On John type ellipsoids

B. Klartag[*]

Tel Aviv University

Abstract

Given an arbitrary convex symmetric body $K \subset \mathbb{R}^n$, we construct a natural and non-trivial continuous map u_K which associates ellipsoids to ellipsoids, such that the Löwner-John ellipsoid of K is its unique fixed point. A new characterization of the Löwner-John ellipsoid is obtained, and we also gain information regarding the contact points of inscribed ellipsoids with K.

1 Introduction

We work in $\mathbb{R}^n$, yet we choose no canonical scalar product. A centrally-symmetric ellipsoid in $\mathbb{R}^n$ is any set of the form

$$\left\{ \sum_{i=1}^{n} \lambda_i u_i \; ; \; \sum_i \lambda_i^2 \le 1, \; u_1, .., u_n \in \mathbb{R}^n \right\}.$$

If $u_1, .., u_n$ are linearly independent, the ellipsoid is non-degenerate. Whenever we mention an "ellipsoid" we mean a centrally-symmetric non-degenerate one. Given an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, denote by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ the unique scalar product such that $\mathcal{E} = \{ x \in \mathbb{R}^n; \langle x, x \rangle_{\mathcal{E}} \le 1 \}$. There is a group $O(\mathcal{E})$ of linear isometries of $\mathbb{R}^n$ (with respect to the metric induced by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$), and a unique probability measure $\mu_{\mathcal{E}}$ on $\partial \mathcal{E}$ which is invariant under $O(\mathcal{E})$. A body in $\mathbb{R}^n$ is a centrally-symmetric convex set with a non-empty interior. Given a body $K \subset \mathbb{R}^n$, denote by $\| \cdot \|_K$ the unique norm on $\mathbb{R}^n$ such that K is its unit ball:

$$\| x \|_K = \inf \{ \lambda > 0; x \in \lambda K \}.$$

Given a body $K \subset \mathbb{R}^n$ and an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, denote

$$M_{\mathcal{E}}^2(K) = M_{\mathcal{E}}^2(\| \cdot \|_K) = \int_{\partial \mathcal{E}} \| x \|_K^2 \, d\mu_{\mathcal{E}}(x).$$

This quantity is usually referred to as M_2. Let us consider the following parameter:

$$J_K(\mathcal{E}) = \inf_{\mathcal{F} \subset K} M_{\mathcal{E}}(\mathcal{F}) \qquad (1)$$

where the infimum runs over all ellipsoids $\mathcal{F}$ that are contained in K. Since the set of all ellipsoids contained in K (including degenerate ellipsoids) is a compact set with respect to the Hausdorff metric, the infimum is actually attained. In addition, the minimizing ellipsoids must be non-degenerate, since otherwise $J_K(\mathcal{E}) = \infty$ which is impossible for a body K. We are not so much interested in the exact value of $J_K(\mathcal{E})$, as in the ellipsoids where the minimum is obtained.

In Section 2 we prove that there exists a unique ellipsoid for which the minimum in (1) is attained. We shall denote this unique ellipsoid by $u_K(\mathcal{E})$, and we show that the map u_K is continuous. A finite measure ν on $\mathbb{R}^n$ is called $\mathcal{E}$-isotropic if for any $\theta \in \mathbb{R}^n$,

$$\int \langle x, \theta \rangle_{\mathcal{E}}^2 d\nu(x) = L_\nu^2 \langle \theta, \theta \rangle_{\mathcal{E}}$$

where L_ν does not depend on θ. One of the important properties of the map u_K is summarized in the following proposition, to be proved in Section 3.

Proposition 1.1 *Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^n$ be ellipsoids such that $\mathcal{F} \subset K$. Then $\mathcal{F} = u_K(\mathcal{E})$ if and only if there exists an $\mathcal{E}$-isotropic measure ν supported on $\partial\mathcal{F} \cap \partial K$.*

In particular, given any Euclidean structure (i.e. scalar product) in $\mathbb{R}^n$, there is always a unique ellipsoid contained in K with an isotropic measure supported on its contact points with K. This unexpected fact leads to a connection with the Löwner-John ellipsoid of K, which is the (unique) ellipsoid of maximal volume contained in K. By the characterization of the Löwner-John ellipsoid due to John [4] and Ball [1] (see also [3]), $u_K(\mathcal{E}) = \mathcal{E}$ if and only if the ellipsoid $\mathcal{E}$ is the Löwner-John ellipsoid of K. Thus, we obtain the following:

Corollary 1.2 *Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E} \subset K$ be an ellipsoid such that for any ellipsoid $\mathcal{F} \subset K$,*

$$M_{\mathcal{E}}(\mathcal{F}) \geq 1.$$

Then $\mathcal{E}$ is the Löwner-John ellipsoid of K.

As a byproduct of our methods, we also obtain an extremality property of the mean width of the Löwner-John ellipsoid (Corollary 5.2). In Section 4 we show that the body K is determined by the map u_K. Further evidence for the naturalness of this map is demonstrated in Section 5, where we discuss optimization problems similar to the optimization problem in (1), and discover connections with the map u_K.

2 Uniqueness

Let D be a minimizing ellipsoid in (1). We will show that it is the only minimizing ellipsoid. We write $|x| = \sqrt{\langle x, x \rangle_D}$. An equivalent definition of $J_K(\mathcal{E})$ is the following:

$$J_K^2(\mathcal{E}) = \min \left\{ \int_{\partial \mathcal{E}} |T^{-1}(x)|^2 d\mu_{\mathcal{E}}(x) \; ; \; \|T : l_2^n \to X_K\| \leq 1 \right\} \quad (2)$$

where $X_K = (\mathbb{R}^n, \|\cdot\|_K)$ is the normed space whose unit ball is K, and where $l_2^n = (\mathbb{R}^n, |\cdot|)$. The definitions are indeed equivalent; $T(D)$ is the ellipsoid from definition (1), as clearly $\|x\|_{T(D)} = |T^{-1}(x)|$. Since D is a minimizing ellipsoid, Id is a minimizing operator in (2). Note that in (2) it is enough to consider linear transformations which are self adjoint and positive definite with respect to $\langle \cdot, \cdot \rangle_D$. Assume on the contrary that T is another minimizer, where $T \neq Id$ is a self adjoint positive definite operator. Let $\{e_1, .., e_n\}$ be the orthogonal basis of eigenvectors of T, and let $\lambda_1, .., \lambda_n > 0$ be the corresponding eigenvalues. Consider the operator $S = \frac{Id + T}{2}$. Then S satisfies the norm condition in (2), and by the strict convexity of the function $x \mapsto \frac{1}{x^2}$ on $(0, \infty)$,

$$\int_{\partial \mathcal{E}} |S^{-1}(x)|^2 d\mu_{\mathcal{E}}(x) = \int_{\partial \mathcal{E}} \sum_{i=1}^n \left(\frac{1}{\frac{1+\lambda_i}{2}} \right)^2 \langle x, e_i \rangle_D^2 d\mu_{\mathcal{E}}(x)$$

$$< \int_{\partial \mathcal{E}} \sum_{i=1}^n \frac{1 + \left(\frac{1}{\lambda_i} \right)^2}{2} \langle x, e_i \rangle_D^2 d\mu_{\mathcal{E}}(x)$$

$$= \frac{\int_{\partial \mathcal{E}} |x|^2 d\mu_{\mathcal{E}}(x) + \int_{\partial \mathcal{E}} |T^{-1}(x)|^2 d\mu_{\mathcal{E}}(x)}{2} = J_K^2(\mathcal{E})$$

since not all the λ_i's equal one, in contradiction to the minimizing property of Id and T. Thus we may define a map u_K which matches to any ellipsoid $\mathcal{E}$, the unique ellipsoid $u_K(\mathcal{E})$ such that $u_K(\mathcal{E}) \subset K$ and $J_K(\mathcal{E}) = M_{\mathcal{E}}(u_K(\mathcal{E}))$. It is easily verified that for any linear operator T, and $t \neq 0$,

$$u_{TK}(T\mathcal{E}) = Tu_K(\mathcal{E}), \quad (3)$$

$$u_K(t\mathcal{E}) = u_K(\mathcal{E}).$$

The second property means that the map u_K is actually defined over the "projective space" of ellipsoids. Moreover, the image of u_K is naturally a "projective ellipsoid" rather than an ellipsoid: If $\mathcal{E}$ and $t\mathcal{E}$ both belong to the image of u_K, then $t = \pm 1$. Nevertheless, we still formally define u_K as a map that matches an ellipsoid to an ellipsoid, and not as a map defined over the "projective space of ellipsoids".

Let us establish the continuity of the map u_K. One can verify that $M_{\mathcal{E}}(\mathcal{F})$ is a continuous function of $\mathcal{E}$ and $\mathcal{F}$ (using an explicit

formula as in (4), for example). Fix a body $K \subset \mathbb{R}^n$, and denote by X the compact space of all (possibly degenerate) ellipsoids contained in K. Then $M_{\mathcal{E}}(\mathcal{F}) : X \times X \to [0, \infty]$ is continuous. The map u_K is defined only on a subset of X, the set of non-degenerate ellipsoids. The continuity of u_K follows from the following standard lemma.

Lemma 2.1 *Let X be a compact metric space, and $f : X \times X \to [0, \infty]$ a continuous function. Let $Y \subset X$, and assume that for any $y \in Y$ there exists a unique $g(y) \in X$ such that*

$$\min_{x \in X} f(x, y) = f(g(y), y).$$

Then $g : Y \to X$ is continuous.

Proof: Assume that $y_n \to y$ in Y. The function $\min_{x \in X} f(x, y)$ is continuous, and therefore

$$\min_{x \in X} f(x, y_n) = f(g(y_n), y_n) \overset{n \to \infty}{\longrightarrow} f(g(y), y) = \min_{x \in X} f(x, y).$$

Since $X \times X$ is compact, f is uniformly continuous and

$$|f(g(y_n), y) - f(g(y), y)|$$
$$\leq \quad |f(g(y_n), y) - f(g(y_n), y_n)| + |f(g(y_n), y_n) - f(g(y), y)| \overset{n \to \infty}{\longrightarrow} 0.$$

Therefore, for any convergent subsequence $g(y_{n_k}) \to z$, we must have $f(z, y) = f(g(y), y)$ and by uniqueness $z = g(y)$. Since X is compact, necessarily $g(y_n) \to g(y)$, and g is continuous. $\qquad\square$

3   Extremality conditions

There are several ways to prove the existence of the isotropic measure announced in Proposition 1.1. One can adapt the variational arguments from [3], or use the Lagrange multiplier technique due to John [4] (as suggested by O. Guedon). The argument we choose involves duality of linear programming (see e.g. [2]). For completeness, we state and sketch the proof of the relevant theorem ($\langle \cdot, \cdot \rangle$ is an arbitrary scalar product in $\mathbb{R}^m$):

Theorem 3.1 *Let $\{u_\alpha\}_{\alpha \in \Omega} \subset \mathbb{R}^m$, $\{b_\alpha\}_{\alpha \in \Omega} \subset \mathbb{R}$ and $c \in \mathbb{R}^m$. Assume that*

$$\langle x^0, c \rangle = \inf\{\langle x, c \rangle; \forall \alpha \in \Omega, \langle x, u_\alpha \rangle \geq b_\alpha\}$$

and also $\langle x^0, u_\alpha \rangle \geq b_\alpha$ for any $\alpha \in \Omega$. Then there exist $\lambda_1, .., \lambda_s > 0$ and $u_1, .., u_s \in \Omega' = \{\alpha \in \Omega; \langle x^0, u_\alpha \rangle = b_\alpha\}$ such that

$$c = \sum_{i=1}^{s} \lambda_i u_i.$$

Proof: $K = \{x \in \mathbb{R}^m; \forall \alpha \in \Omega, \langle x, u_\alpha \rangle \geq 1\}$ is a convex body. x^0 lies on its boundary, and $\{x \in \mathbb{R}^m; \langle x, c \rangle = \langle x^0, c \rangle\}$ is a supporting hyperplane to K at x^0. The vector c is an inner normal vector to K at x^0, hence $-c$ belongs to the cone of outer normal vectors to K at x^0. The crucial observation is that this cone is generated by $-\Omega'$ (e.g. Corollary 8.5 in chapter II of [2]), hence

$$c \in \left\{ \sum_{i=1}^{s} \lambda_i u_i \; ; \; \forall i \; u_i \in \Omega', \lambda_i \geq 0 \right\}.$$

$\square$

Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid. This ellipsoid induces a scalar product in the space of operators: if $T, S : \mathbb{R}^n \to \mathbb{R}^n$ are linear operators, and $\{e_1, .., e_n\} \subset \mathbb{R}^n$ is any orthogonal basis (with respect to $\langle \cdot, \cdot \rangle_\mathcal{E}$), then

$$\langle T, S \rangle_\mathcal{E} = \sum_{i,j} T_{i,j} S_{i,j}$$

where $T_{i,j} = \langle Te_i, e_j \rangle_\mathcal{E}$ and $S_{i,j} = \langle Se_i, e_j \rangle_\mathcal{E}$ are the entries of the corresponding matrix representations of T and S. This scalar product does not depend on the choice of the orthogonal basis. If $\mathcal{F} = \{x \in \mathbb{R}^n; \langle x, Tx \rangle_\mathcal{E} \leq 1\}$ is another ellipsoid, then

$$M_\mathcal{E}^2(\mathcal{F}) = \int_{\partial \mathcal{E}} \langle x, Tx \rangle_\mathcal{E} d\mu_\mathcal{E}(x) \tag{4}$$

$$= \sum_{i,j=1}^{n} T_{i,j} \int_{\partial \mathcal{E}} \langle x, e_i \rangle_\mathcal{E} \langle x, e_j \rangle_\mathcal{E} d\mu_\mathcal{E}(x) = \sum_{i,j=1}^{n} T_{i,j} \frac{\delta_{i,j}}{n} = \frac{1}{n} \langle T, Id \rangle_\mathcal{E}.$$

The ellipsoid $\mathcal{F} = \{x \in \mathbb{R}^n; \langle x, Tx \rangle_\mathcal{E} \leq 1\}$ is contained in K if and only if for any $x \in \partial K$,

$$\langle x, Tx \rangle_\mathcal{E} = \langle x \otimes x, T \rangle_\mathcal{E} \geq 1$$

where $(x \otimes x)(y) = \langle x, y \rangle_\mathcal{E} x$ is a linear operator. Therefore, the optimization problem (1) is equivalent to the following problem:

$$nJ_K^2(\mathcal{E}) = \min\{\langle T, Id \rangle_\mathcal{E} \; ; \; T \text{ is } \mathcal{E}\text{–positive, } \forall x \in \partial K \; \langle x \otimes x, T \rangle_\mathcal{E} \geq 1\}$$

where we say that T is $\mathcal{E}$-positive if it is self adjoint and positive definite with respect to $\langle \cdot, \cdot \rangle_\mathcal{E}$. Actually, the explicit positivity requirement is unnecessary. If K is non-degenerate and $\forall x \in \partial K \; \langle T, x \otimes x \rangle_\mathcal{E} \geq 1$ then T is necessarily positive definite with respect to $\mathcal{E}$. This is a linear optimization problem, in the space $\mathbb{R}^m = \mathbb{R}^{n^2}$. Let T be the unique self adjoint minimizer, and let $\mathcal{F} = \{x \in \mathbb{R}^n; \langle x, Tx \rangle_\mathcal{E} \leq 1\}$ be the corresponding ellipsoid. By Theorem 3.1, there exist $\lambda_1, .., \lambda_s > 0$ and vectors $u_1, .., u_s \in \partial K$ such that

1. For any $1 \leq i \leq s$ we have $\langle u_i \otimes u_i, T \rangle_{\mathcal{E}} = 1$, i.e. $u_i \in \partial K \cap \partial \mathcal{F}$.

2. $Id = \sum_{i=1}^{s} \lambda_i u_i \otimes u_i$. Equivalently, for any $\theta \in \mathbb{R}^n$,

$$\sum_{i=1}^{s} \lambda_i \langle u_i, \theta \rangle_{\mathcal{E}}^2 = \langle \theta, \theta \rangle_{\mathcal{E}}^2.$$

Hence we proved the following:

Lemma 3.2 *Let $K \subset \mathbb{R}^n$ be a body and let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid. If $u_K(\mathcal{E}) = \mathcal{F}$, then there exist contact points $u_1, .., u_s \in \partial K \cap \partial \mathcal{F}$ and positive numbers $\lambda_1, .., \lambda_s$ such that for any $\theta \in \mathbb{R}^n$,*

$$\sum_{i=1}^{s} \lambda_i \langle u_i, \theta \rangle_{\mathcal{E}}^2 = \langle \theta, \theta \rangle_{\mathcal{E}}.$$

The following lemma completes the proof of Proposition 1.1.

Lemma 3.3 *Let $K \subset \mathbb{R}^n$ be a body and let $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^n$ be ellipsoids. Assume that $\mathcal{F} \subset K$ and that there exists a measure ν supported on $\partial K \cap \partial \mathcal{F}$ such that for any $\theta \in \mathbb{R}^n$,*

$$\int \langle x, \theta \rangle_{\mathcal{E}}^2 d\nu(x) = \langle \theta, \theta \rangle_{\mathcal{E}}.$$

Then $u_K(\mathcal{E}) = \mathcal{F}$.

Proof: Since $\int x \otimes x d\nu(x) = Id$, for any operator T,

$$\int \langle Tx, x \rangle_{\mathcal{E}} d\nu(x) = \langle T, Id \rangle_{\mathcal{E}} = n \int_{\partial \mathcal{E}} \langle Tx, x \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x). \qquad (5)$$

where the last equality follows by (4). Let T be such that $\mathcal{F} = \{x \in \mathbb{R}^n; \langle Tx, x \rangle_{\mathcal{E}} \leq 1\}$. By (5),

$$\nu(\partial \mathcal{F}) = \int_{\partial \mathcal{F}} \langle Tx, x \rangle_{\mathcal{E}} d\nu(x) = n \int_{\partial \mathcal{E}} \langle Tx, x \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x) = n M_{\mathcal{E}}^2(\mathcal{F}).$$

Suppose that $u_K(\mathcal{E}) \neq \mathcal{F}$. Then there exists a linear map $S \neq T$ such that $\langle Sx, x \rangle_{\mathcal{E}} \geq 1$ for all $x \in \partial K$ and such that

$$\int_{\partial \mathcal{E}} \langle Sx, x \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x) < M_{\mathcal{E}}^2(\mathcal{F}).$$

Therefore,

$$\nu(\partial K) \leq \int_{\partial K} \langle Sx, x \rangle_{\mathcal{E}} d\nu(x) = n \int_{\partial \mathcal{E}} \langle Sx, x \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x) < n M_{\mathcal{E}}^2(\mathcal{F})$$

6

which is a contradiction, since $\nu(\partial K) = \nu(\partial \mathcal{F}) = nM_{\mathcal{E}}^2(\mathcal{F})$. □

Remark: This proof may be modified to provide an alternative proof of John's theorem. Indeed, instead of minimizing the linear functional $\langle T, Id \rangle$ we need to minimize the concave functional $det^{1/n}(T)$. The minimizer still belongs to the boundary, and minus of the gradient at this point belongs to the normal cone.

4   Different bodies have different maps

Lemma 4.1 *Let $K, T \subset \mathbb{R}^n$ be two closed bodies such that $T \not\subset K$. Then there exists an ellipsoid $\mathcal{F} \subset T$ such that $\mathcal{F} \not\subset K$ and n linearly independent vectors $v_1, .., v_n$ such that for any $1 \leq i \leq n$,*

$$v_i \in \partial \mathcal{F} \cap \partial C$$

where $C = conv(K, \mathcal{F})$ and conv denotes convex hull.

Proof: Let $U \subset T \setminus K$ be an open set whose closure does not intersect K, and let $v_1^*, .., v_n^*$ be linearly independent functionals on $\mathbb{R}^n$ such that for any $y \in K, \ z \in U$,

$$v_i^*(y) < v_i^*(z)$$

for all $1 \leq i \leq n$. Let $\mathcal{F} \subset conv(U, -U)$ be an ellipsoid that intersects U, and let $v_1, .., v_n \in \partial \mathcal{F}$ be the unique vectors such that for $1 \leq i \leq n$,

$$v_i^*(v_i) = \sup_{v \in \mathcal{F}} v_i^*(v).$$

Then $v_1, .., v_n$ are linearly independent. Also, $v_i^*(v_i) = \sup_{v \in C} v_i^*(v)$ and hence $v_1, .., v_n$ belong to the boundary of $C = conv(K, \mathcal{F})$. □

Theorem 4.2 *Let $K, T \subset \mathbb{R}^n$ be two closed bodies, such that for any ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ we have $J_K(\mathcal{E}) = J_T(\mathcal{E})$. Then $K = T$.*

Proof: Assume $K \neq T$. Without loss of generality, $T \not\subset K$. Let $\mathcal{F}$ and $v_1, .., v_n$ be the ellipsoid and vectors from Lemma 4.1. Consider the following bodies:

$$L = conv\{\mathcal{F}, K \cap T\}, \quad C = conv\{\mathcal{F}, K\}.$$

Then $v_1, .., v_n \in \partial C$ and also $v_1, .., v_n \in \partial L$. Let $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ be the scalar product with respect to which these vectors constitute an orthonormal basis. Then the uniform measure on $\{v_1, .., v_n\}$ is $\mathcal{E}$-isotropic. By Proposition 1.1, $u_L(\mathcal{E}) = u_C(\mathcal{E}) = \mathcal{F}$, and

$$J_L(\mathcal{E}) = M_{\mathcal{E}}(u_L(\mathcal{E})) = M_{\mathcal{E}}(u_C(\mathcal{E})) = J_C(\mathcal{E}). \tag{6}$$

Since $K \subset C$, also $u_K(\mathcal{E}) \subset C$. Since $\mathcal{F} = u_C(\mathcal{E}) \not\subset K$, we have $\mathcal{F} \neq u_K(\mathcal{E})$. By the uniqueness of the minimizing ellipsoid for $J_C(\mathcal{E})$,

$$J_C(\mathcal{E}) = M_{\mathcal{E}}(\mathcal{F}) < M_{\mathcal{E}}(u_K(\mathcal{E})) = J_K(\mathcal{E}). \tag{7}$$

Since $L \subset T$ we have $J_T(\mathcal{E}) \leq J_L(\mathcal{E})$. Combining this with (6) and (7), we get

$$J_T(\mathcal{E}) \leq J_L(\mathcal{E}) = J_C(\mathcal{E}) < J_K(\mathcal{E})$$

and therefore $J_K(\mathcal{E}) \neq J_T(\mathcal{E})$. $\qquad\qquad\square$

Corollary 4.3 *Let $K, T \subset \mathbb{R}^n$ be two closed bodies such that $u_K = u_T$. Then $K = T$.*

Proof: For any ellipsoid $\mathcal{E} \subset \mathbb{R}^n$,

$$J_K(\mathcal{E}) = M_{\mathcal{E}}(u_K(\mathcal{E})) = M_{\mathcal{E}}(u_T(\mathcal{E})) = J_T(\mathcal{E})$$

and the corollary follows from Theorem 4.2.

5 Various optimization problems

Given an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ and a body $K \subset \mathbb{R}^n$, define

$$K_{\mathcal{E}}^{\circ} = \{x \in \mathbb{R}^n; \forall x \in K, \ \langle x, y \rangle_{\mathcal{E}} \leq 1\}$$

and also $M_{\mathcal{E}}^*(K) = M_{\mathcal{E}}(K_{\mathcal{E}}^{\circ})$. Consider the following optimization problem:

$$\inf_{K \subset \mathcal{F}} M_{\mathcal{E}}^*(\mathcal{F}) \tag{8}$$

where the infimum runs over all ellipsoids that contain K. Then (8) is simply the dual, equivalent formulation of problem (1) that was discussed above. Indeed, $\mathcal{F}$ is a minimizer in (8) if and only if $u_{K_{\mathcal{E}}^{\circ}}(\mathcal{E}) = \mathcal{F}_{\mathcal{E}}^{\circ}$. An apriori different optimization problem is the following:

$$I_K(\mathcal{E}) = \sup_{K \subset \mathcal{F}} M_{\mathcal{E}}(\mathcal{F}) \tag{9}$$

where the supremum runs over all ellipsoids that contain K. The characteristics of this problem are indeed different. For instance, the supremum need not be attained, as shown by the example of a narrow cylinder (in which there is a maximizing sequence of ellipsoids that tends to an infinite cylinder) and need not be unique, as shown by the example of a cube (any ellipsoid whose axes are parallel to the edges of the cube, and that touches the cube - is a maximizer. See also the proof of Corollary 5.2). Nevertheless, we define

$$\bar{u}_K(\mathcal{E}) = \{\mathcal{F} \subset \mathbb{R}^n \ ; \ \mathcal{F} \text{ is an ellipsoid}, \ K \subset \mathcal{F}, \ M_{\mathcal{E}}(\mathcal{F}) = I_K(\mathcal{E})\}.$$

The dual, equivalent formulation of (9) means to maximize M^* among ellipsoids that are contained in K. Apriori, $u_K(\mathcal{E})$ and $\bar{u}_K(\mathcal{E})$ do not seem to be related. It is not clear why there should be a connection between minimizing M and maximizing M^* among inscribed ellipsoids. The following proposition reveals a close relation between the two problems.

Proposition 5.1 *Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^n$ be ellipsoids. Then*
$$\mathcal{F} \in \bar{u}_K(\mathcal{E}) \iff u_{K_{\mathcal{F}}^\circ}(\mathcal{E}) = \mathcal{F}.$$

Proof:

$\Longrightarrow$: We write $\mathcal{F} = \{x \in \mathbb{R}^n ; \langle x, Tx \rangle_{\mathcal{E}} \leq 1\}$ for an $\mathcal{E}$-positive operator T. Since $\mathcal{F} \in \bar{u}_K(\mathcal{E})$, the operator T is a maximizer of

$$nI_K^2(\mathcal{E}) = \max\{\langle S, Id \rangle_{\mathcal{E}} \; ; \; S \in L(n), \ \forall x \in \partial K \ 0 \leq \langle S, x \otimes x \rangle_{\mathcal{E}} \leq 1\}.$$

where $L(n)$ is the space of linear operators acting on $\mathbb{R}^n$. Note that the requirement $\langle S, x \otimes x \rangle_{\mathcal{E}} \geq 0$ for any $x \in \partial K$ ensures that S is $\mathcal{E}$-non-negative definite. This is a linear optimization problem. Following the notation of Theorem 3.1, we rephrase our problem as follows:

$$-nI_K^2(\mathcal{E}) = \inf\{\langle S, -Id \rangle_{\mathcal{E}}; \forall x \in \partial K, \ \langle S, x \otimes x \rangle_{\mathcal{E}} \geq 0, \langle S, -x \otimes x \rangle_{\mathcal{E}} \geq -1\}.$$

According to Theorem 3.1, since T is a maximizer, there exist $\lambda_1, .., \lambda_s > 0$ and vectors $u_1, .., u_t, u_{t+1}, .., u_s \in \partial K$ such that

1. For any $1 \leq i \leq t$ we have $\langle T, -u_i \otimes u_i, \rangle_{\mathcal{E}} = -1$, i.e. $u_i \in \partial K \cap \partial \mathcal{F}$.
   For any $t + 1 \leq i \leq s$ we have $\langle T, u_i \otimes u_i \rangle_{\mathcal{E}} = 0$.
2. $Id = \sum_{i=1}^{t} \lambda_i u_i \otimes u_i - \sum_{i=t+1}^{s} \lambda_i u_i \otimes u_i$.

Since we assumed that $\mathcal{F}$ is an ellipsoid, T is $\mathcal{E}$-positive, and it is impossible that $\langle Tu_i, u_i \rangle_{\mathcal{E}} = 0$. Hence, $t = s$ and there exists an $\mathcal{E}$-isotropic measure supported on $\partial K \cap \partial \mathcal{F}$. Since $K \subset \mathcal{F}$, then $\mathcal{F} \subset K_{\mathcal{F}}^\circ$ and $\partial K_{\mathcal{F}}^\circ \cap \partial \mathcal{F} = \partial K \cap \partial \mathcal{F}$. Therefore, there exists an $\mathcal{E}$-isotropic measure supported on $\partial K_{\mathcal{F}}^\circ \cap \partial \mathcal{F}$. Since $\mathcal{F} \subset K_{\mathcal{F}}^\circ$ we must have $u_{K_{\mathcal{F}}^\circ}(\mathcal{E}) = \mathcal{F}$, according to Proposition 1.1.

$\Longleftarrow$: Since $u_{K_{\mathcal{F}}^\circ}(\mathcal{E}) = \mathcal{F}$, then $K \subset \mathcal{F}$ and we can write $Id = \int x \otimes x d\nu(x)$ where $supp(\nu) \subset \partial K_{\mathcal{F}}^\circ \cap \partial \mathcal{F} = \partial K \cap \partial \mathcal{F}$. Reasoning as in Lemma 3.3, $\nu(\partial K) = nM_{\mathcal{E}}^2(\mathcal{F})$ and for any admissible operator S,

$$\langle S, Id \rangle_{\mathcal{E}} = \int \langle x, Sx \rangle_{\mathcal{E}} d\nu(x) \leq \nu(\partial K) = nM_{\mathcal{E}}^2(\mathcal{F})$$

since $\langle x, Sx \rangle_{\mathcal{E}} \leq 1$ for any $x \in \partial K$. Hence T is a maximizer, and $\mathcal{F} \in \bar{u}_K(\mathcal{E})$. $\qquad \square$

If $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^n$ are ellipsoids, then $K_{\mathcal{E}}^{\circ}$ is a linear image of $K_{\mathcal{F}}^{\circ}$. By (3), the map $u_{K_{\mathcal{E}}^{\circ}}$ is completely determined by $u_{K_{\mathcal{F}}^{\circ}}$. Therefore, the family of maps $\{u_{K_{\mathcal{F}}^{\circ}}; \mathcal{F} \text{ is an ellipsoid}\}$ is determined by a single map $u_{K_{\mathcal{E}}^{\circ}}$, for any ellipsoid $\mathcal{E}$. By Proposition 5.1, this family of maps determines $\bar{u}_K$. Therefore, for any ellipsoid $\mathcal{E}$, the map $u_{K_{\mathcal{E}}^{\circ}}$ completely determines $\bar{u}_K$. Additional consequence of Proposition 5.1 is the following:

Corollary 5.2 *Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E}$ be its Löwner-John ellipsoid. Then for any ellipsoid $\mathcal{F} \subset K$,*

$$M_{\mathcal{E}}^*(\mathcal{F}) \le 1.$$

Equality occurs for $\mathcal{F} = \mathcal{E}$, yet there may be additional cases of equality.

Proof: If $\mathcal{E}$ is the Löwner-John ellipsoid, then $u_K(\mathcal{E}) = \mathcal{E}$. By Proposition 5.1, $\mathcal{E} \in \bar{u}_{K_{\mathcal{E}}^{\circ}}(\mathcal{E})$. Dualizing, we get that $\mathcal{E} \subset K$, and

$$1 = M_{\mathcal{E}}^*(\mathcal{E}) = \sup_{\mathcal{F} \subset K} M_{\mathcal{E}}^*(\mathcal{F})$$

where the supremum is over all ellipsoids contained in K. This proves the inequality. To obtain the remark about the equality cases, consider the cross-polytope $K = \{x \in \mathbb{R}^n; \sum_{i=1}^n |x_i| \le 1\}$, where $(x_1, .., x_n)$ are the coordinates of x. By symmetry arguments, its Löwner-John ellipsoid is $D = \{x \in \mathbb{R}^n; \sum_i x_i^2 \le \frac{1}{n}\}$. However, for any ellipsoid of the form

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n; \sum_i \frac{x_i^2}{\lambda_i} \le 1 \right\}$$

where the λ_i are positive and $\sum_i \lambda_i = 1$, we get that $\mathcal{E} \subset K$, yet $M_D^*(\mathcal{E}) = M_D^*(D) = 1$. $\qquad\square$

Remarks:

1. If $K \subset \mathbb{R}^n$ is smooth and strictly convex, then $\bar{u}_K$ is always a singleton. Indeed, if K is strictly convex and is contained in an infinite cylinder, it is also contained in a subset of that cylinder which is an ellipsoid, hence the supremum is attained. From the proof of Proposition 5.1 it follows that if $\mathcal{F}_1, \mathcal{F}_2$ are maximizers, then there exists an isotropic measure supported on their common contact points with K. Since K is smooth, it has a unique supporting hyperplane at any of these contact points, which is also a common supporting hyperplane of $\mathcal{F}_1$ and $\mathcal{F}_2$. Since these common contact points span $\mathbb{R}^n$, necessarily $\mathcal{F}_1 = \mathcal{F}_2$. Hence, if K is smooth and strictly convex, only John ellipsoid may cause an equality in Corollary 5.2.

2. If $\mathcal{E}$ is the Löwner-John ellipsoid of K, then for any other ellipsoid $\mathcal{F} \subset K$ we have $M_{\mathcal{E}}(\mathcal{F}) > M_{\mathcal{E}}(\mathcal{E}) = 1$. This follows from our methods, yet it also follows immediately from the fact that $\frac{1}{M_{\mathcal{E}}(\mathcal{F})} \leq \left(\frac{Vol(\mathcal{F})}{Vol(\mathcal{E})}\right)^{1/n}$, and from the uniqueness of the Löwner-John ellipsoid.

References

[1] K. M. Ball, Ellipsoids of maximal volume in convex bodies, Geometriae Dedicata **41** (1992), 241–250

[2] A. Barvinok, A Course in Convexity, Amer. Math. Soc., Graduate Studies in Math., Vol. **54**, Providence, RI (2002).

[3] A. Giannopoulos, V. D. Milman, Extremal problems and isotropic positions of convex bodies. Israel J. Math. **117** (2000) 29–60.

[4] F. John, Extremum problems with inequalities as subsidiary conditions, in Courant Anniversary Volume, Interscience, New York (1948) 187–204.