An isomorphic version of the slicing problem

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Abstract

Here we show that any centrally-symmetric convex body $K \subset \mathbb{R}^n$ has a perturbation $T \subset \mathbb{R}^n$ which is convex and centrally-symmetric, such that the isotropic constant of T is universally bounded. T is close to K in the sense that the Banach-Mazur distance between T and K is $O(\log n)$. If K is a body of a non-trivial type then the distance is universally bounded. The distance is also universally bounded if the perturbation T is allowed to be non-convex. Our technique involves the use of mixed volumes and Alexandrov-Fenchel inequalities. Some additional applications of this technique are presented here.

1 Introduction

Let $K \subset \mathbb{R}^n$ be a centrally-symmetric (i.e. $K = -K$) convex set with a non-empty interior. Such sets are referred to here as "bodies". We denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the standard scalar product and Euclidean norm in $\mathbb{R}^n$. We also define D as the unit Euclidean ball and $S^{n-1} = \partial D$. The body K has a linear image $\tilde{K}$ with $Vol(\tilde{K}) = 1$ such that

$$\int_{\tilde{K}} \langle x, \theta \rangle^2 dx \qquad (1)$$

does not depend on the choice of $\theta \in S^{n-1}$. We say that $\tilde{K}$ is an isotropic linear image of K or that $\tilde{K}$ is in isotropic position. The isotropic linear image of K is unique, up to orthogonal transformations (e.g. [MP1]). The quantity in (1), for any $\theta \in S^{n-1}$ and any $\tilde{K}$ an isotropic linear image of K, is usually referred to as L_K^2 or as the square

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of the isotropic constant of K. An equivalent definition of L_K is the following:

$$nL_K^2 = \inf_T \int_K |Tx|^2 dx \tag{2}$$

where the infimum is over all matrices T such that $det(T) = 1$. For a comprehensive discussion of the isotropic position and the isotropic constant we refer the reader to [MP1].

L_K is an important linearly invariant parameter associated with K. A major conjecture is whether there exists a universal constant $c > 0$ such that $L_K < c$ for all convex centrally-symmetric bodies in all dimensions. A proof of this conjecture will have various consequences. Among others (see [MP1]), it will establish the fact that any body of volume one has at least one $n-1$ dimensional section whose volume is greater than some positive universal constant. This conjecture is known as the slicing problem or the hyperplane conjecture. The best estimate known to date is $L_K < cn^{1/4} \log n$ for $K \subset \mathbb{R}^n$ and is due to Bourgain [Bou2] (see also the presentation in [D]). In addition, the conjecture was verified for large classes of bodies (some examples of references are [Ba2], [Bou1], [J], [KMP], [MP1]).

In this note we deal with a known relaxation of this conjecture, which we call the "isomorphic slicing problem". It was suggested to the author by V. Milman. For two sets $K, T \subset \mathbb{R}^n$, we define their "geometric distance" as

$$d_G(K,T) = \inf \left\{ ab; \ \frac{1}{a}K \subset T \subset bK, \ a, b > 0 \right\}.$$

The Banach-Mazur distance between K and T is

$$d_{BM}(K,T) = \inf\{ d_G(K, L(T)) \ ; \ L \textit{ is a linear operator} \}.$$

Let $K_n, T_n \subset \mathbb{R}^n$ for $n = 1, 2, ...$ be a sequence of bodies such that $d_{BM}(K_n, T_n) < Const$ independent of the dimension n. In this case we say that the families $\{K_n\}$ and $\{T_n\}$ are uniformly isomorphic. Indeed, the norms defined by K_n and T_n are uniformly isomorphic. The isomorphic slicing problem asks whether the slicing problem is correct, at least up to a uniform isomorphism. Formally:

Question 1.1 *Do there exist constants $c_1, c_2 > 0$ such that for any dimension n, for any centrally-symmetric convex body $K \subset \mathbb{R}^n$, there exists a centrally-symmetric convex body $T \subset \mathbb{R}^n$ with $d_{BM}(K,T) < c_1$ and $L_T < c_2$?*

In this note we answer this question affirmatively, up to a logarithmic factor. The following is proven here:

Theorem 1.2 *For any centrally-symmetric convex body $K \subset \mathbb{R}^n$ there exists a centrally-symmetric convex body $T \subset \mathbb{R}^n$ with $d_{BM}(K,T) < c_1 \log n$ and*

$$L_T < c_2$$

where $c_1, c_2 > 0$ are numerical constants.

The $\log n$ factor in Theorem 1.2 stems from the use of the l-position and Pisier's estimate for the norm of the Rademacher projection (see [P]). In fact, in the notation of Theorem 1.2 we prove that $d_{BM}(K,T) < c_1 M(K) M^*(K)$ (see definitions in Section 3). Therefore we verify the validity of the isomorphic slicing conjecture for bodies that have a linear image with bounded MM^*. This large class of bodies includes all bodies of a non trivial type (e.g. [MS]). In addition, Proposition 5.2 and Proposition 5.3 provide other classes of bodies for which Question 1.1 has a positive answer.

There exist some connections between the slicing problem and its isomorphic versions. An example is provided in the following lemma.

Lemma 1.3 *Assume that there exist $c_1, c_2 > 0$ such that for any integer n and an isotropic body $K \subset \mathbb{R}^n$ there exists an isotropic body $T \subset \mathbb{R}^n$ with $d_G(K,T) < c_1$ and $L_T < c_2$. Then there exists $c_3 > 0$ such that for any integer n and body $K \subset \mathbb{R}^n$, we have $L_K < c_3$.*

Proof: $L_T < c_2$, therefore T is in M-position (as observed by K. Ball, see definitions and proofs in [MP1]). Since $d_G(K,T) < c_1$, then K is also in M-position. Using Proposition 1.4 from [BKM] we obtain a universal bound for the isotropic constant. $\square$

A set $K \subset \mathbb{R}^n$ is star-shaped if for any $0 \le t \le 1$ and $x \in K$ we have $tx \in K$. A star shaped set $K \subset \mathbb{R}^n$ is quasi-convex with constant $C > 0$ if $K + K \subset CK$, where $K + T = \{k + t; k \in K, t \in T\}$ for any $K, T \subset \mathbb{R}^n$. For centrally-symmetric quasi-convex sets, the isomorphic slicing problem has an affirmative answer. Formally, as is proven in Section 4,

Theorem 1.4 *For any $C > 1$ there exist $c_1, c_2 > 0$ with the following property: If $K \subset \mathbb{R}^n$ is centrally-symmetric and quasi-convex with constant C, then there exists a centrally-symmetric $T \subset \mathbb{R}^n$ such that $d_{BM}(K,T) < c_1$ and $L_T < c_2$. (Note that T is necessarily $c_1 C$-quasi convex).*

Our proof has a number of consequences which are formulated and proved in Section 5. Among these are an improvement of an estimate from [BKM], and a connection between the isotropic position and an M-position of order α for bodies with a small isotropic constant. Throughout this paper the letters $c, C, c', c_1, c_2, Const$ etc. denote positive numerical constants, whose value may differ in various appearances. The same goes for $c(\varphi), C(\varphi)$ etc. which denote some positive functions that depend purely on their arguments. We ignore measurability issues as they are not essential to our discussion. All sets and functions used here are assumed to be measurable.

2 Log concave functions

In this section we mention some facts regarding log-concave functions, most of which are known and appear in [Ba1] or [MP1], yet our versions are slightly different. $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if $\log f$ is concave on its support. f is s-concave, for $s > 0$, if $f^{1/s}$ is concave on its support. Any s-concave function is also log-concave (see e.g. [Bo], also for the connection with log-concave measures). Given a non-negative function f on $\mathbb{R}^n$ we define for $x \in \mathbb{R}^n$,

$$\|x\|_f = \left(\int_0^\infty f(rx) r^{n+1} dr \right)^{-1/n+2}.$$

We also define $K_f = \{x \in \mathbb{R}^n ; \|x\|_f \leq 1\}$. The following Busemann-type theorem appears in [Ba1] (see also [MP1]):

Theorem 2.1 *Let f be an even log-concave function on $\mathbb{R}^n$. Then K_f is convex and centrally-symmetric and $\|\cdot\|_f$ is a norm.*

In what follows we repeatedly use two well known facts. The first is that for any $1 \leq k \leq n$,

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} < \left(e\frac{n}{k}\right)^k. \tag{3}$$

The second is that for any integers $a, b \geq 0$,

$$\int_0^1 s^a (1-s)^b ds = \frac{1}{(a+b+1)\binom{a+b}{a}}. \tag{4}$$

Lemma 2.2 *Let $f : \mathbb{R}^n \to [0, \infty)$ be an even function whose restriction to any straight line through the origin is s-concave. If $s > n$ then*

$$d_G(K_f, Supp(f)) < c\frac{s}{n}$$

where $c > 0$ is a numerical constant, and $Supp(f) = \{x ; f(x) > 0\}$.

Proof: Multiplying f by a constant if necessary, we may assume that $f(0) = 1$. Fix $\theta \in S^{n-1}$. Denote $M_\theta = \sup\{r > 0; f(r\theta) > 0\}$. Since $f|_{\theta\mathbb{R}}$ is s-concave and $f(0) = 1$, for all $0 \le r \le M_\theta$,

$$f(r\theta) \ge \left(1 - \frac{r}{M_\theta}\right)^s.$$

By the definition of $\|\theta\|_f$ and by (4),

$$\|\theta\|_f^{-(n+2)} \ge \int_0^{M_\theta} \left(1 - \frac{r}{M_\theta}\right)^s r^{n+1} dr = \frac{M_\theta^{n+2}}{(n+s+2)\binom{n+s+1}{n+1}}.$$

In addition, since $f|_{\theta\mathbb{R}}$ is even, its maximum is $f(0) = 1$ and

$$\|\theta\|_f^{-(n+2)} \le \int_0^{M_\theta} r^{n+1} dr = \frac{1}{n+2} M_\theta^{n+2}.$$

Combining this with the estimate (3),

$$\frac{(n+2)^{1/(n+2)}}{M_\theta} \le \|\theta\|_f \le \frac{e(n+s+2)^{1/n+2}\left(\frac{n+s+1}{n+1}\right)^{\frac{n+1}{n+2}}}{M_\theta}$$

and since $s > n$,

$$\forall \theta \in S^{n-1},\ \frac{c_1}{M_\theta} < \|\theta\|_f < \frac{c_2}{M_\theta}\frac{s}{n} \quad \Rightarrow \quad \frac{n}{c_2 s} Supp(f) \subset K_f \subset \frac{1}{c_1} Supp(f)$$

and the lemma is proven. $\square$

The isotropic constant and the isotropic position may also be defined for arbitrary measures or densities, not only for convex bodies. Let $f : \mathbb{R}^n \to [0, \infty)$ be an even function with $0 < \int_{\mathbb{R}^n} f < \infty$. The entries of its covariance matrix with respect to a fixed orthonormal basis $\{e_1, .., e_n\}$ are defined as

$$M_{i,j} = \frac{1}{\int_{\mathbb{R}^n} f(x)dx} \int_{\mathbb{R}^n} f(x)\langle x, e_i\rangle\langle x, e_j\rangle dx.$$

We define $L_f = \left(\frac{f(0)}{\int_{\mathbb{R}^n} f}\right)^{\frac{1}{n}} det(M)^{\frac{1}{2n}}$. One can verify that if $f = 1_K$ is the characteristic function a body $K \subset \mathbb{R}^n$, then $L_f = L_K$. Our next lemma claims that if f is log-concave, then the body K_f shares the isotropic constant of the function f, up to a universal constant. This fact appears in [MP1] and in [Ba1], but our formulation is slightly different. For completeness we present a proof here.

Lemma 2.3 *Let f be an even function on $\mathbb{R}^n$ whose restriction to any straight line through the origin is log-concave. Assume that $\int_{\mathbb{R}^n} f < \infty$. Then,*

$$c_1 L_f < L_{K_f} < c_2 L_f$$

where $c_1, c_2 > 0$ are universal constants.

Proof: We may assume that $f(0) = 1$. Integrating in polar coordinates, for any $y \in \mathbb{R}^n$,

$$\begin{aligned}
&\int_{K_f} \langle x, y\rangle^2 dx \\
&= \int_{S^{n-1}} \int_0^{1/\|\theta\|_f} \langle y, r\theta\rangle^2 r^{n-1} dr d\theta = \frac{1}{n+2} \int_{S^{n-1}} \langle y, \theta\rangle^2 \frac{1}{\|\theta\|_f^{n+2}} d\theta \\
&= \frac{1}{n+2} \int_0^\infty \int_{S^{n-1}} f(r\theta) \langle y, \theta\rangle^2 r^{n+1} dr d\theta = \frac{1}{n+2} \int_{\mathbb{R}^n} \langle x, y\rangle^2 f(x) dx
\end{aligned}$$

where $d\theta$ is the induced surface area measure on S^{n-1}. Denote by $M(f)$ and $M(K_f)$ the inertia matrices of f and of 1_{K_f}, respectively. We conclude that $Vol(K_f) M(K_f) = \frac{1}{n+2} \left(\int_{\mathbb{R}^n} f\right) M(f)$. To compare the isotropic constants, we need to estimate $\frac{\int f}{Vol(K_f)}$. Now,

$$Vol(K_f) = \frac{1}{n} \int_{S^{n-1}} \left(\int_0^\infty f(r\theta) r^{n+1} dr \right)^{\frac{n}{n+2}} d\theta. \tag{5}$$

We shall use the following one-dimensional lemma, which is proven at the end of this section (see also [Ba1], [BKM] or [MP1]).

Lemma 2.4 *Let $g : [0, \infty) \to [0, \infty)$ be a non-increasing log-concave function with $g(0) = 1$ and $\int_0^\infty g(t) t^{n-1} dt < \infty$. Then, for any integer $n \geq 1$,*

$$\frac{n^{\frac{n+2}{n}}}{n+2} \leq \frac{\int_0^\infty g(t) t^{n+1} dt}{\left(\int_0^\infty g(t) t^{n-1} dt\right)^{\frac{n+2}{n}}} \leq \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}}.$$

(the left-most inequality - which is more important to us - holds also without the log-concavity assumption).

Since f is even and log-concave on any line through the origin, it is non-increasing on any ray that starts at the origin. From the left-most inequality in Lemma 2.4, for any $\theta \in S^{n-1}$ (except for a set of measure zero where the integral diverges),

$$\int_0^\infty f(r\theta) r^{n+1} dr \geq \frac{n^{\frac{n+2}{n}}}{n+2} \left(\int_0^\infty f(r\theta) r^{n-1} dr \right)^{\frac{n+2}{n}}$$

and according to (5),

$$Vol(K_f) \geq \frac{1}{n}\frac{n^{\frac{n+2}{n}}}{n+2}\int_{S^{n-1}}\int_0^\infty f\left(r\theta\right)r^{n-1}drd\theta = \frac{n^{2/n}}{n+2}\int_{\mathbb{R}^n} f.$$

Since $M(K_f) = \frac{1}{n+2}\frac{\int_{\mathbb{R}^n} f}{Vol(K_f)}M(f)$,

$$\frac{L_{K_f}^2}{L_f^2} = \frac{1}{n+2}\left(\frac{\int_{\mathbb{R}^n} f}{Vol(K_f)}\right)^{1+\frac{2}{n}} \leq \frac{1}{n+2}\left(\frac{n+2}{n^{2/n}}\right)^{\frac{n+2}{n}} < c_2.$$

This completes the proof of one part of the lemma. The proof of the other inequality is similar. Using the right-most inequality in Lemma 2.4,

$$\frac{L_{K_f}^2}{L_f^2} = \frac{1}{n+2}\left(\frac{\int_{\mathbb{R}^n} f}{Vol(K_f)}\right)^{1+\frac{2}{n}} \geq \frac{1}{n+2}\left(\frac{n\left((n-1)!\right)^{\frac{n+2}{n}}}{(n+1)!}\right)^{\frac{n+2}{n}} > c_1$$

and the lemma is proven. □

Proof of Lemma 2.4: Begin with the left-most inequality. Define $A > 0$ such that $\int_0^\infty g(t)t^{n-1}dt = \int_0^A t^{n-1}dt$. Then,

$$\begin{aligned} &\int_0^A (1-g(t))t^{n+1}dt - \int_A^\infty g(t)t^{n+1}dt \\ &\quad\leq \quad A^2\left[\int_0^A (1-g(t))t^{n-1}dt - \int_A^\infty g(t)t^{n-1}dt\right] = 0. \end{aligned}$$

Since $\int_0^A t^{n+1}dt = \frac{n^{\frac{n+2}{n}}}{n+2}\left(\int_0^A t^{n-1}dt\right)^{\frac{n+2}{n}}$, we get that

$$\int_0^\infty g(t)t^{n+1}dt \geq \int_0^A t^{n+1}dt = \frac{n^{\frac{n+2}{n}}}{n+2}\left(\int_0^\infty g(t)t^{n-1}dt\right)^{\frac{n+2}{n}}.$$

To obtain the other inequality we need to use the log-concavity of the function. Define $B > 0$ such that $h(t) = e^{-Bt}$ satisfies

$$\int_0^\infty g(t)t^{n-1}dt = \int_0^\infty h(t)t^{n-1}dt.$$

It is impossible that $g < h$ always or $g > h$ always, hence necessarily $t_0 = \inf\{t > 0; h(t) \geq g(t)\}$ is finite. $-\log g$ is convex and vanishes at zero, so $\tilde{g}(t) = \frac{-\log g(t)}{t}$ is non-decreasing. Thus $(B - \tilde{g}(t))(t - t_0) \geq 0$ or equivalently $(h(t) - g(t))(t - t_0) \geq 0$ for all $t > 0$. Therefore,

$$\begin{aligned} &\int_0^{t_0} (g(t)-h(t))t^{n+1}dt - \int_{t_0}^\infty (h(t)-g(t))t^{n+1}dt \\ &\quad\leq \quad t_0^2\left[\int_0^{t_0} (g(t)-h(t))t^{n-1}dt - \int_{t_0}^\infty (h(t)-g(t))t^{n-1}dt\right] = 0. \end{aligned}$$

Since $\int_0^\infty e^{-tB}t^{n+1}dt = \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}}\left(\int_0^\infty e^{-tB}t^{n-1}dt\right)^{\frac{n+2}{n}}$,

$$\int_0^\infty g(t)t^{n+1}dt \leq \int_0^\infty h(t)t^{n+1}dt = \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}}\left(\int_0^\infty g(t)t^{n-1}dt\right)^{\frac{n+2}{n}}.$$

$\square$

3 Constructing a function on K

Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. In this section we find an αn-concave function F supported on K whose isotropic constant is bounded. From Lemma 2.3 it follows that $L_{K_F} < Const.$ According to Lemma 2.1, K_F is a convex body, and by Lemma 2.2 we get that $d_G(K, K_F) < c\alpha$. If good estimates on α were obtained, Theorem 1.2 would follow. Let $\|\cdot\|$ be the norm for which K is its unit ball, and denote by σ the unique rotation invariant probability measure on S^{n-1}. The median of $\|x\|$ on S^{n-1} with respect to σ is referred to as $M'(K)$. We abbreviate $M' = M'(K)$ and define the following function on K:

$$f_K(x) = \inf\left\{0 \leq t \leq 1; x \in (1-t)\left[K \cap \frac{1}{M'}D\right] + tK\right\}.$$

Then f_K is a convex function which equals zero on $K \cap \frac{1}{M}D$. Define also

$$M(K) = \int_{S^{n-1}} \|x\| d\sigma(x), \quad M^*(K) = \int_{S^{n-1}} \|x\|_* d\sigma(x)$$

where $\|x\|_* = \sup_{y\in K}\langle x, y\rangle$ is the dual norm.

Proposition 3.1 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body, and let $\alpha = cM(K)M^*(K)$. Then,*

$$\int_K (1 - f_K(x))^{\alpha n}\, dx < 2Vol\left(K \cap \frac{1}{M'}D\right)$$

where $c > 0$ is some numerical constant.

Proof: We denote $F(x) = (1 - f(x))^{\alpha n}$. Then,

$$\begin{aligned}\int_K F(x)dx &= \int_0^1 Vol\{x \in K; F(x) \geq t\}dt \\ &= \int_0^1 Vol\{x \in K; f(x) \leq 1 - t^{\frac{1}{\alpha n}}\}dt\end{aligned}$$

and substituting $s = 1 - t^{\frac{1}{\alpha n}}$ yields

$$\int_K F(x)dx = \alpha n \int_0^1 (1-s)^{\alpha n - 1} Vol\left((1-s)\left[K \cap \frac{1}{M'}D\right] + sK\right) ds.$$

Expand the volume term into a polynomial whose coefficients are mixed volumes (see e.g. [Sch]):

$$Vol\left((1-s)\left[K \cap \frac{1}{M'}D\right] + sK\right) = \sum_{i=0}^{n} \binom{n}{i} V_i s^i (1-s)^{n-i}$$

where $V_i = V(K, i; \left[K \cap \frac{1}{M'}D\right], n-i)$. Then,

$$\int_K F(x)dx = \alpha n \sum_{i=0}^{n} V_i \binom{n}{i} \int_0^1 s^i (1-s)^{(\alpha+1)n-i-1} ds$$

and by (4),

$$\int_K F(x)dx = \frac{\alpha}{\alpha+1} V_0 \sum_{i=0}^{n} \frac{\binom{n}{i}}{\binom{(1+\alpha)n-1}{i}} \frac{V_i}{V_0}.$$

Using (3) we may write

$$\int_K F(x)dx = \frac{\alpha}{\alpha+1} V_0 \left[1 + \sum_{i=1}^{n} \left(c_{n,i} \frac{n}{(1+\alpha)n-1} \left(\frac{V_i}{V_0}\right)^{1/i}\right)^i\right] \quad (6)$$

where $\frac{1}{e} \le c_{n,i} \le e$. By Alexandrov-Fenchel inequalities, $V_i^2 \ge V_{i-1}V_{i+1}$ for $i \ge 1$ (e.g. [Sch]). It follows that for $1 \le i \le j$,

$$\left(\frac{V_i}{V_0}\right)^{1/i} \ge \left(\frac{V_j}{V_0}\right)^{1/j}. \quad (7)$$

In particular, if $\alpha + 1 > 4e\frac{V_1}{V_0}$, then by (7),

$$c_{n,i} \frac{n}{(1+\alpha)n-1} \left(\frac{V_i}{V_0}\right)^{1/i} < \frac{2e}{1+\alpha} \frac{V_1}{V_0} \le \frac{1}{2}.$$

Substituting into (6) we obtain

$$\int_K F(x)dx < V_0 \sum_{i=0}^{n} \frac{1}{2^i} < 2V_0 = 2Vol\left(K \cap \frac{1}{M'}D\right).$$

We still need to show that our $\alpha = cM(K)M^*(K)$ is greater than $4e\frac{V_1}{V_0}$. Since $\frac{1}{M'}D \cap K \subset \frac{1}{M'}D$,

$$\begin{aligned} V_1 &= V(K, 1; \left[K \cap \frac{1}{M'}D\right], n-1) \\ &\leq V\left(K, 1; \frac{1}{M'}D, n-1\right) = \frac{1}{(M')^{n-1}} Vol(D) M^*(K) \end{aligned}$$

because $Vol(D)M^*(K) = V(K, 1; D, n-1)$ (see e.g. [Sch]). Regarding V_0, since M' is the median,

$$\sigma\left(M'K \cap S^{n-1}\right) \geq \frac{1}{2} \quad \Rightarrow \quad Vol\left(K \cap \frac{1}{M'}D\right) \geq \frac{Vol\left(\frac{1}{M'}D\right)}{2}.$$

In conclusion,

$$\frac{V_1}{V_0} \leq \frac{1}{(M')^{n-1}} Vol(D) M^*(K) \frac{2}{\frac{1}{(M')^n} Vol(D)} = 2M'(K)M^*(K).$$

The median of a positive function is not larger than twice its expectation. Therefore, $M'(K) \leq 2M(K)$, and we get that for $\alpha = cM(K)M^*(K)$, it is true that $\alpha + 1 > 4e\frac{V_1}{V_0}$ for a suitable numerical constant $c > 0$. □

Corollary 3.2 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body, $\alpha = cM(K)M^*(K)$ and denote $F(x) = (1 - f_K(x))^{\alpha n}$. Then,*

$$L_F < c'$$

where $c, c' > 0$ are universal constants.

Proof: Consider F as a density on K, i.e. consider the probability measure $\mu_F(A) = \frac{\int_A F(x)dx}{\int_K F(x)dx}$. Since $F \equiv 1$ on $K \cap \frac{1}{M'}D$, by Proposition 3.1,

$$\mu\left(K \cap \frac{1}{M'}D\right) > \frac{1}{2}.$$

In other words, the median of the Euclidean norm with respect to μ is not larger than $\frac{1}{M'}$. Since F is αn-concave,

$$\mathbb{E}_\mu |x|^2 < \frac{c}{(M')^2}$$

by standard concentration inequalities for the Euclidean norm with respect to log-concave measures (it follows, e.g. from Theorem III.3 in [MS], due to Borell). Combining definition (2) and the fact that

$L_F^2 = \left(\frac{F(0)}{\int_K F}\right)^{\frac{2}{n}} det(M_F)^{\frac{1}{n}}$ where M_F is the covariance matrix, we get that

$$\left(\frac{\int_K F(x)dx}{F(0)}\right)^{\frac{2}{n}} nL_F^2 \leq \mathbb{E}_\mu |x|^2 < \frac{c}{(M')^2}.$$

Since $\int_K F(x)dx \geq Vol\left(\frac{1}{M'}D \cap K\right) \geq \frac{1}{2}Vol(\frac{1}{M'}D)$ and $F(0) = 1$, we obtain that $L_F^2 < \frac{c'}{nVol(D)^{2/n}} < Const.$ $\square$

Proof of Theorem 1.2: We shall use the notion of l-ellipsoid, and Pisier's estimate for $M(K)M^*(K)$. We refer the reader to [P] or [MS] for definitions and proofs. Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. There exists a linear image $\tilde{K}$ of K such that its l-ellipsoid is the standard Euclidean ball. By Pisier's estimate,

$$M^*(\tilde{K})M(\tilde{K}) < c \log d_{BM}(K, D) < c' \log n.$$

According to Corollary 3.2, there exists an αn-concave function F supported exactly on $\tilde{K}$, with $\alpha = cM(\tilde{K})M^*(\tilde{K})$ and $L_F < c_1$. By Lemma 2.3 we get that $L_{K_F} < c_2$. From Lemma 2.2,

$$d_{BM}(K, K_F) \leq d_G(\tilde{K}, K_F) < c\alpha < c'M(\tilde{K})M^*(\tilde{K}) < C \log n.$$

This completes the proof. $\square$

4 The quasi-convex case

We define the covering number of $K \subset \mathbb{R}^n$ by $T \subset \mathbb{R}^n$ as

$$N(K, T) = \min \left\{ N > 0; \exists x_1, .., x_N \in \mathbb{R}^n,\ K \subset \bigcup_{i=1}^{N} x_i + T \right\}.$$

Every convex body $K \subset \mathbb{R}^n$ is associated with a special ellipsoid, called a Milman ellipsoid or an M-ellipsoid. An M-ellipsoid may be defined by the following theorem, which was proved for the convex case in [M1] (see also chapter 7 in [P]). The extension to the quasi convex case appears in [BBS].

Theorem 4.1 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric quasi-convex body with constant β. Then there exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ with $Vol(\mathcal{E}) = Vol(K)$ such that*

$$N(K, \mathcal{E}) < e^{cn}, \quad N(\mathcal{E}, K) < e^{cn}$$

where $c = c(\beta) > 0$ depends solely on β. We say that $\mathcal{E}$ is an M-ellipsoid of K (with constant c).

If a Euclidean ball of appropriate radius is an M-ellipsoid of K, we say that K is in M-position (with some constant). The following lemma is standard:

Lemma 4.2 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric quasi-convex body with constant β such that $Vol(K) = 1$, and which is in M-position with constant $c = c(\beta)$. Then,*

1. $Vol(K \cap \sqrt{n}D)^{1/n} > c' Vol(D)^{1/n}$.
2. $K \subset e^{\tilde{c}n} D$

where $c' = c'(\beta) > 0$, $\tilde{c} = \tilde{c}(\beta) > 0$ depend solely on β.

Proof: All constants in this proof depend on β. Let $\mathcal{D}_n$ be a Euclidean ball of volume one in $\mathbb{R}^n$. Then $N(K, \mathcal{D}_n) < e^{\bar{c}n}$. Since $c < Vol(\sqrt{n}D)^{1/n} < C$, then also $N(K, \sqrt{n}D) < e^{cn}$ (e.g. Lemma 7.5 in [P]). Hence there exists a point $x \in \mathbb{R}^n$ such that $Vol\left(K \cap (x + \sqrt{n}D)\right) > e^{-cn}$. Since K is centrally-symmetric, $K \cap (-x + \sqrt{n}D) \neq \emptyset$. By quasi-convexity,

$$\emptyset \neq \left[K \cap \left(x + \sqrt{n}D\right)\right] + \left[K \cap \left(-x + \sqrt{n}D\right)\right] \subset \beta K \cap 2\sqrt{n}D$$

and hence $Vol(\beta K \cap 2\sqrt{n}D) > e^{-cn}$, as it contains a translation of $K \cap (x + \sqrt{n}D)$. Since $\beta \geq 2$,

$$Vol(K \cap \sqrt{n}D) \geq \frac{1}{\beta^n} Vol(\beta K \cap 2\sqrt{n}D) > e^{-(c + \log\beta)n}.$$

To obtain that $K \subset e^{\tilde{c}n}D$, we just use the fact that K is a star body, and that a segment of length larger than $2\sqrt{n}e^{cn}$ cannot be covered by e^{cn} balls of radius $\sqrt{n}$. $\square$

Let $K \subset \mathbb{R}^n$ be a centrally-symmetric quasi-convex body with constant β (in short "a β-quasi-body"). Assume that $Vol(K) = 1$ and that K is in M-position. Let us construct the following function on K:

$$F_K(x) = \begin{cases} 1 & |x| \leq \sqrt{n} \\ \left(1 - \frac{|x| - \sqrt{n}}{M_x - \sqrt{n}}\right)^{\alpha n} & |x| > \sqrt{n} \end{cases}$$

for some $\alpha > 0$ to be determined later, where

$$M_x = \sup\left\{r > 0; r\frac{x}{|x|} \in K\right\}.$$

F_K is not log-concave, yet we may still consider the centrally-symmetric set $K_{F_K} \subset \mathbb{R}^n$, defined in Section 2. Note that the restriction of F_K to any straight line through the origin is αn-concave on its support, hence it is possible to apply Lemma 2.2 or Lemma 2.3. We begin with a one-dimensional lemma.

Lemma 4.3 *Let $0 < a < b$ and $\alpha > 1$ be such that $b > 2a\left(1+\frac{\alpha}{e}\right)$. Let n be a positive integer. Then,*

$$\int_a^b \left(1 - \frac{t-a}{b-a}\right)^{\alpha n} t^n dt < \left(\frac{c_1}{\alpha}\right)^n \int_a^b t^n dt$$

where $c_1 > 0$ is a numerical constant.

Proof: Denote the integral on the left by I and the integral on the right by $J = \frac{1}{n+1}\left[b^{n+1} - a^{n+1}\right]$. Substituting $s = \frac{t-a}{b-a}$ obtains

$$\begin{aligned} I &= (b-a)\int_0^1 (1-s)^{\alpha n}\left(a + (b-a)s\right)^n ds \\ &= (b-a)\sum_{i=0}^n \binom{n}{i} a^{n-i}(b-a)^i \int_0^1 (1-s)^{\alpha n} s^i ds \end{aligned}$$

and using (4),

$$I = (b-a)a^n \sum_{i=0}^n \frac{\binom{n}{i}}{(\alpha n + i + 1)\binom{\alpha n + i}{i}} \left(\frac{b-a}{a}\right)^i.$$

The estimate (3) along with some trivial inequalities, yields that

$$I \leq \frac{b-a}{\alpha n} a^n \sum_{i=0}^n \left(\frac{e}{\alpha}\right)^i \left(\frac{b-a}{a}\right)^i = \frac{b-a}{\alpha n} a^n \frac{q^{n+1}-1}{q-1}$$

where $q = \frac{e(b-a)}{\alpha a}$. We assumed that $q \geq 2$, and hence

$$I \leq \frac{2}{en}(aq)^{n+1} = \frac{2}{en}\left(\frac{e}{\alpha}\right)^n (b-a)^{n+1} < \left(\frac{c}{\alpha}\right)^n J.$$

$\square$

Next we show that for a suitable value of α, which is just a numerical constant, most of the mass of F_K is not far from the origin.

Lemma 4.4 *For any $\alpha > 1$,*

$$\int_{\mathbb{R}^n \setminus c_2 \alpha \sqrt{n} D} F_K(x)dx < \left(\frac{c_1}{\alpha}\right)^{n-1} Vol(K)$$

where c_1 is the constant from Lemma 4.3 and $0 < c_2 \leq 2 + \frac{2}{e}$ is a numerical constant.

Proof: Note that

$$\int_{\mathbb{R}^n\setminus\sqrt{n}D} F_K(x)dx = \int_{S^{n-1}} \int_{\sqrt{n}}^{\max\{M_\theta,\sqrt{n}\}} \left(1 - \frac{r-\sqrt{n}}{M_\theta - \sqrt{n}}\right)^{\alpha n} r^{n-1} dr d\theta$$

where $d\theta$ is the induced surface area measure on the sphere. Let $E = \{\theta \in S^{n-1}; M_\theta > c_2\alpha\sqrt{n}\}$. By Lemma 4.3,

$$\begin{aligned} & \int_{\mathbb{R}^n\setminus c_2\alpha\sqrt{n}D} F_K(x)dx \\ < \quad & \int_E \int_{\sqrt{n}}^{M_\theta} \left(1 - \frac{r-\sqrt{n}}{M_\theta - \sqrt{n}}\right)^{\alpha n} r^{n-1} dr d\theta \\ < \quad & \left(\frac{c_1}{\alpha}\right)^{n-1} \int_E \int_{\sqrt{n}}^{M_\theta} r^{n-1} dr d\theta < \left(\frac{c_1}{\alpha}\right)^{n-1} Vol(K). \end{aligned}$$

□

Lemma 4.5 *Assume that $K \subset \mathbb{R}^n$ is a β-quasi-body of volume one in M-position. Then for $\alpha = c_3(\beta)$,*

$$L_{F_K} < c_4(\beta)$$

where $c_3(\beta), c_4(\beta)$ depend solely on β, not on K or on n.

Proof: By Lemma 4.2,

$$Vol\left(K \cap \sqrt{n}D\right)^{1/n} > c'(\beta).$$

If $\alpha = c_3(\beta)$ is suitably chosen, then by Lemma 4.4,

$$\int_{\mathbb{R}^n\setminus c_2\alpha\sqrt{n}D} F_K(x)dx < \left(\frac{c_1}{\alpha}\right)^{n-1} < \frac{\alpha}{c_1}\left(\frac{1}{e^{2\tilde{c}(\beta)}}\right)^n Vol\left(K \cap \sqrt{n}D\right).$$

Define a measure by $\mu(E) = \frac{\int_E F_K(x)dx}{\int_{\mathbb{R}^n} F_K(x)dx}$. Since F_K equals 1 on $K \cap \sqrt{n}D$, we get that

$$\mu(\mathbb{R}^n \setminus c_2\alpha\sqrt{n}D) < \frac{\alpha}{c_1}\left(\frac{1}{e^{2\tilde{c}(\beta)}}\right)^n.$$

Since $K \subset e^{\tilde{c}(\beta)n}D$, then

$$\mathbb{E}_\mu|x|^2 < (c_2\alpha)^2 n + \frac{\alpha}{c_1}\left(\frac{1}{e^{2\tilde{c}(\beta)}}\right)^n \cdot e^{2\tilde{c}(\beta)n} < c(\beta)n.$$

Therefore, as in Corollary 3.2, $L_{F_K}^2 < c(\beta)\left(\frac{F_K(0)}{\int F_K}\right)^{\frac{2}{n}}$. Note that $F_K(0) = 1$. Since $\int F_K \geq Vol(K \cap \sqrt{n}D)$, we conclude that

$$L_{F_K}^2 < c_4(\beta).$$

□

Proof of Theorem 1.4: Let $K \subset \mathbb{R}^n$ be a C-quasi-body. Let $\tilde{K}$ be a linear image of K such that $Vol(\tilde{K}) = 1$ and $\tilde{K}$ is in M-position (with a constant that depends only on C). Consider the function $F_{\tilde{K}}$ for $\alpha = c_3(C)$. By Lemma 2.2, the body $T = K_{F_{\tilde{K}}}$ satisfies

$$d_G(\tilde{K}, T) < c'(C)$$

for some function $c'(C) > 0$. Also, by Lemma 2.3 and Lemma 4.5,

$$L_T < \tilde{c} L_{F_{\tilde{K}}} < \bar{c}(C)$$

for some $\bar{c}(C)$, a function of C. This completes the proof. □

Remark: There exist quasi-bodies with large isotropic constants. For example, fix $\{e_1, .., e_n\}$ an orthonormal basis in $\mathbb{R}^n$, and let $K = B_1^n \cup \bigcup_{i=1}^n e_i + B_1^n$ where $B_1^n = \{x; \sum_i |\langle x, e_i \rangle| \leq 1\}$. The quasi-convex body K has an isotropic constant of order $\sqrt{n}$, the largest possible order. However, if a quasi-body is close to an ellipsoid, then its isotropic constant is controlled by the distance to the ellipsoid. Also, a quasi-body with a small outer volume ratio has a universally bounded isotropic constant.

5 Consequences of the proof

Here we present a few results which are byproducts of our methods. Our first two propositions enrich the family of convex bodies for which Question 1.1 has an affirmative answer. In this section $Vol(T)$ denotes the volume of a set $T \subset \mathbb{R}^n$ relative to its affine hull.

Lemma 5.1 *Let $K \subset \mathbb{R}^n$ be an isotropic centrally-symmetric convex body of volume one, $0 < \lambda < 1$ and $L_K < A$ for some $A > 1$. Then for any subspace E of dimension λn,*

$$Vol(K \cap E)^{\frac{1}{n}} < c(A)$$

where $c(A)$ depends solely on A, and is independent of the body K and of the dimension n.

Proof: Since $\mathbb{E}_K |x|^2 < nA^2$, the median of the function $|x|$ on K is smaller than $2\sqrt{n}A$. Then $K' = K \cap 2\sqrt{n}AD$ satisfies $Vol(K') > \frac{1}{2}$. Also, given any subspace $E \subset \mathbb{R}^n$ of dimension λn,

$$Vol(K' \cap E) \leq Vol(2\sqrt{n}AD \cap E) \leq \left(c \frac{A}{\sqrt{\lambda}} \right)^{\lambda n}.$$

Since K' is symmetric, $Vol(K') \le Vol(K' \cap E) Vol(Proj_{E^\perp} K')$, where $E^\perp$ is the orthogonal complement of E and $Proj_{E^\perp}$ is the orthogonal projection onto $E^\perp$ in $\mathbb{R}^n$. Therefore,

$$Vol\left(Proj_{E^\perp} K\right) \ge Vol\left(Proj_{E^\perp} K'\ \right) \ge \frac{Vol(K')}{Vol(K' \cap E)} \ge \left(c\frac{\sqrt{\lambda}}{A}\right)^{\lambda n}.$$

We denote the polar body of K by $K^\circ = \{y \in \mathbb{R}^n; \forall x \in K, \langle x, y\rangle \le 1\}$. By Santaló's inequality [Sa] and reverse Santaló [BM] (recall that projection and section are dual operations),

$$\begin{aligned} & Vol(K \cap E) Vol\left(Proj_{E^\perp} K\right) && (8) \\ < & \left(\frac{c}{\lambda n}\right)^{\lambda n} \left(\frac{c}{(1-\lambda)n}\right)^{(1-\lambda)n} \frac{1}{Vol\left(Proj_E K^\circ\right) Vol(K^\circ \cap E^\perp)} \\ < & \left(\frac{c'}{n}\right)^n \frac{1}{Vol(K^\circ)} < \left(\frac{c''}{n}\right)^n \frac{1}{Vol(D)^2} Vol(K) < \tilde{c}^n Vol(K). \end{aligned}$$

Hence,

$$Vol(K \cap E)^{\frac{1}{n}} < \tilde{c} \frac{Vol(K)^{\frac{1}{n}}}{Vol\left(Proj_{E^\perp} K\right)^{\frac{1}{n}}} < \tilde{c}\left(c\frac{A}{\sqrt{\lambda}}\right)^{\lambda} < c' A^{\lambda}$$

and the lemma is proven, with $c(A) = cA > cA^\lambda$. □

The next proposition states that the isomorphic slicing conjecture holds for all projections to proportional dimension of bodies with a bounded isotropic constant.

Proposition 5.2 *Let $K \subset \mathbb{R}^n$ be a body with $L_K < A$, and let $0 < \lambda < 1$. Then for any subspace E of dimension λn, there exists a convex body $T \subset E$ such that*

$$d_{BM}(Proj_E(K), T) < c'(\lambda), \quad L_T < c(\lambda, A)$$

where $Proj_E$ is the orthogonal projection onto E in $\mathbb{R}^n$, and $c'(\lambda), c(\lambda, A)$ are independent of K and of n.

Proof: We may assume that K is of volume one and in isotropic position. For $x \in E$, define

$$f(x) = Vol(K \cap [E^\perp + x]).$$

For any $\theta_1, \theta_2 \in E$,

$$\int_E \langle x, \theta_1\rangle\langle x, \theta_2\rangle f(x)dx = \int_K \langle x, \theta_1\rangle\langle x, \theta_2\rangle dx.$$

Hence by Lemma 5.1,

$$L_f = (f(0))^{\frac{1}{\lambda n}} L_K < Vol(K \cap E^\perp)^{\frac{1}{\lambda n}} A < c(A)^{\frac{1}{\lambda}} A = c'(\lambda, A).$$

Set $T = K_f$. By Lemma 2.3 we know that $L_T < \tilde{c} L_f < c''(\lambda, A)$. Also, by Brunn-Minkowski (e.g. [Sch]) f is $(1-\lambda)n$-concave. By Lemma 2.2 $d_G(T, Proj_E(K)) < c\frac{1-\lambda}{\lambda}$, and the proof is complete. □

Our next proposition verifies the isomorphic slicing conjecture under the condition that at least a small portion of K (say, of volume larger than $e^{-\sqrt{n}}$) is located not too far from the origin.

Proposition 5.3 *Let $K \subset \mathbb{R}^n$ be a body of volume one, such that $K \subset \beta n D$. Assume that $Vol(K \cap \gamma\sqrt{n}D) > e^{-\delta\sqrt{n}}$. Then there exists a body $T \subset \mathbb{R}^n$ such that*

$$d_{BM}(K, T) < c\left(1 + \frac{\beta\delta}{\gamma}\right), \quad L_T < c'\gamma$$

where $c, c' > 0$ are numerical constants.

Proof: If $K \subset 2\gamma\sqrt{n}D$, the proposition is trivial since $L_K < c'\gamma$. Assume the contrary, and denote $C = K \cap 2\gamma\sqrt{n}D$. As in Section 3, we define

$$f(x) = \inf\{0 \le t \le 1; x \in (1-t)C + tK\}$$

and consider the density $F(x) = (1 - f(x))^{\alpha n}$ on K for $\alpha = c'\frac{V(K,1;C,n-1)}{Vol(C)}$. As in Proposition 3.1, we get that $\int_C F(x)dx > \frac{1}{2}\int_K F(x)dx$. The same argument used in Corollary 3.2 shows that

$$L_{K_F} < c'\gamma, \quad d_G(K_F, K) < c\frac{V(K,1;C,n-1)}{Vol(C)}.$$

Hence, it remains to show that $\frac{V(K,1;C,n-1)}{Vol(C)} \le 1 + \frac{\beta\delta}{\gamma}$. Define $f(t) = Vol(K \cap tD)$. According to our assumption, $\log f(\gamma\sqrt{n}) > -\delta\sqrt{n}$ and $\log f(2\gamma\sqrt{n}) < 0$. We conclude that there exists $\gamma\sqrt{n} < t_0 < 2\gamma\sqrt{n}$ with $(\log f(t_0))' < \frac{\delta}{\gamma}$. By Brunn-Minkowski inequality, $\log f$ is concave and $(\log f)'$ is decreasing. Therefore, for $t = 2\gamma\sqrt{n} \ge t_0$,

$$(\log f(t))' = \frac{Vol(K \cap tS^{n-1})}{Vol(K \cap tD)} < \frac{\delta}{\gamma}.$$

For $x \in \partial C$, we denote by ν_x the outer unit normal to C at x, if it is unique (it is unique except for a set of measure zero, see [Sch]). Let

$h_K(x) = \sup_{y\in K}\langle x, y\rangle$. Then (see [Sch]),

$$\begin{aligned} V(K,1;C,n-1) &= \frac{1}{n}\int_{\partial C} h_K(\nu_x)dx \\ &= \frac{1}{n}\int_{K\cap tS^{n-1}} h_K(x)dx + \frac{1}{n}\int_{\partial C\setminus tS^{n-1}} h_C(\nu_x)dx \\ &\leq \frac{1}{n}\left(\frac{\delta}{\gamma}Vol(C)\right)\beta n + Vol(C) = \left(1+\frac{\beta\delta}{\gamma}\right)Vol(C) \end{aligned}$$

where we used the fact that $h_K \leq \beta n$ and that $Vol(C) = \frac{1}{n}\int_{\partial C} h_C(\nu_x)dx$. This completes the proof. $\square$

Following Pisier (e.g. [P]), we say that K is in M-position of order α with constants c_α, c'_α if $Vol(K) = Vol(rD)$ and for all $t > 1$

$$max\{N(K, tc_\alpha rD), N(rD, tc_\alpha K)\} < e^{c'_\alpha \frac{n}{t^\alpha}}. \tag{9}$$

By a duality theorem [AMS], if K is in M-position of order α, then also

$$max\left\{N\left(K^\circ, c'c_\alpha t\frac{1}{r}D\right), N\left(\frac{1}{r}D, c'c_\alpha tK^\circ\right)\right\} < e^{\tilde{c}_\alpha \frac{n}{t^\alpha}}$$

for some numerical constant $c' > 0$. A fundamental theorem of Pisier [P] states that for any $\alpha < 2$, a centrally-symmetric convex body has a linear image in M-position of order α, with some constants that depend solely on α. Next, we show that bodies with a relatively small isotropic constant satisfy half of the requirements of Pisier's M-position of order 1.

Proposition 5.4 *Let $K \subset \mathbb{R}^n$ be a convex isotropic body whose volume is one and such that $L_K < A$ for some number A. Then for any $t > 1$,*

$$N(K, ctA\sqrt{n}D) < exp\left(c'\frac{n}{t}\right)$$

where $c, c' > 0$ are numerical cosntants.

Proof: If $K \subset 4A\sqrt{n}D$, then trivially $N(K, 4At\sqrt{n}D) = 1$ and there is nothing to prove. Otherwise, denote $f(t) = Vol(K \cap tD)$. The median of the Euclidean norm on K is smaller than $2\sqrt{n}A$, hence $f(2\sqrt{n}A) \geq \frac{1}{2}$. Also, $f(4\sqrt{n}A) < 1$. Therefore, there exists a point $t_0 \in [2\sqrt{n}A, 4\sqrt{n}A]$ such that

$$\frac{Vol_{n-1}(K\cap t_0 S^{n-1})}{Vol_n(K\cap t_0 D)} = (\log f(t_0))' < \frac{\log 2}{4\sqrt{n}A - 2\sqrt{n}A} = \frac{c}{\sqrt{n}A}.$$

Denote $T = K \cap t_0 D$. For $x \in \partial T$, denote by ν_x the outer unit normal to T at x, if it is unique. Since K is isotropic, $K \subset \tilde{c}nAD$ (see [MP1]), and

$$\int_{K \cap t_0 S^{n-1}} h_K(\nu_x) dx \tag{10}$$
$$\leq \; Vol_{n-1}(K \cap t_0 S^{n-1}) \tilde{c} n A \leq \frac{c}{\sqrt{n}A} Vol(T) \tilde{c} n A = c' \sqrt{n} Vol(T).$$

Because $Vol(T) = \frac{1}{n} \int_{\partial T} h_T(\nu_x) dx$,

$$\int_{\partial T \setminus t_0 S^{n-1}} h_K(\nu_x) dx = \int_{\partial T \setminus t_0 S^{n-1}} h_T(\nu_x) dx \leq n Vol(T). \tag{11}$$

Since $\partial T = \partial T \setminus t_0 S^{n-1} \cup [K \cap t_0 S^{n-1}]$, adding (10) to (11) obtains

$$nV(T, n-1; K, 1) = \int_{\partial T} h_K(\nu_x) dx \leq n Vol(T) \left[1 + \frac{c'}{\sqrt{n}} \right].$$

Therefore $V(T, n-1; T + \varepsilon K, 1) \leq Vol(T) \left[1 + \varepsilon \left(1 + \frac{c'}{\sqrt{n}} \right) \right]$ for any $\varepsilon > 0$. By Minkowsi inequality (e.g. [Sch]),

$$Vol(T)^{\frac{n-1}{n}} Vol(T + \varepsilon K)^{\frac{1}{n}} \leq V(T, n-1; T + \varepsilon K, 1)$$

and hence

$$Vol(T + \varepsilon K)^{\frac{1}{n}} \leq Vol(T)^{\frac{1}{n}} \left[1 + \varepsilon \left(1 + \frac{c'}{\sqrt{n}} \right) \right].$$

Denote $t = \frac{1}{\varepsilon}$. Then for any $t > 0$ (see e.g. Lemma 4.16 in [P]),

$$N(K, 2tT) \leq \frac{Vol(K + tT)}{Vol(tT)} \leq \left[1 + \frac{1}{t} \left(1 + \frac{c'}{\sqrt{n}} \right) \right]^n < e^{c_1 \frac{n}{t}}$$

where $c_1 < 1 + \frac{c'}{\sqrt{n}}$ is in fact very close to one. For $t \geq 1$,

$$N(K, 4At\sqrt{n}D) \leq N(K, 2tt_0 D) \leq N(K, 2t[K \cap t_0 D]) \leq e^{c_1 \frac{n}{t}}$$

since $t_0 \geq 2\sqrt{n}A$ and the proposition is proven. $\square$

Remark: As is evident from the proof, Proposition 5.4 also holds for any $A > 0$ that satisfies $Vol(K \cap 2\sqrt{n}A) > e^{-\sqrt{n}}$. This is a much weaker requirement than $L_K < A$.

The next Proposition follows immediately from Proposition 2.2 in [KM] and Theorem 5.2 in [P] (due to Carl [C]).

Proposition 5.5 *Assume that there exists $c > 0$ such that for any dimension n and for any centrally symmetric convex body $K \subset \mathbb{R}^n$ we have $L_K < c$. Then for any centrally symmetric isotropic convex body $K \subset \mathbb{R}^n$ of volume one,*

$$N(\sqrt{n}D, c'tK) < exp\left(c'\frac{n}{t^{\frac{1}{3}}}\right)$$

where $c' = c'(c)$ depends only on c. Furthermore, the exponent "$\frac{1}{3}$" may be replaced by number smaller than $\frac{1}{2}$.

Proposition 5.4 and Proposition 5.5 together imply that if the hyperplane conjecture is correct, then the isotropic position is an M-position of order α for any $\alpha < \frac{1}{2}$. This information adds to the result of K. Ball, which states that the isotropic position is an M-position under the slicing hypothesis.

For $K \subset \mathbb{R}^n$, the volume ratio of K is defined as

$$v.r.(K) = \sup_{\mathcal{E} \subset K} \left(\frac{Vol(K)}{Vol(\mathcal{E})}\right)^{\frac{1}{n}}$$

where the supremum is over all ellipsoids contained in K. We denote

$$L_n = \sup\{L_K \ ; \ K \subset \mathbb{R}^n \ is \ a \ centrally - symmetric \ convex \ body\},$$

$$L_n(a) = \sup\{L_K \ ; \ K \subset \mathbb{R}^n, \ v.r.(K) \leq a\}.$$

In [BKM] it is proven that for any $\delta > 0$,

$$L_n < c(\delta) \ L_n(v(\delta))^{1+\delta} \tag{12}$$

where $c(\delta), v(\delta) \approx e^{\frac{c}{1-\delta}}$. Next, we improve the dependence in (12).

Corollary 5.6 *There exist $c_1, c_2 > 0$, such that for all n,*

$$L_n < c_1 L_n(c_2).$$

Proof: Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body of volume one. Assume that K is in M-position. Then there exists a rotation $U \in O(n)$ such that the body $K + UK$ satisfies $v.r.(K + UK) < c$, for some numerical constant $c > 0$ (see [M2]). Define the following function:

$$f(x) = (1_K * 1_{UK})(x) = \int_{\mathbb{R}^n} 1_K(t)1_{UK}(x-t)dt = Vol(K \cap (x + UK))$$

where $1_K, 1_{UK}$ are the characteristic functions of K and UK. It is straightforward to validate that $\int_{\mathbb{R}^n} f = 1$ and that $supp(f) = K+UK$. For any $\theta \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \langle x, \theta\rangle^2 f(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle t + x - t, \theta\rangle^2 1_K(t) 1_{UK}(x-t)dtdx$$

$$= \int_K \langle x, \theta\rangle^2 dx + \int_{UK} \langle x, \theta\rangle^2 dx$$

and hence $M(f) = M(K) + M(UK)$. In addition, since $det(M(K)) = det(M(U(K))$ and the matrices are positive,

$$det(M(f))^{1/n} \geq det(M(K))^{1/n} + det(M(UK))^{1/n} = 2det(M(K))^{1/n}.$$

Since $f(0) = Vol(K \cap UK) > c^n$ (e.g. [M2]), it follows that $L_K < c' L_f$. The function f is also n-concave, for it is a convolution of characteristic functions of convex bodies (e.g. the appendix of [GrM]). Therefore, the body $T = K_f$ satisfies $d_G(T, K+UK) < c$, and $v.r.(T) < c_2$. Since $L_K < cL_f < c_1 L_T$, the corollary follows. □

Remarks.

1. At present, there is no good proven bound for $M(K)M^*(K)$ in the non-symmetric case, and hence the central symmetry assumption of the body is crucial to the proof of Theorem 1.2. However, some of the statements in this paper may be easily generalized to non-symmetric bodies. In particular, Theorem 1.4, Propositions 5.2–5.5 and Corollary 5.6 also hold in the non-symmetric case.
2. The proof of Corollary 5.6 reduces the problem of bounding the isotropic constant of K, to the problem of bounding the isotropic constant of a body close to $K+UK$, where $U \in O(n)$ and K is in M-position. If K is not centrally-symmetric, yet its barycenter is at the origin, then $Vol(K \cap (-K)) > c^n$ (see [MP2]). Choosing $U = -Id$ we find a centrally-symmetric body T, close to $K - K$, with $L_K < cL_T$. Hence, universal boundness of the isotropic constant of convex, centrally-symmetric bodies would imply the universal boundness of the isotropic constant of non-symmetric convex bodies as well. We also conclude Bourgain's estimate $L_K < cn^{1/4} \log n$ for $K \subset \mathbb{R}^n$ being a non-symmetric convex body. This was previously proved in [Pa].

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