat all deviation regimes *simultaneously* in a single unified framework. We point out that Corollary 1.4 by itself is not new. The *large* positive-deviation estimate:

$$P(|X| \geq (1+t)\sqrt{n}) \leq \exp(-cn^{\frac{\alpha}{2}} t) \quad \forall t \geq C \ ,$$

was first obtained by G. Paouris in [22]; it is known to be sharp, up to the value of the constants. The more general deviation estimate (1.9) was obtained in [9], improving when $t \in [0, C]$ all previously known results due to the first named author and to Fleury [11, 12, 5] (we refer to [9] for a more detailed account of these previous estimates). In that work, the convolution with Gaussian preprocessing was used, and so it was not possible to independently deduce the small-ball estimate (1.10). The latter estimate was first obtained by Paouris in [23], using the reverse Blaschke–Santaló inequality of J. Bourgain and V. Milman [4]. In comparison, our main tool in the proof of Theorem 1.1 is a covering argument in the spirit of V. Milman's M-position [17, 19, 18] (see also [25]), together with a recent lower-bound on the volume of Z_p bodies obtained in our previous joint work [14].

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2 Key Proposition

In this section, we prove the following key proposition:

Proposition 2.1. *Let X, X' be as in Theorem 1.1, let U be uniformly distributed on $O(n)$, and set:*

$$Y := \frac{X + U(X')}{\sqrt{2}} \ .$$

Then there exists a $c > 0$, so that:

$$\forall C_1 > 0 \ \ \exists c_1 > 0 \ \ \forall p \in [2, cn^{\alpha/2}] \ \ \mathbb{P}(Z_p^+(Y) \supset c_1 \sqrt{p} B_2^n) \geq 1 - \exp(-C_1 n) \ .$$

Here, as elsewhere, "uniformly distributed on $O(n)$" is with respect to the probability measure $\mu_{O(n)}$.

We begin with the following estimate due to Grünbaum [8] (see also [6, Formula (10)] or [2, Lemma 3.3] for simplified proofs):

Lemma 2.2 (Grünbaum)**.** *Let X_1 denote a random variable on $\mathbb{R}$ with log-concave density and barycenter at the origin. Then $\frac{1}{e} \leq \mathbb{P}(X_1 \geq 0) \leq 1 - \frac{1}{e}$.*

Recall that the Minkowski sum $K + L$ of two compact sets $K, L \subset \mathbb{R}^n$ is defined as the compact set given by $\{x + y; x \in K, y \in L\}$. When K, L are convex, the support functional satisfies $h_{K+L} = h_K + h_L$.

Lemma 2.3. *With the same notations as in Proposition 2.1:*

$$Z_p^+(Y) \supset \frac{1}{2\sqrt{2} e^{1/p}} (Z_p^+(X) + U(Z_p^+(X))) \ .$$

Proof. Given $\theta \in S^{n-1}$, denote $Y_1 = \langle Y, \theta \rangle$, $X_1 = \langle X, \theta \rangle$ and $X_1' = \langle U(X'), \theta \rangle$. By the Prékopa–Leindler theorem (e.g. [7]), all these one-dimensional random variables have log-concave densities, and since their barycenter is at the origin, we obtain by Lemma 2.2:

$$h^p_{Z_p^+(Y)}(\theta) = 2\mathbb{E}(Y_1)_+^p = \frac{2}{2^{p/2}} \mathbb{E}\left(X_1 + X_1'\right)_+^p \geq \frac{2}{2^{p/2}} \mathbb{E}(X_1)_+^p \mathbb{P}(X_1' \geq 0) \geq \frac{2}{e 2^{p/2}} \mathbb{E}(X_1)_+^p \ .$$

Exchanging the roles of X_1 and X_1' above, we obtain:

$$h^p_{Z_p^+(Y)}(\theta) \geq \frac{1}{e2^{p/2}} \max\left(h^p_{Z_p^+(X)}(\theta), h^p_{Z_p^+(U(X'))}(\theta) \right) .$$

Consequently:

$$h_{Z_p^+(Y)}(\theta) \geq \frac{1}{\sqrt{2}e^{1/p}} \frac{h_{Z_p^+(X)}(\theta) + h_{Z_p^+(U(X'))}(\theta)}{2} ,$$

and since $Z_p^+(U(X')) = U(Z_p^+(X')) = U(Z_p^+(X))$, the assertion follows. $\square$

Next, recall that given two compact subsets $K, L \subset \mathbb{R}^n$, the covering number $N(K,L)$ is defined as the minimum number of translates of L required to cover K. The volume-radius of a compact set $K \subset \mathbb{R}^n$ is defined as:

$$\mathrm{V.Rad.}(K) = \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B_2^n)} \right)^{\frac{1}{n}} ,$$

measuring the radius of the Euclidean ball whose volume equals the volume of K. A convex compact set with non-empty interior is called a convex body, and given a convex body K with the origin in its interior, its polar K° is the convex body given by:

$$K^\circ := \{ y \in \mathbb{R}^n ; \langle x, y \rangle \leq 1 \quad \forall x \in K \} .$$

Finally, the mean-width of a convex body K, denoted $W(K)$, is defined as $W(K) = 2\int_{S^{n-1}} h_K(\theta) d\mu_{S^{n-1}}(\theta)$, where $\mu_{S^{n-1}}$ denotes the Haar probability measure on S^{n-1}. The following two lemmas are certainly well-known; we provide a proof for completeness.

Lemma 2.4. *Let $K \subset \mathbb{R}^n$ be a convex body with barycenter at the origin, so that:*

$$N(K, B_2^n) \leq \exp(A_1 n) \quad \textit{and} \quad \textit{V.Rad.}(K) \geq a_1 > 0 .$$

Then:

$$N(K^\circ, B_2^n) \leq \exp(A_2 n) ,$$

where $A_2 \leq A_1 + \log(C/a_1)$, and $C > 0$ is a universal constant.

Proof. Set $K_s = K \cap -K$. By the covering estimate of H. König and V. Milman [15], it follows that:

$$N(K^\circ, B_2^n) \leq N(K_s^\circ, B_2^n) \leq C^n N(B_2^n, K_s) .$$

Using standard volumetric covering estimates (e.g. [25, Chapter 7]), we deduce:

$$N(K^\circ, B_2^n) \leq C^n \left(\frac{\mathrm{Vol}(B_2^n + K_s/2)}{\mathrm{Vol}(K_s/2)} \right) \leq C^n N(K_s/2, B_2^n) \frac{\mathrm{Vol}(2B_2^n)}{\mathrm{Vol}(K_s/2)} .$$

By a result of V. Milman and A. Pajor [20], it is known that $\mathrm{Vol}(K_s) \geq 2^{-n}\mathrm{Vol}(K)$, and hence:

$$N(K^\circ, B_2^n) \leq (8C)^n N(K, B_2^n) \mathrm{V.Rad.}(K)^{-n} \leq (8C/a_1)^n \exp(A_1 n) ,$$

as required. $\square$

Lemma 2.5. *Let L denote any compact set in $\mathbb{R}^n$ ($n \geq 2$), so that $N(L, B_2^n) \leq \exp(A_1 n)$. If U is uniformly distributed on $O(n)$, then:*

$$P(L \cap U(L) \subset A_3 B_2^n) \geq 1 - \exp(-A_2 n) \ ,$$

where $A_2 = A_1 + (\log 2)/2$ and $A_3 = C' \exp(6A_1)$, for some universal constant $C' > 0$.

Proof Sketch. Assume that $L \subset \cup_{i=1}^{\exp(A_1 n)} (x_i + B_2^n)$. Set $R = 4C \exp(6A_1)$, for some large enough constant $C > 0$, and without loss of generality, assume that among all translates $\{x_i\}$, $\{x_i\}_{i=1}^N$ are precisely those points lying outside of RB_2^n. Observe that for each $i = 1, \ldots, N$, the cone $\{t(x_i + B_2^n); t \geq 0\}$ carves a spherical cap of Euclidean radius at most $1/R$ on S^{n-1}. By the invariance of the Haar measures on S^{n-1} and $O(n)$ under the action of $O(n)$, it follows that for every $i, j \in \{1, \ldots, N\}$:

$$P(U(x_i + B_2^n) \cap (x_j + B_2^n) \neq \emptyset) \leq \mu_{S^{n-1}}(B_{2/R}) \ ,$$

where B_ε denotes a spherical cap on S^{n-1} of Euclidean radius ε, and recall $\mu_{S^{n-1}}$ denotes the normalized Haar measure on S^{n-1}. When $\varepsilon < 1/(2C)$, it is easy to verify that:

$$\mu_{S^{n-1}}(B_\varepsilon) \leq (C\varepsilon)^{n-1} \ ,$$

and so it follows by the union-bound that:

$$P(L \cap U(L) \subset (R+1)B_2^n) \geq P(\forall i, j \in \{1, \ldots, N\} \quad U(x_i + B_2^n) \cap (x_j + B_2^n) = \emptyset) \geq 1 - N^2 (2C/R)^{n-1} \ .$$

Since $N \leq \exp(2A_1(n-1))$, our choice of R yields the desired assertion with $C' = 5C$. □

It is also useful to state:

Lemma 2.6. *For any density g on $\mathbb{R}^n$ and $p \geq 1$:*

$$Z_p^+(g) \subset 2^{1/p} Z_p(g) \subset Z_p^+(g) - Z_p^+(g) \ . \tag{2.1}$$

Proof. The first inclusion is trivial. The second follows since $a^{1/p} + b^{1/p} \geq (a+b)^{1/p}$ for $a, b \geq 0$, and hence for all $\theta \in S^{n-1}$:

$$h_{Z_p^+(g) - Z_p^+(g)}(\theta) = h_{Z_p^+(g)}(\theta) + h_{Z_p^+(g)}(-\theta) \geq 2^{1/p} h_{Z_p(g)}(\theta) \ .$$

□

The next two theorems play a crucial role in our argument. The first is due to Paouris [22], and the second to the authors [14]:

Theorem (Paouris)**.** *With the same assumptions as in Theorem 1.1:*

$$W(Z_p(X)) \leq C\sqrt{p} \quad \forall p \in [2, cn^{\alpha/2}] \ . \tag{2.2}$$

Theorem (Klartag–Milman)**.** *With the same assumptions as in Theorem 1.1:*

$$V.Rad.(Z_p(X)) \geq c\sqrt{p} \quad \forall p \in [2, cn^{\alpha/2}] \; . \tag{2.3}$$

We are finally ready to provide a proof of Proposition 2.1:

Proof of Proposition 2.1. Let $p \in [2, cn^{\alpha/2}]$, where $c > 0$ is some small enough constant so that (2.2) and (2.3) hold. We will ensure that $c \leq 1$, so there is nothing to prove if $n = 1$. By (2.1), Sudakov's entropy estimate (e.g. [25]) and (2.2), we have:

$$N(Z_p^+(X)/\sqrt{p}, B_2^n) \leq N(2^{1/p} Z_p(X)/\sqrt{p}, B_2^n) \leq \exp(\tilde{C} n W(2^{1/p} Z_p(X)/\sqrt{p})^2) \leq \exp(Cn) \; . \tag{2.4}$$

Note that by (2.1) and the Rogers–Shephard inequality [26], we have:

$$2^{n/p}\mathrm{Vol}(Z_p(X)) \leq \mathrm{Vol}(Z_p^+(X) - Z_p^+(X)) \leq 4^n \mathrm{Vol}(Z_p^+(X)) \; .$$

Consequently, the volume bound in (2.3) also applies to $Z_p^+(X)$:

$$\mathrm{V.Rad.}(Z_p^+(X)) \geq c_1\sqrt{p} \; . \tag{2.5}$$

By Lemma 2.4, (2.4) and (2.5) imply that:

$$N(\sqrt{p}(Z_p^+(X))^\circ, B_2^n) \leq \exp(C_2 n) \; .$$

Consequently, Lemma 2.5 implies that if U is uniformly distributed on $O(n)$, then for any $C_1 \geq C_2 + (\log 2)/2$, there exists a $C_3 > 0$, so that:

$$\mathbb{P}\left(Z_p^+(X)^\circ \cap U(Z_p^+(X)^\circ) \subset \frac{C_3}{\sqrt{p}} B_2^n \right) \geq 1 - \exp(-C_1 n) \; ,$$

or by duality (since $T(K)^\circ = (T^{-1})^*(K^\circ)$ for any linear map T of full rank), that:

$$\begin{aligned} & \mathbb{P}\left(Z_p^+(X) + U(Z_p^+(X)) \supset C_3^{-1}\sqrt{p} B_2^n\right) \\ \geq \; & \mathbb{P}\left(\mathrm{conv}(Z_p^+(X) \cup U(Z_p^+(X))) \supset C_3^{-1}\sqrt{p} B_2^n\right) \geq 1 - \exp(-C_1 n) \; . \end{aligned}$$

Lemma 2.3 now concludes the proof. □

3 Remaining Details

We now complete the remaining (standard) details in the proof of Theorem 1.1.

Proof of Theorem 1.1.

1. For any $U \in O(n)$ and $t \geq 0$, observe that:

$$
\begin{aligned}
& 2\max\left(\mathbb{P}\left(\left|\frac{X+U(X')}{\sqrt{2}}\right| \leq t\right), \mathbb{P}\left(\left|\frac{X-U(X')}{\sqrt{2}}\right| \leq t\right)\right) \\
\geq \; & \mathbb{P}\left(\left|\frac{X+U(X')}{\sqrt{2}}\right| \leq t\right) + \mathbb{P}\left(\left|\frac{X-U(X')}{\sqrt{2}}\right| \leq t\right) \\
= \; & \mathbb{P}\left(\frac{|X|^2+|X'|^2}{2} + \langle X, U(X')\rangle \leq t^2\right) + \mathbb{P}\left(\frac{|X|^2+|X'|^2}{2} - \langle X, U(X')\rangle \leq t^2\right) \\
\geq \; & \mathbb{P}\left(|X| \leq t \text{ and } |X'| \leq t \text{ and } \langle X, U(X')\rangle \leq 0\right) \\
& + \mathbb{P}\left(|X| \leq t \text{ and } |X'| \leq t \text{ and } \langle X, U(X')\rangle > 0\right) \\
= \; & \mathbb{P}\left(|X| \leq t \text{ and } |X'| \leq t\right) = \mathbb{P}\left(|X| \leq t\right)^2 \; .
\end{aligned}
$$

Similarly:

$$
2\max\left(\mathbb{P}\left(\left|\frac{X+U(X')}{\sqrt{2}}\right| \geq t\right), \mathbb{P}\left(\left|\frac{X-U(X')}{\sqrt{2}}\right| \geq t\right)\right) \geq \mathbb{P}\left(|X| \geq t\right)^2 \; .
$$

This is precisely the content of the first assertion of Theorem 1.1.

2. Given $\theta \in S^{n-1}$, denote $Y_1 = P_\theta Y_+^U$, $X_1 = P_\theta X$ and $X_2 = P_\theta U(X')$, where P_θ denotes orthogonal projection onto the one-dimensional subspace spanned by θ. We have:

$$
\begin{aligned}
& h_{Z_p(Y_+^U)}(\theta) = (\mathbb{E}|Y_1|^p)^{\frac{1}{p}} = \left(\mathbb{E}\left|\frac{X_1+X_2}{\sqrt{2}}\right|^p\right)^{\frac{1}{p}} \\
\leq \; & \frac{1}{\sqrt{2}}\left((\mathbb{E}|X_1|^p)^{\frac{1}{p}} + (\mathbb{E}|X_2|^p)^{\frac{1}{p}}\right) = \frac{1}{\sqrt{2}}\left(h_{Z_p(X)}(\theta) + h_{Z_p(U(X)))}(\theta)\right) \; .
\end{aligned}
$$

Employing in addition (2.1), it follows that:

$$
Z_p^+(Y_+^U) \subset 2^{1/p} Z_p(Y_+^U) \subset \frac{2^{1/p}}{\sqrt{2}}\left(Z_p(X) + U(Z_p(X))\right) \; ,
$$

and the second assertion for Y_+^U follows since $Z_p(X) \subset C p^{\frac{1}{\alpha}} B_2^n$ by assumption. Similarly for Y_-^U.

3. Given a natural number i, set $p_i = 2^i$. Proposition 2.1 ensures the existence of a constant $c > 0$, so that for any $C_1 > 0$, there exists a constant $c_1 > 0$, so that for any $p_i \in [2, cn^{\frac{\alpha}{2}}]$, there exists a subset $A_i \subset O(n)$ with:

$$
\mu_{O(n)}(A_i) \geq 1 - \exp(-C_1 n) \; ,
$$

so that:

$$
\forall U \in A_i \quad Z_{p_i}(Y_+^U) \supset c_1 \sqrt{p_i} B_2^n \; .
$$

Denoting $A_0 := \cap \left\{ A_i \, ; \, p_i \in [2, cn^{\frac{\alpha}{2}}] \right\}$, and setting $A = A_0 \cap -A_0$, where $-A_0 := \{-U \in O(n); U \in A_0\}$, it follow by the union-bound that:

$$\mu_{O(n)}(A) \geq 1 - 2\log(C_2 + n)\exp(-C_1 n) \ .$$

By choosing the constant $C_1 > 0$ large enough, we conclude that:

$$\mu_{O(n)}(A) \geq 1 - \exp(-C_3 n) \ .$$

By construction, the set A has the property that:

$$\forall U \in A \quad \forall p_i \in [2, cn^{\frac{\alpha}{2}}] \quad Z_{p_i}(Y_\pm^U) \supset c_1 \sqrt{p_i} B_2^n \ .$$

Using (1.6), it follows that:

$$\forall U \in A \quad \forall p \in [2, cn^{\frac{\alpha}{2}}] \quad Z_p(Y_\pm^U) \supset \frac{c_1}{\sqrt{2}} \sqrt{p} B_2^n \ ,$$

thereby concluding the proof of the third assertion.

□

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