Interpolations, convexity and geometric inequalities

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Abstract

We survey some interplays between spectral estimates of Hörmander-type, degenerate Monge-Ampère equations and geometric inequalities related to log-concavity such as Brunn-Minkowski, Santaló or Busemann inequalities.

1 Introduction

The Brunn-Minkowski inequality has an L^2 interpretation, an observation that can be traced back to the proof provided by Hilbert. More recently, it has been noted that the Brunn-Minkowski inequality for convex bodies is related, in its local form, to spectral inequalities. In fact, the Prékopa theorem, which is the function form of the Brunn-Minkowski inequality for convex sets, is *equivalent* to spectral inequalities of Brascam-Lieb type. The local derivation of Prékopa's theorem from spectral L^2 inequalities was described in the more general complex setting in [13] and then extended further in [6, 7].

Let $K_0, K_1 \subset \mathbb{R}^n$ be two convex bodies (i.e., compact convex sets with non-empty interior) and denote, for $t \in [0,1]$,

$$K(t) := (1-t)K_0 + tK_1 = \{z \in \mathbb{R}^n \ ; \ \exists (a,b) \in K_0 \times K_1, \ z = (1-t)a + tb\}. \tag{1}$$

The Brunn-Minkowski inequality is central in the theory of convex bodies. Denoting the Lebesgue measure by $|\cdot|$, it states that

$$|K(t)| \geq |K_0|^{1-t}\,|K_1|^t,$$

with equality if and only if $K_0 = K_1 + x_0$ for $x_0 \in \mathbb{R}^n$. Introducing the convex body

$$K := \bigcup_{t\in[0,1]} \{t\} \times K(t) \subset \mathbb{R}^{n+1},$$

then $K(t)$ is the section over t, and the Brunn-Minkowski inequality expresses the log-concavity of the marginal measure. Namely, it shows that the function

$$\alpha(t) := -\log|K(t)|$$

is convex. The Brunn-Minkowski inequality for convex bodies admits the following useful functional form, which states that marginals of log-concave functions are log-concave.

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Theorem 1 (Prékopa)**.** *Let* $F : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ *be convex with* $\int \exp(-F) < \infty$ *and define* $\alpha : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ *by*

$$e^{-\alpha(t)} = \int_{\mathbb{R}^n} e^{-F(t,x)}\, dx.$$

Then α *is convex.*

The Brunn-Minkowski inequality then follows by considering, for a given convex set $K \subset \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, the convex function F defined by

$$e^{-F(t,x)} = \mathbf{1}_K(t,x) = \mathbf{1}_{K(t)}(x). \tag{2}$$

The standard proofs of Brunn-Minkowski rely on parameterization or mass transport techniques between K_0 and K_1, with the parameter $t \in [0,1]$ being fixed. A natural question is whether one can provide a direct local approach by proving $\alpha''(t) \geq 0$? The answer is affirmative and this was shown recently by Ball, Barthe and Naor [4]. As mentioned earlier, this local approach was put forward in an L^2 framework, for analogous complex versions, in Cordero-Erausquin [13] and in subsequent far-reaching works by Berndtsson [6, 7].

Another essential concept in the theory of convex bodies is duality. This requires us to fix a center and a scalar product. Let $x \cdot y$ stand for the standard scalar product of $x, y \in \mathbb{R}^n$. We write $|x|^2 = x \cdot x$ and $B_2^n = \{x \in \mathbb{R}^n \ ; \ x \cdot x \leq 1\}$, the associated unit ball. Recall that $K \subset \mathbb{R}^n$ is a centrally-symmetric convex body if and only if K is the unit ball for some norm $\|\cdot\|$ on $\mathbb{R}^n$, a relation denoted by $K = B_{\|\cdot\|} := \{x \in \mathbb{R}^n \ ; \ \|x\| \leq 1\}$. The polar of K is defined as the unit ball of the dual norm $\|\cdot\|_*$,

$$K^\circ = B_{\|\cdot\|_*} = \{y \in \mathbb{R}^n \ ; \ x \cdot y \leq 1, \ \forall x \in K\}.$$

We have the following beautiful result:

Theorem 2 (Blaschke-Santaló inequality)**.** *For every centrally-symmetric convex body* $K \subset \mathbb{R}^n$*, we have*

$$|K|\,|K^\circ| \leq |B_2^n|^2$$

with equality holding true if and only if K *is an ellipsoid (i.e. a linear image of* B_2^n*).*

The corresponding functional form reads as follows (see [1, 2]): for an even function $f : \mathbb{R}^n \to \mathbb{R}$ with $0 < \int e^{-f} < \infty$, if $\mathcal{L}f$ denotes its Legendre transform, then

$$\int e^{-f} \int e^{-\mathcal{L}f} \leq \Big(\int e^{-|x|^2/2}\, dx \Big)^2 = (2\pi)^n. \tag{3}$$

Note that the Brunn-Minkowski inequality entails

$$\sqrt{|K|\,|K^\circ|} \leq \left| \frac{K + K^\circ}{2} \right|.$$

However, in general we have $\frac{K+K^\circ}{2} \not\supseteq B_2^n$. For instance, take $K = T(B_2^n)$, where $T \neq \mathrm{Id}_{\mathbb{R}^n}$ is a positive-definite symmetric operator. Then $K^\circ = T^{-1}(B_2^n)$. Observe that $\frac{K+K^\circ}{2} \supset \frac{T+T^{-1}}{2}(B_2^n)$ and

$$\frac{T + T^{-1}}{2} > \sqrt{T\,T^{-1}} = \mathrm{Id}_{\mathbb{R}^n}$$

in the sense of symmetric matrices. This suggest that instead of taking convex combinations, as in the Brunn-Minkowski theory, we would like to consider geometric means of convex bodies. It turns out that this is exactly what complex interpolation does, and it is a challenging question to understand real analogues of this procedure.

In this note we will consider several ways of going from K_0 to K_1, or equivalently from a norm $\|\cdot\|_0$ to another norm $\|\cdot\|_1$. There are many ways to recover the volume of K from the associated norm $\|\cdot\|$. Let $p > 0$ and $n \geq 1$. There exists an explicit constant $c_{n,p} > 0$ such that for every centrally-symmetric convex body $K \subset \mathbb{R}^n$, with associated norm $\|\cdot\|_K$, we have

$$\int_{\mathbb{R}^n} e^{-\|x\|_K^p/p}\, dx = c_{n,p}\, |K|. \tag{4}$$

Note that the procedure (2) corresponds to the case $p \to +\infty$.

We aim to find ways of interpolating between norms in order to recover, among other things, the Brunn-Minkowski and the Santaló inequalities.

Let us next put forward some notation as well as a formula that we shall use throughout the paper.

Notation 3. *For a function $F : \mathbb{R}^n \to \mathbb{R}$ such that $\int e^{-F(x)}\, dx < +\infty$, we denote by μ_F the probability measure on $\mathbb{R}^n$ given by*

$$d\mu_F(x) := \frac{e^{-F(x)}}{\int e^{-F}}\, dx.$$

For a function of $n+1$ variables $F : I \times \mathbb{R}^n \to \mathbb{R}$, where I is an interval of $\mathbb{R}$, we denote, for a fixed $t \in I$, $F_t := F(t, \cdot) : \mathbb{R}^n \to \mathbb{R}$ and then by μ_{F_t} the corresponding probability measure on $\mathbb{R}^n$. We also set

$$\alpha(t) = -\log \int_{\mathbb{R}^n} e^{-F_t(x)} dx.$$

The variance with respect to a *probability* measure μ of a function $u \in L^2(\mu)$ – where, depending on the context, we consider either real-valued or complex-valued functions – is defined as the L^2 norm of the projection of u onto the space of functions orthogonal to constant functions, i.e.

$$\mathrm{Var}_\mu(u) := \int \left|u - \textstyle\int u\, d\mu\right|^2\, d\mu = \int |u|^2\, d\mu - \left|\int u\, d\mu\right|^2.$$

A straightforward computation yields:

Fact 4. *With Notation 3, we have for every $t \in I$,*

$$\begin{aligned} \alpha''(t) &= \int_{\mathbb{R}^n} \partial^2_{tt} F\, d\mu_{F_t}(x) - \left[\int_{\mathbb{R}^n} \big(\partial_t F(t,x)\big)^2\, d\mu_{F_t}(x) - \left(\int_{\mathbb{R}^n} \partial_t F(t,x)\, d\mu_{F_t}(x)\right)^2\right] \\ &= \int_{\mathbb{R}^n} \partial^2_{tt} F\, d\mu_{F_t} - \mathrm{Var}_{\mu_{F_t}}\big(\partial_t F\big), \end{aligned} \tag{5}$$

assuming that F is sufficiently regular to allow for the differentiations under the integral sign.

Our goal is to understand for which families of functions F the function α is convex, by looking at α''. Actually, we will first discuss the complex case, where convexity is replaced by plurisubharmonicity. We will recover the fact that families given by complex interpolation, or equivalently by degenerate Monge-Ampre equations, lead to subharmonic functions α. Then we will try to see, at a very heuristic level, what can be said in the real case. A final section proposes a local L^2 approach, to the Busemann inequality, similar to that used in the preceding sections.

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2 The complex case

Let K_0 and K_1 be two unit balls of $\mathbb{C}^n$ associated with the (complex vector space) norms $\|\cdot\|_0$ and $\|\cdot\|_1$. Note that here we are working with the class of convex bodies K of $\mathbb{R}^{2n}$ that are *circled*, meaning that $e^{i\theta}K = K$ for every $\theta \in \mathbb{R}$. We think of a normed space as a triplet consisting of a vector space, a norm and its unit ball. Consider the complex normed spaces $X_0 = (\mathbb{C}^n, \|\cdot\|_0, K_0)$ and $X_1 = (\mathbb{C}^n, \|\cdot\|_1, K_1)$ and write

$$X_z = (\mathbb{C}^n, \|\cdot\|_z, K_z)$$

for the complex Calderón interpolated space at

$$z \in C := \{w \in \mathbb{C}\, ; \Re(w) \in [0,1]\}$$

where $\Re(w)$ is the real part of $w \in \mathbb{C}$. Recall that $X_z = X_{\Re(z)}$ and therefore $K_z = K_t$ with $t = \Re(z) \in [0,1]$. We have:

Theorem 5 ([12])**.** *The function $t \to |K_t|$ is log-concave on $[0,1]$ and so*

$$|K_0|^{1-t}\,|K_1|^t \le |K_t|. \tag{6}$$

In the case of complex unit balls, this result improves upon the Brunn-Minkowski inequality since it can be verified, by using the Poisson kernel on $[0,1] \times \mathbb{C}^n$ and the definition of the interpolated norm, that

$$K_t \subset (1-t)K_0 + tK_1 = K(t).$$

In this setting, it also gives the Santaló inequality. Indeed, for a given complex unit ball $K \subset \mathbb{C}^n$, let X_0 be the associated complex normed space, and let X_1 be the dual conjugate space which has $K^\circ \subset \mathbb{C}^n$ as its unit ball. Then it is well known that

$$X_{1/2} = \ell_2^n(\mathbb{C}) = \ell_2^{2n}(\mathbb{R}) \tag{7}$$

and therefore we obtain

$$\sqrt{|K|\,|K^\circ|} \le |B_2^{2n}|.$$

(Let us mention here that the conjugation bar in the statements of [12] is superfluous according to standard definitions).

In order to have a better grasp on complex interpolation, let us write an explicit formula in the specific case of Reinhardt domains. A subset $K \subset \mathbb{C}^n$ is *Reinhardt* if for any $z = (z_1, \dots, z_n) \in \mathbb{C}^n$,

$$(z_1, \dots, z_n) \in K \quad \Leftrightarrow \quad (|z_1|, \dots, |z_n|) \in K.$$

Note that a Reinhardt convex set is necessarily circled. In the case where $X_0 = (\mathbb{C}^n, \|\cdot\|_0, K_0)$ and $X_1 = (\mathbb{C}^n, \|\cdot\|_1, K_1)$ are such that K_0 and K_1 are Reinhardt, the interpolated space $X_z = (\mathbb{C}^n, \|\cdot\|_z, K_z)$ satisfies

$$K_z = \left\{ z \in \mathbb{C}^n \,;\, \exists (a, b) \in K_0 \times K_1,\ |z_j| = |a_j|^{1-t} |b_j|^t \text{ for } j = 1, \dots, n \right\}$$

with $t = \Re(z)$. The case of Reinhardt unit balls is particularly simple and easy to analyze, but it has its limitations. Still, the idea is that in general, K_t should be understood as a "geometric mean" of the bodies K_0 and K_1, whereas the Minkowski sum (1) reminds us of an arithmetic mean.

Theorem 5 was proved using the complex version of the Prékopa theorem obtained by Berndtsson [5], which was derived in [13] using a local computation and L^2 spectral inequalities of Hördmander type. Here, we would like to provide a different direct proof, by combining the results of Rochberg and Hörmander's *a priori* L^2-estimates. Let $\|\cdot\|_z$ be a family of interpolated norms on $\mathbb{C}^n$ and $K_z = B_{\|\cdot\|_z}$. We assume for simplicity that these norms are smooth and strictly convex, so that we will not have to worry about justification of the differentiations under the integral signs. In fact, by approximation we can assume that $1/R \leq \mathrm{Hess}\|\cdot\|_k^2 \leq R$ (for some large constant $R > 1$) for $k = 1, 2$, and these bounds remain valid for the interpolated norms. Introduce the function $F : C \times \mathbb{C}^n \to \mathbb{R}$,

$$F(z, w) := \frac{1}{2} \|w\|_z^2.$$

Denote the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ by λ, and introduce, in view of (4),

$$\alpha(z) = -\log \int_{\mathbb{C}^n} e^{-F(z,w)}\, d\lambda(w) = -\log |K_z| - \log(c_{2n,2})$$

for $z \in C$. Our goal is to prove that $t \to \alpha(t)$ is convex on $[0, 1]$. Since $\alpha(z) = \alpha(\Re(z))$, this is equivalent to proving that α is subharmonic on the strip C. The following analogue of (5) is also straightforward:

$$\frac{1}{4} \Delta \alpha(z) = \partial^2_{z\bar{z}} \alpha(z) = \int_{\mathbb{C}^n} \partial^2_{z\bar{z}} F \, d\mu_{F_z} - \int_{\mathbb{C}^n} \left| \partial_z F(w) - \textstyle\int \partial_z F \, d\mu_{F_z} \right|^2 d\mu_{F_z}(w),$$

where μ_{F_z} is the probability measure on $\mathbb{C}^n$ given by $d\mu_{F_z}(w) = \dfrac{e^{-F(z,w)}}{\int e^{-F(z,\zeta)} d\lambda(\zeta)} d\lambda(w)$.

It was explained by Rochberg [17] that complex interpolation is characterized by the following differential equation:

$$\partial^2_{z\bar{z}} F = \sum_{j,k=1}^{n} F^{j\bar{k}}(z, w) \partial_{\overline{w_j}}(\partial_z F) \overline{\partial_{\overline{w_k}}(\partial_z F)} \tag{8}$$

where $(F^{j\bar{k}})_{j,k \leq n}$ is the inverse of the complex Hessian in the w-variables of $F(z, w)$, that is

$$\left(F^{j\bar{k}} \right)_{j,k \leq n} = \left(\mathrm{Hess}_w^{\mathbb{C}} F \right)^{-1} := \left[\left(\partial^2_{w_j \overline{w_k}} F \right)_{j,k \leq n} \right]^{-1}.$$

Actually, the function F is plurisubharmonic on $C \times \mathbb{C}^n \subset \mathbb{C}^{n+1}$ and (8) expresses the fact that it is a solution of the degenerate Monge-Ampère equation

$$\det\left(\mathrm{Hess}^{\mathbb{C}}_{z,w} F\right) = 0$$

where $\mathrm{Hess}^{\mathbb{C}}_{z,w} F$ is the full complex Hessian of $F(z,w)$, an $(n+1)\times(n+1)$ matrix.

As a consequence of the previous discussion, we have that, for a fixed $z \in C$ and setting $u := \partial_z F(z,\cdot) : \mathbb{C}^n \to \mathbb{C}$,

$$\Delta\alpha(z)/4 = \int_{\mathbb{C}^n} \sum_{j,k=1}^n F^{j\overline{k}} \partial_{\overline{w_j}} u \, \overline{\partial_{\overline{w_k}} u} \, d\mu_{F_z} - \int \left|u - \textstyle\int u \, d\mu_{F_z}\right|^2 \, d\mu_{F_z}. \tag{9}$$

Of course, it is now irresistible to appeal to Hörmander's *a priori* estimate (see e.g. [15]). It states that if $F : \mathbb{C}^n \to \mathbb{R}$ is a (strictly) plurisubharmonic function and if u is a (smooth enough) function, then

$$\int_{\mathbb{C}^n} |u - P_H u|^2 \, d\mu_F \leq \int_{\mathbb{C}^n} \sum_{j,k=1}^n F^{j\overline{k}} \partial_{\overline{w_j}} u \, \overline{\partial_{\overline{w_k}} u} \, d\mu_F \tag{10}$$

where $d\mu_F(w) = \dfrac{e^{-F(w)}}{\int e^{-F} d\lambda} d\lambda(w)$ and $P_H : L^2(\mu_F) \to L^2(\mu_F)$ is the orthogonal projection onto the closed space $H = \{h \in L^2(\mu_F) \ ; \ \overline{\partial} h = 0\}$ of holomorphic functions. Actually, this *a priori* estimate on $\mathbb{C}^n$ is rather easy to prove by duality and integration by parts. We now apply this result to $F = F(z,\cdot)$, $\mu_F = \mu_{F_z}$ and $u = \partial_z F$. Note that F (and thus μ_F) and u are invariant under the action of S^1: $F(z, e^{i\theta} w) = F(z,w)$ and the same is true for $\partial_z F$. This implies that the function $P_H u$ has the same invariance, but since it is a holomorphic function on $\mathbb{C}^n$, it has to be constant. Therefore $P_H u = \int u d\mu_{F_z}$ and we indeed obtain that $\Delta\alpha(z) \geq 0$ by combining (9) and (10), as desired.

Here, we reproved (6) without using explicitly [5], but rather by combining the local computations of [13] and the degenerate Monge-Ampère equation satisfied by the complex interpolation. In fact, this computation also appears, in a much more general and deep form, in recent works by Berndtsson [6, 7]. The reason is that complex interpolation corresponds to a geodesic in the space of metrics, and therefore enters Berndtsson's abstract theorems. Also, it can be noticed that complex interpolation corresponds to an extremal construction (for given boundary data), in the sense that it can be viewed as a plurisubharmonic hull. Equivalently, plurisubharmonic functions may be viewed as sub-solutions of degenerate Monge-Ampère equations.

Following our presentation, it is very tempting to develop an analogous presentation for convex bodies in $\mathbb{R}^n$. However, the real case is more complex, as we shall now see.

3 Real interpolations

The concept of interpolation and the basic properties we present here are due to Semmes [18], building on previous work by Rochberg [17]. Semmes indeed raised the question of whether such interpolations (which are not interpolations in the operator sense) could be used to prove inequalities, by showing that certain functionals are convex along the interpolation.

Our main contribution here is to explain that this is indeed the case, by connecting this interpolation with some well-known spectral inequalities. However, some discussions will remain at a heuristic level, as it is not the purpose of this note to discuss existence, unicity and regularity of solutions to the partial differential equations we refer to.

Definition 1 (Rochberg-Semmes interpolation [18])**.** *Let I be an interval of $\mathbb{R}$ and $p \in [1,+\infty]$. We say that a smooth function $F : I \times \mathbb{R}^n \to \mathbb{R}$ is a family of p-interpolation if for any $t \in I$, the function $F(t,\cdot)$ is (strongly) convex on $\mathbb{R}^n$ and for $(t,x) \in I \times \mathbb{R}^n$*

$$\partial^2_{tt} F = \frac{1}{p}\big(\mathrm{Hess}_x F\big)^{-1}\nabla\partial_t F \cdot \nabla\partial_t F. \tag{11}$$

Accordingly, when $\partial^2_{tt} F \geq \frac{1}{p}\big(\mathrm{Hess}_x F\big)^{-1}\nabla\partial_t F \cdot \nabla\partial_t F$, we say that F is a sub-family of p-interpolation.

In Definition 1, we denote by ∇F the gradient of $F(t,x)$ in the x variables, and a function is strongly convex when $\mathrm{Hess}_x F > 0$. By standard linear algebra we have the following equivalent formulation in terms of the degenerate Monge-Ampère equation:

Proposition 6 (Interpolation and degenerate Monge-Ampère equation)**.** *Let $F : I \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $F(t,\cdot)$ is (strongly) convex on $\mathbb{R}^n$ and introduce, for $(t,x) \in I \times \mathbb{R}^n$, the $(n+1)\times(n+1)$ matrix*

$$H = H_p F(t,x) := \begin{pmatrix} \partial^2_{tt} F & (\nabla_x \partial_t F)^* \\ \nabla_x \partial_t F & p\mathrm{Hess}_x F \end{pmatrix}. \tag{12}$$

Then, F is a family (resp. a sub-family) of p-interpolation if and only if $\det H = 0$ (resp. $\det H \geq 0$) on $I \times \mathbb{R}^n$.

In particular, 1-interpolation corresponds exactly to the degenerate Monge-Ampère equation on $I \times \mathbb{R}^n$. In fact, we see p-interpolation as a (Dirichlet) boundary value problem.

Definition 2. *Let F_0 and F_1 be two smooth convex functions on $\mathbb{R}^n$. We say that $\{F_t : \mathbb{R}^n \to \mathbb{R}\}_{\in[0,1]}$ is a p-interpolated family associated with $\{F_0, F_1\}$ if $F(t,x) = F_t(x)$ is a family of p-interpolation on $[0,1]\times\mathbb{R}^n$ with boundary value $F(0,\cdot) = F_0$ and $F(1,\cdot) = F_1$.*

As we said above, we will not discuss in this exposition questions related to existence, uniqueness and regularity of solutions to this Dirichlet problem (except for the easy case $p = 1$, explained below). However, it is reasonable to expect that generalized solutions, which are sufficient for our purposes, can be constructed by using Perron processes, as mentioned by Semmes [18].

Using Notation 3, given a family or a sub-family of p-interpolation F, we aim to understand the convexity of the function on I,

$$\alpha(t) = -\log \int_{\mathbb{R}^n} e^{-F(t,x)}\, dx. \tag{13}$$

In view of (5), we see that for every fixed $t \in I$ we have the implication

$$\mathrm{Var}_{\mu_{F_t}}(\partial_t F) \leq \frac{1}{p}\int_{\mathbb{R}^n} \big(\mathrm{Hess}_x F\big)^{-1}\nabla\partial_t F \cdot \nabla\partial_t F\, d\mu_{F_t} \quad \Longrightarrow \quad \alpha''(t) \geq 0, \tag{14}$$

under some mild regularity assumptions. The left-hand side is of course reminiscent of the real version of Hörmander's estimate (10), which is known as the Brascamp-Lieb from [9]. Recall that this inequality states that if $F : \mathbb{R}^n \to \mathbb{R}$ is a (strongly) convex function and if $u \in L^2(\mu_F)$ is a locally Lipschitz function, then

$$\mathrm{Var}_{\mu_F}(u) \leq \int_{\mathbb{R}^n} \big(\mathrm{Hess}_x F\big)^{-1} \nabla u \cdot \nabla u \, d\mu_F, \tag{15}$$

with our notation $d\mu_F(x) = \dfrac{e^{-F(x)}}{\int e^{-F}}\, dx$. Again, this inequality can easily be proven along the lines of Hörmander's approach (see below).

Applying the Brascamp-Lieb inequality (15) to $F = F(t, \cdot)$ and $u = \partial_t F$ when F is a 1-interpolation sub-family, we obtain, in view of (14), the following statement:

Proposition 7. *If F is a sub-family of 1-interpolation, then α is convex.*

The first comment is that we have not proved anything new! Indeed, it is directly verified below that for any C^2-smooth function F,

$$F \text{ is a sub-family of 1-interpolation} \quad \Longleftrightarrow \quad F \text{ is convex on } I \times \mathbb{R}^n. \tag{16}$$

Therefore, we have reproduced Prékopa's Theorem 1. In order to demonstrate (16), observe that the positive semi-definiteness of the matrix $H_1 F(t, x)$ amounts to the inequality

$$(\mathrm{Hess}_x F) y \cdot y + 2\nabla_x(\partial_t F) \cdot y + \partial^2_{tt} F \geq 0 \qquad \text{for all } \ y \in \mathbb{R}^n,$$

or equivalently,

$$\partial^2_{tt} F \geq \sup_{y \in \mathbb{R}^n} \left[2\nabla_x(\partial_t F) \cdot y - (\mathrm{Hess}_x) F y \cdot y \right] = \big(\mathrm{Hess}_x F\big)^{-1} \nabla_x \partial_t F \cdot \nabla_x \partial_t F,$$

as $\mathrm{Hess}_x F$ is positive definite. Let us note that if F_0 and F_1 are given, then the associated family of 1-interpolation – equivalently, the unique solution to the degenerate Monge-Ampère equation on $[0, 1] \times \mathbb{R}^n$ with $F(t, x)$ convex in x – is

$$F(t, w) = \inf_{w = (1-t)x + ty} \big\{ (1 - t) F_0(x) + t F_1(y) \big\}. \tag{17}$$

Every sub-family of 1-interpolation is *above* this F, and thus the statement of Prékopa's Theorem reduces to 1-interpolation families (an argument that is standard in the study of functional Brunn-Minkowski inequalities). One way to recover the Brunn-Minkowski inequality directly from this family F of 1-interpolation, is to take, as in the derivation from Prékopa's theorem, something like $F_0(x) = \|x\|^q_{K_0}/q$, $F_1(y) := \|y\|^q_{K_1}/q$ and let $q \to +\infty$.

We have just shown that Prékopa's theorem reduces, locally, to the Brascamp-Lieb inequality. This is parallel to the complex setting, i.e to the local L^2-proof of the complex Prékopa theorem of Berndtsson given in [13] and extended in [6, 7]. The converse procedure was known, starting from the work of Brascamp and Lieb; more explicitely, Bobkov and Ledoux [8] noted that the Prékopa-Leindler inequality (an extension of Prékopa's result to the case fibers are not convex) indeed implies the Brascamp-Lieb inequality. We also emphasize Colesanti's work [11], where, starting from the Brunn-Minkowski inequality, spectral inequalities of Brascamp-Lieb type on the boundary ∂K of a convex body $K \subset \mathbb{R}^n$ are obtained. This can also be recovered by applying the Brascamp-Lieb inequality to homogeneous functions. The conclusion is that all of these results are the global/local versions of the same phenomena. At the local level, we have reduced the problem to the inequality (15) which expresses a spectral bound in $L^2(\mu_F)$ for the elliptic operator associated with the Dirichlet form on the right-hand side of (15).

For completeness, we would like to briefly recall here Hörmander's original approach to (15). Consider the Laplace-type operator on $L^2(\mu_F)$,

$$L := \Delta - \nabla F \cdot \nabla,$$

that we define, say, on C^2-smooth compactly supported functions. First, recall the integration by parts formulae, $\int uL\varphi \, d\mu_F = -\int \nabla u \cdot \nabla \varphi \, d\mu_F$ and

$$\int_{\mathbb{R}^n} (L\varphi)^2 \, d\mu_F = \int_{\mathbb{R}^n} (\mathrm{Hess}_x F)\nabla\varphi \cdot \nabla\varphi \, d\mu_F + \int_{\mathbb{R}^n} \|\mathrm{Hess}_x \varphi\|_2^2 \, d\mu_F, \tag{18}$$

where $\|\mathrm{Hess}\varphi\|_2^2 = \sum_{i,j\le n} (\partial^2_{i,j}\varphi)^2$. Let u be a locally-Lipschitz function on $\mathbb{R}^n$. We use the (rather weak) standard observation that the image by L of the C^2-smooth compactly supported functions is dense in the space of $L^2(\mu_F)$ functions orthogonal to constants (see e.g. [14]). For $\varepsilon > 0$ let φ be a C^2-smooth, compactly-supported function such that $L\varphi - (u - \int u d\mu_F)$ has $L^2(\mu_F)$-norm smaller than ε. Then, by integration by parts and using (18) we get

$$\begin{aligned}
\mathrm{Var}_{\mu_F}(u) &= 2\int \left(u - \textstyle\int u \, d\mu_F\right) L\varphi \, d\mu_F - \int (L\varphi)^2 \, d\mu_F + \int \left(L\varphi - \left(u - \textstyle\int u \, d\mu_F\right)\right)^2 d\mu_F \\
&\le -2\int \nabla u \cdot \nabla\varphi \, d\mu_F - \int (\mathrm{Hess}_x F)\nabla\varphi \cdot \nabla\varphi \, d\mu_F - \int \|\mathrm{Hess}_x\varphi\|_2^2 \, d\mu_F + \varepsilon^2 \\
&\le -2\int \nabla u \cdot \nabla\varphi - \int (\mathrm{Hess}_x F)\nabla\varphi \cdot \nabla\varphi \, d\mu_F + \varepsilon^2 \\
&\le \int \left(\mathrm{Hess}_x F\right)^{-1} \nabla u \cdot \nabla u \, d\mu_F + \varepsilon^2,
\end{aligned}$$

and (15) follows by letting ε tend to zero.

Let us go back to interpolation families. As we said, 1-sub-interpolation corresponds to a function F that is convex on $I \times \mathbb{R}^n$. More generally, we have the following characterization, proved by Semmes:

Proposition 8. *For a smooth function $F : I \times \mathbb{R}^n \to \mathbb{R}$, the following are equivalent:*

- *F is a sub-family of p-interpolation.*
- *With the notation (12), we have, $\forall (t,x) \in I \times \mathbb{R}^n$, $H_p F(t,x) \ge 0$.*
- *For all $x_0, y_0 \in \mathbb{R}^n$, the function*

$$(s,t) \longrightarrow F\left(t, x_0 + (t + \sqrt{p-1}\, s) y_0\right)$$

is subharmonic on the subset of $\mathbb{R}^2$ where it is defined.

Note that the third condition in Proposition 8 needs only a minimal level of smoothness. We may thus speak of a sub-family F of p-interpolation even when F is not very smooth.

We turn now to duality, which was part of the motivation of Semmes. We shall denote by $\mathcal{L}$ the Legendre transform in space, i.e. on $\mathbb{R}^n$. In particular, for $F : I \times \mathbb{R}^n$, we shall write

$$\mathcal{L}F(t,x) = \mathcal{L}(F_t)(x) = \sup_{y \in \mathbb{R}^n} \{x \cdot y - F(t,y)\}.$$

It is classical that if F is the family of 1-interpolation given by (17), then $\mathcal{L}F$ is a family of ∞-interpolation, meaning that $\mathcal{L}F$ is affine in t:

$$\mathcal{L}F_t(x) = (1-t)\mathcal{L}F_0(x) + t\mathcal{L}F_1(x).$$

So in this case, when we move to the dual setting, Brunn-Minkowski or Prékopa's inequality is replaced by the trivial fact that $\alpha(t) = -\log \int e^{-\mathcal{L}_t F(x)}\, dx$ is *concave* by Hölder's inequality.

More general duality relations hold for p-interpolations. Suppose $F(t,x) = F_t(x)$ is convex in x, and denote $G(t,y) = \mathcal{L}F_t(y)$. We have the identity (proved below):

$$\partial^2_{tt} F + \partial^2_{tt} G = (\mathrm{Hess}_x F)^{-1} \nabla \partial_t F \cdot \nabla \partial_t F = (\mathrm{Hess}_y G)^{-1} \nabla \partial_t G \cdot \nabla \partial_t G, \tag{19}$$

where F and its derivatives are evaluated at (t,x), while G and its derivatives are evaluated at $(t,y) = (t, \nabla F(x))$. From this identity, we immediately conclude

Proposition 9. *If F is a family of p-interpolation, then $\mathcal{L}F$ is a family of p'-interpolation, where $\frac{1}{p'} + \frac{1}{p} = 1$.*

We now present the details of the straightforward proof of (19). From the definition,

$$\begin{aligned} G(t, \nabla F(t,x)) &= \langle x, \nabla F(t,x) \rangle - F(t,x), & (20)\\ \nabla G_t(\nabla F_t(x)) = (\nabla G)(t, \nabla F(x)) &= x & (21)\\ \mathrm{Hess}_y G(t, \nabla F(x,t)) &= (\mathrm{Hess}_x F(t,x))^{-1}. & (22) \end{aligned}$$

where the gradients and the hessians refer only to the space variables x, y. By differentiating (21) with respect to t, we see that

$$\nabla \partial_t G = -(\mathrm{Hess}_y G)(\nabla \partial_t F) \tag{23}$$

where G and its derivatives are evaluated at $(t,y) = (t, \nabla F(x))$, while F and its derivatives are evaluated at (t,x). From (22) and (23),

$$-\nabla \partial_t G \cdot \nabla \partial_t F = (\mathrm{Hess}_x F)^{-1} \nabla \partial_t F \cdot \nabla \partial_t F = (\mathrm{Hess}_y G)^{-1} \nabla \partial_t G \cdot \nabla \partial_t G. \tag{24}$$

Differentiating (20) with respect to t and using (21) we get that $\partial_t G(t, \nabla F(x)) = -\partial_t F(t,x)$. If we differentiate this last equality one more time with respect to t, we find

$$\partial^2_{tt} G + \nabla \partial_t G \cdot \nabla \partial_t F = -\partial^2_{tt} F,$$

which combined with (24) yields the desired formula (19).

As a consequence of Proposition 8, we see that 2-interpolation families satisfy an interpolation duality theorem. Let f be a convex function on $\mathbb{R}^n$, and suppose that $F_t(x) = F(t,x)$ is the 2-interpolation family F with $F_0 = f$ and $F_1 = \mathcal{L}f$. Then,

$$F(t,x) = \mathcal{L}F(1-t,x)$$

provided we have unicity for the 2-interpolation problem, and therefore we have

$$F\left(\frac{1}{2}, x\right) = \frac{|x|^2}{2}.$$

If we take $f(x) = \|x\|_K^2/2$, then $\mathcal{L}f(x) = \|x\|_{K^\circ}^2/2$. Thus, if we could prove that for a 2-interpolation family F, the associated function α from (13) is convex, as it is for 1-interpolations, then we would recover Santaló's inequality. This would be the case if we had a Brascamp-Lieb inequality with a factor $1/2$ on the right-hand side of (15) for every convex function $F : \mathbb{R}^n \to \mathbb{R}$. However, this is of course false in general. Recall that even for the Santaló inequaliy, some "center" must be fixed or some symmetry must be assumed. Therefore, a more reasonable question to ask, is whether α is convex when the initial data f is even. This guarantees that F_t is even for all $t \in [0,1]$. However, it is again false in general that the Brascamp-Lieb inequality holds with factor $1/2$ in the right-hand side of (15) when F and u are even, as can be shown by taking a perturbation of the Gaussian measure. This suggests that the answer to the question could be negative in general. A reasonable conjecture, perhaps, is:

Conjecture 10. *Assume F_0 and F_1 are even, convex and 2-homogeneous (i.e. $F_i(x) = \lambda_i \|x\|_{K_i}^2$ for some centrally-symmetric convex bodies $K_i \subset \mathbb{R}^n$), properties that propagate along the interpolation. Then, the function α associated with the 2-interpolation family is convex.*

Here is a much more modest result:

Fact 11. *Assume that f is convex and even, and let F be a 2-interpolation family with $F_0 = f$ and $F_1 = \mathcal{L}f$, with the associated function α as in* (13)*. Then, one has*

$$\alpha''(1/2) \geq 0.$$

Proof. Since $F(\frac{1}{2}, x) = |x|^2/2$, the probability measure $\mu_{F_{1/2}}$ is exactly the Gaussian measure on $\mathbb{R}^n$, which we denote by γ. Note also that $\mathrm{Hess}_x F_{1/2} = \mathrm{Id}_{\mathbb{R}^n}$. Therefore, if we denote $u = \partial_t F(\frac{1}{2}, \cdot)$, we need to check that

$$\mathrm{Var}_\gamma(u) \leq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, d\gamma.$$

The function $v := u - \int u \, d\gamma$ is by construction orthogonal to constant functions in $L^2(\gamma)$. But since u is even (because F_t is even for all t, and so is $\partial_t F$), this function v is also orthogonal to linear functions. Recall that the Hermite (or Ornstein-Uhlenbeck) operator $L = \Delta - x \cdot \nabla$ has non-positive integers as eigenvalues, and that the eigenspaces (generated by Hermite polynomials) associated with the eigenvalues 0 and -1 are formed by the constant and linear functions. Therefore, v belongs to the subspace where $-L \geq 2\,\mathrm{Id}$ and so

$$\mathrm{Var}_\gamma(u) = \int |v|^2 \, d\gamma \leq -\frac{1}{2} \int vLv \, d\gamma = \frac{1}{2} \int |\nabla u|^2 \, d\gamma.$$

$\square$

We conclude this section by mentioning that we have analogous formulas in the case where we work with some fixed measure ν on $\mathbb{R}^n$, in place of the Lebesgue measure. Then, for a function $F : \mathbb{R}^n \to \mathbb{R}$ such that $\int e^{-F}\, d\nu < +\infty$, we denote by $\mu_{\nu,F}$ the probability measure on $\mathbb{R}^n$ given by

$$d\mu_{\nu,F}(x) := \frac{e^{-F(x)}}{\int e^{-F}\, d\nu}\, d\nu(x).$$

For a function of $n+1$ variables $F : I \times \mathbb{R}^n \to \mathbb{R}$, we denote as before $F_t := F(t,\cdot) : \mathbb{R}^n \to \mathbb{R}$ and then μ_{ν,F_t} is the corresponding probability measure on $\mathbb{R}^n$. We are then interested in the convexity of the function

$$\alpha_\nu(t) := -\log \int_{\mathbb{R}^n} e^{-F(t,x)}\, d\nu(x) = -\log \int_{\mathbb{R}^n} e^{-F_t}\, d\nu.$$

The computation is identical:

$$\alpha_\nu''(t) = \int_{\mathbb{R}^n} \partial^2_{tt} F\, d\mu_{\nu,F_t} - \mathrm{Var}_{\mu_{\nu,F_t}}\big(\partial_t F\big).$$

Here is an illustration. Let ν be a symmetric log-concave measure on $\mathbb{R}^n$: $d\nu(x) = e^{-W(w)}\, dx$ with W being convex and even on $\mathbb{R}^n$, and consider the family

$$F(t,x) = e^t\, |x|^2/2.$$

This is a typical example of a 2-interpolation family. Then, the fact that the corresponding α_ν is convex is equivalent to the B-conjecture proved in [14]. The argument there begins with the computation above. It turns out that for this particular family F, the required Brascamp-Lieb inequality reduces to a Poincaré inequality for the measure μ_{ν,F_t}, which holds precisely with a constant $1/2$ when restricted to even functions.

Let us also mention in this direction that the Santaló inequality in its functional form (3) also holds if the Lebesgue measure is, in the three integrals, replaced by an even log-concave measure of $\mathbb{R}^n$, as noted in Klartag [16]. Several examples of this type suggest that the Lebesgue measure can often be replaced by a more general log-concave measure.

4 The Busemann Inequality

We conclude this survey with a proof of the Busemann inequality via L^2 inequalities. The Busemann inequality [10] is concerned with non-parallel hyperplane sections of a convex body $K \subset \mathbb{R}^n$. In the particular case where K is centrally-symmetric, the Busemann inequality states that

$$g(x) = \frac{|x|}{|K \cap x^\perp|} \qquad\qquad (x \in \mathbb{R}^n)$$

is a norm on $\mathbb{R}^n$. Here $|K \cap x^\perp|$ is the $(n-1)$-dimensional volume of the hyperplane section $K \cap x^\perp = \{y \in K; y \cdot x = 0\}$, and $g(0) = 0$ as interpreted by continuity. The convexity of the function g is a non-trivial fact. Using the Brunn-Minkowski inequality, the convexity of g reduces to a statement about log-concave functions in the plane, as observed by Busemann. Indeed, the convexity of g has to be checked along affine lines, and therefore on 2-dimensional vector subspaces. Specifically, let $E \subset \mathbb{R}^n$ be a two-dimensional plane, which we conveniently identify with $\mathbb{R}^2$. For $y \in \mathbb{R}^2 = E$ set

$$e^{-w(y)} = |K \cap (y + E^{\perp})|,$$

the $(n-2)$-dimensional volume of the the section of K. Then $w : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ is a convex function, according to the Brunn-Minkowski inequality. For $p > 0$ and $t \in \mathbb{R}$ define

$$\alpha_p(t) = \int_0^\infty e^{-w(ts,s)} s^{p-1} ds. \tag{25}$$

Note that when K is centrally-symmetric, $2\sqrt{1+t^2}\alpha_1(t) = |K \cap (1,-t)^{\perp}|$. We therefore see that Busemann's inequality amounts to the convexity of the function $1/\alpha_1(t)$ on $\mathbb{R}$. Next we will prove the following more general statement, which is due to Ball [3] when $p \geq 1$:

Theorem 12. *Let X be an n-dimensional real linear space and let $w : X \to \mathbb{R}$ be a convex function with $\int e^{-w} < \infty$. For $p > 0$ and $0 \neq x \in X$ denote*

$$h(x) = \left(\int_0^\infty e^{-w(sx)} s^{p-1} ds \right)^{-1/p}$$

with $h(0) = 0$. Then h is a convex function on X.

Busemann's proof of the case $p = 1$ of Theorem 12, and the generalization to $p \geq 1$ by Ball, rely on transportation of measure in one dimension. The proof we present below may be viewed as an infinitesimal version of Busemann's transportation argument. This is reminiscent of the proof given in Ball, Barthe and Naor [4] of the Prékopa inequality, which may be viewed as an infinitesimal version of the transportation proof of the latter inequality.

Proof of Theorem 12: By a standard approximation argument, we may assume that w is smooth and $1/R \leq \mathrm{Hess}(w) \leq R$ at all points of $\mathbb{R}^n$, for some large constant $R > 1$. Therefore h is a continuous function, smooth outside the origin, and homogeneous of degree one. Since convexity of a function involves three collinear points contained in a two-dimensional subspace, we may assume that $n = 2$. Thus, selecting a point $0 \neq z \in X$ and a direction $\theta \in X$, our goal is to show that $\partial^2_{\theta\theta} h(z) \geq 0$ (since h is homogeneous of degree one, it suffices to consider the case $z \neq 0$). If θ is proportional to z, then the second derivative vanishes as h is homogeneous of degree one. We may therefore select coordinates $(t, x) \in \mathbb{R}^2 = X$, and identify $z = (0, 1)$ and $\theta = (1, 0)$. With this identification, in order to prove the theorem we need to show that

$$\left(\alpha_p^{-1/p}\right)''(0) \geq 0,$$

where α_p is defined in (25). Equivalently, we need to prove that at the origin,

$$\partial^2_{tt} \alpha_p \leq \left(1 + \frac{1}{p}\right) (\partial_t \alpha_p)^2 / \alpha_p. \tag{26}$$

We denote by μ the probability measure on $[0, \infty)$ whose density is proportional to the integrable function $\exp(-w(0,x))x^{p-1}$. Similarly to Fact 5 above with $F(t,x) = w(tx,x)$, the desired inequality (26) is equivalent to

$$\mathrm{Var}_\mu(x\partial_t w) \leq \int_0^\infty x^2 (\partial^2_{tt} w) d\mu(x) + \frac{1}{p} \left(\int_0^\infty x(\partial_t w) d\mu(x) \right)^2. \tag{27}$$

We will use the convexity of $w(t,x)$ via the inequality $\partial^2_{tt}w \geq \left(\partial^2_{tx}w\right)^2/\partial^2_{xx}w$, which expresses the fact that $w_t(x) = w(t,x)$ is a sub-family of 1-interpolation. Denote $u(x) = x\partial_t w(0,x)$ and compute that $x\partial^2_{tx}w = u' - u(x)/x$ for $x > 0$. Hence, in order to prove (27), it suffices to show that

$$\mathrm{Var}_\mu(u) \leq \int_0^\infty \frac{1}{\partial^2_{xx}w}\left(u'(x) - \frac{u(x)}{x}\right)^2 d\mu(x) + \frac{1}{p}\left(\int_0^\infty u d\mu(x)\right)^2. \tag{28}$$

We will prove (28) for any smooth function $u \in L^2(\mu)$ (it is clear that the function $x\partial_t w(0,x)$ grows at most polynomially at infinity, and hence belongs to $L^2(\mu)$). By approximation, it suffices to restrict our attention to smooth functions such that $u - \int u d\mu$ is compactly-supported in $[0,\infty)$. Consider the Laplace-type operator

$$L\varphi = \varphi'' - \Big(\partial_x w(0,x) - \frac{p-1}{x}\Big)\varphi' = \varphi'' - \partial_x\Big(w(0,x) - (p-1)\log(x)\Big)\varphi'.$$

Integrating the ordinary differential equation, we find a smooth function φ, with $\varphi'(0) = 0$ and φ' compactly-supported in $[0,\infty)$, such that $L\varphi = u - \int u d\mu$. As before, we have the integration by parts $\int (L\varphi)u\, d\mu = -\int \varphi' u'\, d\mu$ and

$$\int_0^\infty (L\varphi)^2 d\mu = -\int_0^\infty \varphi'(x)u'(x)d\mu = \int_0^\infty (\varphi''(x))^2 d\mu + \int_0^\infty \left(\partial^2_{xx}w + \frac{p-1}{x^2}\right)(\varphi'(x))^2 d\mu.$$

Let us abbreviate $w'' = \partial^2_{xx}w(0,x), E = \int u d\mu$ and also $\langle f\rangle = \int_0^\infty f(x)d\mu(x)$. Then, by using the above identities and by completing three squares (marked by wavy underline),

$$\begin{aligned}
\mathrm{Var}_\mu(u) &= -2\langle u'\varphi'\rangle - \langle (L\varphi)^2\rangle \\
&= \underline{\left\langle -2\varphi'\left(u' - \frac{u}{x}\right)\right\rangle} - \left\langle \frac{2\varphi' u}{x}\right\rangle - \left\langle (\varphi'')^2 + \underline{w''(\varphi')^2} + \frac{p-1}{x^2}(\varphi')^2\right\rangle \\
&\leq \underline{\left\langle \frac{1}{w''}\left(u' - \frac{u}{x}\right)^2\right\rangle} - 2\left\langle \frac{\varphi'(L\varphi + E)}{x}\right\rangle - \left\langle (\varphi'')^2 + \frac{p-1}{x^2}(\varphi')^2\right\rangle \\
&= \left\langle \frac{1}{w''}\left(u' - \frac{u}{x}\right)^2\right\rangle + \underline{\langle 2\varphi''\varphi'/x\rangle} - \left\langle \frac{2\varphi' E}{x} + \underline{(\varphi'')^2} + (p+1)\frac{(\varphi')^2}{x^2}\right\rangle \\
&\leq \left\langle \frac{1}{w''}\left(u' - \frac{u}{x}\right)^2\right\rangle - \left\langle \frac{2\varphi' E}{x} + p\underline{\frac{(\varphi')^2}{x^2}}\right\rangle \leq \left\langle \frac{1}{w''}\left(u' - \frac{u}{x}\right)^2\right\rangle + \frac{E^2}{p},
\end{aligned}$$

and (28) is proven. □

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