Poisson processes and a log-concave Bernstein theorem

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Abstract

We discuss interplays between log-concave functions and log-concave sequences. We prove a Bernstein-type theorem, which characterizes the Laplace transform of log-concave measures on the half-line in terms of log-concavity of the alternating Taylor coefficients. We establish concavity inequalities for sequences inspired by the Prékopa-Leindler and the Walkup theorems. One of our main tools is a stochastic variational formula for the Poisson average.

1 Introduction

Let $\varphi \colon [0, \infty) \to \mathbb{R}$ be a continuous function that is C^∞-smooth on $(0, \infty)$. Its alternating Taylor coefficients are

$$a_t(n) = (-1)^n \frac{\varphi^{(n)}(t)}{n!} \qquad\qquad (n \geq 0, t > 0). \qquad (1)$$

A function whose alternating Taylor coefficients are non-negative is called an absolutely monotone function. Bernstein's theorem asserts that the alternating Taylor coefficients are non-negative if and only if there exists a finite, non-negative Borel measure μ on $[0, \infty)$ with

$$\varphi(t) = \int_0^\infty e^{-tx} d\mu(x) \qquad\qquad (t \geq 0). \qquad (2)$$

In other words, φ is the Laplace transform of the measure μ. See Widder [12] for proofs of Bernstein's theorem. We say that the alternating Taylor coefficients are log-concave if the sequence $(a_t(n))_{n \geq 0}$ is a *log-concave sequence* for any $t > 0$. This means that this sequence consists of non-negative numbers and for any $m, n \geq 0$ and $\lambda \in (0, 1)$ such that $\lambda n + (1-\lambda)m$ is an integer,

$$a_t(\lambda n + (1 - \lambda)m) \geq a_t(n)^\lambda a_t(m)^{1-\lambda}.$$

Equivalently, $a_t(n)^2 \geq a_t(n-1)a_t(n+1)$ for every $n \geq 1$ and the set of integers n for which $a_t(n) > 0$ is an interval of integers.

A measure μ on $[0, \infty)$ is log-concave if it is either a delta measure at a certain point, or else an absolutely-continuous measure whose density $f : [0, \infty) \to \mathbb{R}$ is a log-concave

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function. Recall that a function $f : K \to \mathbb{R}$ for some convex set K is log-concave if f is non-negative and

$$f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda} \qquad \text{for all } x, y \in K, 0 < \lambda < 1.$$

Theorem 1.1 ("Log-concave Bernstein theorem"). *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a continuous function that is C^∞-smooth on $(0, \infty)$. Then the alternating Taylor coefficients of φ are log-concave if and only if φ takes the form (2) for a certain finite, log-concave measure μ.*

In Section 2 we prove Theorem 1.1 by using an inversion formula for the Laplace transform as well as the Berwald-Borell inequality [1, 2]. The latter inequality states that if one divides the Mellin transform of a log-concave measure on $[0, \infty)$ by the Gamma function, then a log-concave function is obtained. It directly implies the "if" part of Theorem 1.1. Our theorem admits the following corollary:

Corollary 1.2. *Let μ be a finite, non-negative Borel measure on $[0, \infty)$ and let φ be given by (2). Then μ is log-concave if and only if the function $\left|\varphi^{(n-1)}(t)\right|^{-1/n}$ is convex in $t \in (0, \infty)$ for every $n \geq 1$.*

In fact, in Theorem 1.1 it suffices to verify that the sequence $(a_t(n))_{n \geq 0}$ is log-concave for a sufficiently large t, as follows from the following:

Proposition 1.3. *Let $\varphi : (0, \infty) \to \mathbb{R}$ be real-analytic, and define $a_t(n)$ via (1). Assume that $0 < r < s$ and that the sequence $(a_s(n))_{n \geq 0}$ is log-concave. Then also the sequence $(a_r(n))_{n \geq 0}$ is also log-concave.*

Proposition 1.3 is proven in Section 3, alongside concavity inequalities related to log-concave sequences in spirit of the Walkup theorem [11]. While searching for a Prékopa-Leindler type inequality for sequences, we have found the following:

Theorem 1.4. *Let $f, g, h, k : \mathbb{Z} \to \mathbb{R}$ satisfy*

$$f(x) + g(y) \leq h\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) + k\left(\left\lceil \frac{x+y}{2} \right\rceil\right), \quad \forall x, y \in \mathbb{Z},$$

where $\lfloor x \rfloor$ is the lower integer part of $x \in \mathbb{R}$ and $\lceil x \rceil$ is the upper integer part. Then

$$\left(\sum_{x \in \mathbb{Z}} e^{f(x)}\right) \left(\sum_{x \in \mathbb{Z}} e^{g(x)}\right) \leq \left(\sum_{x \in \mathbb{Z}} e^{h(x)}\right) \left(\sum_{x \in \mathbb{Z}} e^{k(x)}\right).$$

Our proof of Theorem 1.4 which is presented in Section 5 involves probabilistic techniques, and it would be interesting to find a direct proof. However, we believe that the probabilistic method is not without importance in itself, and perhaps it is deeper than other components of this paper. The argument is based on a stochastic variational formula for the expectation of a given function with respect to the Poisson distribution. It is analogous to

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Borell's formula from [4] which is concerned with the Gaussian distribution. The stochastic variational formula is discussed in Section 4.

The Berwald-Borell inequality (or Theorem 1.1) imply that when μ is a finite, log-concave measure on $[0, \infty)$ and $k \leq \ell \leq m \leq n$ are non-negative integers with $k + n = \ell + m$,

$$a_t(\ell)a_t(m) - a_t(k)a_t(n) \geq 0, \tag{3}$$

where $a_t(n) = \int_0^\infty (x^n/n!)e^{-tx}d\mu(x)$ is defined via (1) and (2). The following theorem shows that the left-hand side of (3) is not only non-negative, but it is in fact an absolutely-monotone function of t:

Theorem 1.5. *Let μ be a finite, log-concave measure on $[0, \infty)$. Then for any non-negative integers $k \leq \ell \leq m \leq n$ with $k + n = \ell + m$ there exists a finite, non-negative measure $\nu = \nu_{k,\ell,m,n}$ on $[0, \infty)$, such that for any $t > 0$,*

$$\int_0^\infty e^{-tx}d\nu(x) = a_t(\ell)a_t(m) - a_t(k)a_t(n), \tag{4}$$

where as usual $a_t(n) = \int_0^\infty (x^n/n!)e^{-tx}d\mu(x)$ is defined via (1) and (2).

Theorem 1.5 is proven in Section 3. Let us apply this theorem in a few examples. In the case where μ is an exponential measure, whose density is $e^{-\alpha t}$ on $[0, \infty)$, the measures ν from Theorem 1.5 vanish completely. In the case where μ is proportional to a Gamma distribution, also the measures ν are proportional to Gamma distributions. When μ is the uniform measure on the interval $[1, 2]$, the density of the measure $\nu = \nu_{0,1,1,2}$ from Theorem 1.5 is depicted in Figure 1. This log-concave density equals the convex function $(t-1)(t-2)/2$ in the interval $[2, 3]$, and it equals $(t-2)(4-t)$ in $[3, 4]$.
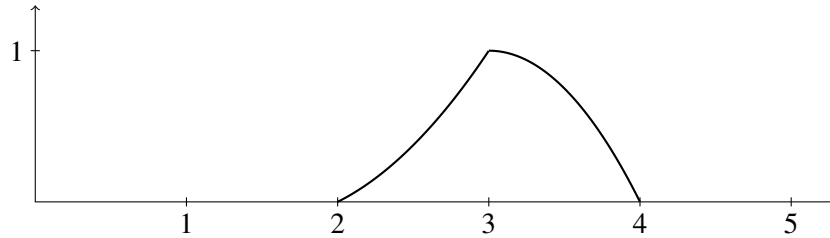

Figure 1: The density of $\nu_{0,1,1,2}$ where μ is uniform on the interval $[1, 2]$.

We suggest to refer to the measure ν from Corollary 1.5 as the Berwald-Borell transform of μ with parameters (k, ℓ, m, n). All Berwald-Borell transforms of log-concave measures that we have encountered so far were log-concave by themselves. It is a curious problem to characterize the family of measures ν which could arise as the Berwald-Borell transform of a log-concave measure on $[0, \infty)$. Such a characterization could lead to new constraints on the moments of log-concave measures on $[0, \infty)$ beyond the constraints given by the Berwald-Borell inequality.

2 Proof of the log-concave Bernstein theorem

The proof of Theorem 1.1 combines ideas of Berwald from the 1940s with the earlier Post inversion formula for the Laplace transform. The "if" direction of Theorem 1.1 follows from:

Lemma 2.1. *Let μ be a finite, log-concave measure on $[0, \infty)$. Assume that φ is given by (2). Then the alternating Taylor coefficients of φ are log-concave.*

Proof. In the case where $\mu = c\,\delta_{x_0}$ we have that $\varphi(t) = c\,\mathrm{e}^{-tx_0}$ and hence

$$a_t(n) = \frac{c\,\mathrm{e}^{-tx_0} x_0^n}{n!}.$$

Since $a_t(n) > 0$ for every n and $a_t(n+1)/a_t(n) = x_0/(n+1)$ is non-increasing, this is indeed a log-concave sequence. In the case where μ has a log-concave density f, we denote $f_t(x) = e^{-tx} f(x)$ and observe that

$$\varphi^{(k)}(t) = \int_0^\infty (-x)^k e^{-tx} f(x) dx = \int_0^\infty (-x)^k f_t(x) dx \qquad (k \geq 0, t > 0). \quad (5)$$

The function f_t is log-concave, and hence we may apply the Berwald-Borell inequality [1, 2], see also Theorem 2.2.5 in [6] or Theorem 5 in [9] for different proofs. This inequality states that the sequence

$$k \to \frac{\int_0^\infty x^k f_t(x) dx}{k!} \qquad (k \geq 0) \quad (6)$$

is log-concave, completing the proof. $\qquad \square$

We now turn to the proof of the "only if" direction of Theorem 1.1, which relies on the Post inversion formula for the Laplace transform, see Feller [7, Section VII.6] or Widder [12, Section VII.1]. Suppose that φ is continuous on $[0, \infty)$ and C^∞-smooth on $(0, \infty)$, and that the alternating Taylor coefficients $a_t(n)$ are log-concave. In particular, the alternating Taylor coefficients are non-negative. We use Bernstein's theorem to conclude that there exists a finite, non-negative Borel measure μ on $[0, \infty)$ such that (2) holds true. All that remains is to prove the following:

Proposition 2.2. *The measure μ is log-concave.*

The proof of Proposition 2.2 requires some preparation. First, it follows from (1) and (2) that for any $R, t > 0$,

$$\sum_{n=0}^{\lfloor Rt \rfloor} t^n a_t(n) = \sum_{n=0}^{\lfloor Rt \rfloor} \frac{t^n}{n!} \int_0^\infty x^n e^{-tx} d\mu(x) = \int_0^\infty \mathbb{P}\left(N_{tx} \leq Rt\right) d\mu(x), \quad (7)$$

where N_s is a Poisson random variable with parameter s, i.e.,

$$\mathbb{P}(N_s = n) = e^{-s} \frac{s^n}{n!}, \qquad \text{for } n = 0, 1, 2, \ldots$$

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The random variable N_s has expectation s and standard-deviation $\sqrt{s}$. From the Markov-Chebyshev inequality, for any $\alpha > 0$,

$$\mathbb{P}(N_s \leq \alpha s) \overset{s \to \infty}{\longrightarrow} \begin{cases} 1 & \alpha > 1 \\ 1/2 & \alpha = 1 \\ 0 & \alpha < 1 \end{cases} \tag{8}$$

where the case $\alpha = 1$ is slightly more difficult, it is left as an exercise in Feller [7, Chapter VII], that may be solved via the central limit theorem for the Poisson distribution. The left-hand side of (8) is always between zero and one. Therefore we may use the bounded convergence theorem, and conclude from (7) that for any $R > 0$,

$$\lim_{t \to \infty} \sum_{n=0}^{\lfloor Rt \rfloor} t^n a_t(n) = \mu([0, R)) + \frac{1}{2} \cdot \mu(\{R\}). \tag{9}$$

For $t > 0$ define $g_t : [0, \infty) \to \mathbb{R}$ via

$$g_t(x) = \begin{cases} t^{n+1} \cdot a_t(n) & x = n/t \text{ for some integer } n \geq 0 \\ t^{x+1} \cdot a_t(n)^{1-\lambda} \cdot a_t(n+1)^\lambda & x = (n+\lambda)/t \text{ for } \lambda \in (0,1), n \geq 0. \end{cases}$$

Write μ_t for the measure on $[0, \infty)$ whose density is g_t. We think about μ_t as an approximation for the discrete measure on $[0, \infty)$ that has an atom at n/t of weight $t^n a_t(n)$ for any $n \geq 0$.

Lemma 2.3. *For any $t > 0$ the measure μ_t is log-concave on $[0, \infty)$. Moreover, if $\mu(\{0\}) = 0$ then for any $R > 0$,*

$$\mu_t([0, R)) \overset{t \to \infty}{\longrightarrow} \mu([0, R)) + \frac{1}{2} \cdot \mu(\{R\}).$$

Proof. We may assume that $\mu \not\equiv 0$, as otherwise the lemma is trivial. Since $\mu(\{0\}) = 0$, for any $t > 0$ and $n \geq 0$,

$$a_t(n) = \int_0^\infty \frac{x^n}{n!} e^{-tx} d\mu(x) > 0.$$

The density g_t is locally-Lipschitz, and for any integer $n \geq 0$ and $x \in (n/t, (n+1)/t)$,

$$(\log g_t)'(x) = \log t + t \log \frac{a_t(n+1)}{a_t(n)}.$$

The sequence $(a_t(n))_{n \geq 0}$ is log-concave, hence $a_t(n+1)/a_t(n)$ is non-increasing in n. We conclude that $(\log g_t)'(x)$, which exists for almost any $x > 0$, is a non-increasing function of $x \in [0, \infty)$. This shows that μ_t is a log-concave measure. In particular, the density g_t is unimodular, meaning that for some $x_0 \geq 0$, the function g_t is non-decreasing in $(0, x_0)$ and non-increasing in (x_0, ∞). We claim that for any $R, t > 0$ we have the Euler-Maclaurin type bound:

$$\left| \int_0^R g_t(x) dx - \sum_{n=0}^{\lfloor Rt \rfloor} \frac{1}{t} \cdot g_t\left(\frac{n}{t}\right) \right| \leq \frac{3}{t} \cdot \sup_{x>0} g_t(x). \tag{10}$$

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Indeed, the sum in (10) is a Riemann sum corresponding to the integral of g_t on the interval $I = [0, \lfloor tR+1 \rfloor/t]$. This Riemann sum corresponds to a partition of I into segments of length $1/t$, and by unimodularity, this Riemann sum can deviate from the actual integral by at most $2/t \cdot \sup_{x>0} g_t(x)$. Since the symmetric difference between I and $[0, R]$ is an interval of length at most $1/t$, the relation (10) follows. According to (10), for any $R, t > 0$,

$$\left| \mu_t([0, R)) - \sum_{n=0}^{\lfloor Rt \rfloor} t^n a_t(n) \right| \leq \frac{3}{t} \cdot \sup_{x>0} g_t(x) = 3 \cdot \sup_{n \geq 0} t^n a_t(n). \tag{11}$$

Next we use our assumption that $\mu(\{0\}) = 0$ and also the fact that $\sup_n e^{-s} s^n/n!$ tends to zero as $s \to \infty$, as may be verified routinely. This shows that for $t > 0$,

$$\sup_{n \geq 0} t^n a_t(n) = \sup_{n \geq 0} \int_0^\infty \frac{(tx)^n}{n!} e^{-tx} d\mu(x) \leq \int_0^\infty \left(\sup_{n \geq 0} \frac{(tx)^n}{n!} e^{-tx} \right) d\mu(x) \xrightarrow{t \to \infty} 0,$$

where we used the dominated convergence theorem in the last passage. The lemma now follows from (9) and (11). $\qquad \square$

The following lemma is due to Borell, and its short proof is contained in [3, Lemma 3.3] and the last paragraph of the proof of [3, Theorem 2.1]. For $A, B \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$ we write $A + B = \{x + y \,;\, x \in A, y \in B\}$ and $\lambda A = \{\lambda x \,;\, x \in A\}$.

Lemma 2.4. *Let μ be a finite Borel measure on $\mathbb{R}$ such that for any intervals $I, J \subseteq \mathbb{R}$ and $0 < \lambda < 1$,*

$$\mu(\lambda I + (1 - \lambda) J) \geq \mu(I)^\lambda \mu(J)^{1-\lambda}. \tag{12}$$

Then μ is log-concave (i.e., either $\mu = c\delta_{x_0}$ for some $c \geq 0, x_0 \in \mathbb{R}$ or else μ has a log-concave density).

Proof of Proposition 2.2. We may assume that $\mu((0, \infty)) > 0$ as otherwise $\mu = c\delta_0$ and the conclusion trivially holds. Therefore $a_t(n) > 0$ for all t and n. By log-concavity of the sequence of alternating Taylor coefficients,

$$\frac{\left(\int_0^\infty x e^{-tx} d\mu(x) \right)^2}{\int_0^\infty (x^2/2) e^{-tx} d\mu(x)} = \frac{a_t(1)^2}{a_t(2)} \geq a_t(0) = \int_0^\infty e^{-tx} d\mu(x) \xrightarrow{t \to \infty} \mu(\{0\}). \tag{13}$$

Write ν_t for the measure on $(0, \infty)$ whose density with respect to μ equals $x \mapsto e^{-tx}$. Then by the Cauchy-Schwartz inequality,

$$\frac{\left(\int_0^\infty x e^{-tx} d\mu(x) \right)^2}{\int_0^\infty x^2 e^{-tx} d\mu(x)} = \frac{\left(\int_0^\infty x d\nu_t(x) \right)^2}{\int_0^\infty x^2 d\nu_t(x)} \leq \nu_t((0, \infty)) \xrightarrow{t \to \infty} 0. \tag{14}$$

From (13) and (14) we see that $\mu(\{0\}) = 0$, which is required for the application of the second part of Lemma 2.3. Let $I, J \subseteq [0, \infty)$ be intervals and $0 < \lambda < 1$. Thanks to Lemma

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2.4, it suffices to prove the inequality (12). Since μ is a finite measure, it suffices to prove that

$$\mu(U) \geq \mu(I_1)^\lambda \mu(J_1)^{1-\lambda} \tag{15}$$

where U is an arbitrary open interval containing $\lambda I + (1 - \lambda)J$, where I_1 is a compact interval contained in the interior of I and J_1 is a compact interval contained in the interior of J. However, by Lemma 2.3,

$$\mu(U) \geq \limsup_{t \to \infty} \mu_t \left(\lambda I + (1 - \lambda)J \right), \quad \mu(I_1) \leq \liminf_{t \to \infty} \mu_t(I), \quad \mu(J_1) \leq \liminf_{t \to \infty} \mu_t(J). \tag{16}$$

Since μ_t is log-concave, the Prékopa-Leindler inequality (see, e.g., [6, Theorem 1.2.3]) implies that for all $t > 0$,

$$\mu_t \left(\lambda I + (1 - \lambda)J \right) \geq \mu_t(I)^\lambda \mu_t(J)^{1-\lambda}. \tag{17}$$

Now (15) follows from (16) and (17), and the proposition is proven. $\qquad\square$

The proof of Theorem 1.1 is complete.

Proof of Corollary 1.2. If μ is of the form $c\delta_0$, then the corollary is trivial. Otherwise, the alternating Taylor coefficients $a_t(n) = (-1)^n \varphi^{(n)}(t)/n!$ are positive for every $t > 0$ and $n \geq 0$. The measure μ is log-concave if and only if the sequence of alternating Taylor coefficients is log-concave for any $t > 0$, which happens if and only if

$$\left(\frac{\varphi^{(n)}(t)}{n!} \right)^2 - \frac{\varphi^{(n-1)}(t)}{(n-1)!} \cdot \frac{\varphi^{(n+1)}(t)}{(n+1)!} \geq 0 \qquad\qquad (n \geq 1, t > 0). \tag{18}$$

Denote by $b_n(t)$ the expression on the left-hand side of (18) multiplied by $(n!)^2$. Then,

$$\frac{d^2}{dt^2} \left| \varphi^{(n-1)}(t) \right|^{-1/n} = \frac{d^2}{dt^2} \left((-1)^{n-1} \cdot \varphi^{(n-1)}(t) \right)^{-1/n} = \frac{n+1}{n^2} \left| \varphi^{(n-1)}(t) \right|^{-(2n+1)/n} \cdot b_n(t).$$

Hence $b_n(t) \geq 0$ for all $t > 0$ if and only if the function $\left| \varphi^{(n-1)}(t) \right|^{-1/n}$ is convex in $(0, \infty)$.
$\qquad\square$

3   The log-concavity measurements are absolutely monotone

In this section we prove Proposition 1.3 and Theorem 1.5. Let $k \leq \ell \leq m \leq n$ be non-negative integers with $k + n = \ell + m$ and let φ be a continuous function on $[0, \infty)$ that is C^∞-smooth in $(0, \infty)$. Define

$$c_{k,\ell,m,n}(t) = a_t(\ell)a_t(m) - a_t(k)a_t(n) = (-1)^{\ell+m} \left[\frac{\varphi^{(\ell)}(t)}{\ell!} \frac{\varphi^{(m)}(t)}{m!} - \frac{\varphi^{(k)}(t)}{k!} \frac{\varphi^{(n)}(t)}{n!} \right],$$

where $a_t(n) = (-1)^n \varphi^{(n)}(t)/n!$ as before. Clearly, the sequence $(a_t(n))_{n \geq 0}$ is log-concave if and only if $c_{k,\ell,m,n}(t) \geq 0$ for all non-negative integers $k \leq \ell \leq m \leq n$ with $k + n = \ell + m$. We call the functions $c_{k,\ell,m,n} : (0, \infty) \to \mathbb{R}$ the log-concavity measurements of φ.

Lemma 3.1. *The derivative of each log-concavity measurement is a linear combination with non-positive coefficients of a finite number of log-concavity measurements.*

Proof. Differentiating (1) we obtain

$$\frac{d}{dt}a_t(n) = -(n+1)a_t(n+1) \qquad\qquad (t > 0, n \geq 0).$$

Abbreviate $b_j = a_t(j)$. Then,

$$-c'_{k,\ell,m,n}(t) = (\ell+1)b_{\ell+1}b_m + (m+1)b_\ell b_{m+1} - (k+1)b_{k+1}b_n - (n+1)b_k b_{n+1}. \qquad (19)$$

Assume first that $\ell < m$. In this case we may rewrite the right-hand side of (19) as

$$(k+1)\left[b_{\ell+1}b_m - b_{k+1}b_n\right] + (\ell-k)\left[b_{\ell+1}b_m - b_k b_{n+1}\right] + (m+1)\left[b_\ell b_{m+1} - b_k b_{n+1}\right].$$

Therefore, in the case $\ell < m$, we expressed $-c'_{k,\ell,m,n}(t)$ as a linear combination with non-negative coefficients of three log-concavity measurements. From now on, we consider the case $\ell = m$. If $k = \ell$, then necessarily $n = m$ and the log-concavity measurement $c_{k,\ell,m,n}(t)$ vanishes. If $k < \ell$, then necessarily $m < n$ and we rewrite the right-hand side of (19) as

$$(k+1)\left[b_\ell b_{m+1} - b_{k+1}b_n\right] + (n+1)\left[b_\ell b_{m+1} - b_k b_{n+1}\right].$$

Consequently, in the case $\ell = m$, we may express $-c'_{k,\ell,m,n}(t)$ as a linear combination with non-negative coefficients of two log-concavity measurements. The proof is complete. $\qquad\square$

Corollary 3.2. *Let $t > 0$ be such that $(a_t(n))_{n\geq 0}$ is a log-concave sequence. Assume that $k \leq \ell \leq m \leq n$ are non-negative integers with $k+n = \ell+m$. Abbreviate $f(t) = c_{k,\ell,m,n}(t)$. Then for all $j \geq 0$,*

$$(-1)^j f^{(j)}(t) \geq 0.$$

Proof. Any log-concavity measurement is non-negative at any $t > 0$. It follows from Lemma 3.1 that $(-1)^j f^{(j)}(t)$ is a finite linear combination with non-negative coefficients of certain log-concavity measurements. Therefore $(-1)^j f^{(j)}(t) \geq 0$. $\qquad\square$

Proof of Proposition 1.3. Write $A \subseteq (0,\infty)$ for the set of all $t > 0$ for which $(a_t(n))_{n\geq 0}$ is a log-concave sequence. Since φ is C^∞-smooth, the set A is closed in $(0,\infty)$. From our assumption, $r \in A$. Define

$$t_0 = \inf\left\{t > 0\,;\, [t,r] \subseteq A\right\}.$$

Then $t_0 \leq r$. Our goal is to prove that $t_0 = 0$. Assume by contradiction that $t_0 > 0$. Since A is a closed set, necessarily $t_0 \in A$. Since φ is real-analytic, the Taylor series of φ converges to φ in $(t_0 - \varepsilon, t_0 + \varepsilon)$ for a certain $\varepsilon > 0$. Assume that $k \leq \ell \leq m \leq n$ are non-negative integers with $k + n = \ell + m$. Then also the Taylor series of $f(t) = c_{k,\ell,m,n}(t)$ converges to f in the same interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. From Corollary 3.2 we thus deduce that for all $t \in (t_0 - \varepsilon, t_0]$,

$$c_{k,\ell,m,n}(t) \geq 0.$$

Consequently, $(t_0 - \varepsilon, t_0] \subseteq A$, in contradiction to the definition of t_0. $\qquad\square$

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We proceed with yet another proof of Proposition 1.3, which is more in spirit of the Walkup theorem which we now recall:

Theorem 3.3 (Walkup theorem [9, 11]). *If $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are log-concave sequences then the sequence $(c_n)_{n \geq 0}$ given by*

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}, \qquad (n \geq 0)$$

is also log-concave.

By Taylor's theorem whenever $0 < s < t$,

$$(t-s)^k a_s(k) = \sum_{n=k}^{\infty} \binom{n}{k} (t-s)^n a_t(n),$$

assuming that φ is real-analytic and that the Taylor series of φ at t converges in $(s - \varepsilon, t)$ for some $\varepsilon > 0$. Therefore Proposition 1.3 is equivalent to the following Walkup-type result:

Proposition 3.4. *If $(a_k)_{k \geq 0}$ is a log-concave sequence then the sequence $(c_k)_{k \geq 0}$ defined by*

$$c_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n, \qquad (k \geq 0)$$

is log-concave as well.

We do not know of a formal derivation of Theorem 3.3 from Proposition 3.4 or vice versa, yet we provide a direct proof of Proposition 3.4 which shares some similarities with the proof of Walkup's theorem and the Borell-Berwald inequality given in [9, 11]. For integers $a \leq b$ we write $[\![a, b]\!] = \{n \in \mathbb{Z}; \ a \leq n \leq b\}$. We begin the direct proof of Proposition 3.4 with the following:

Lemma 3.5. *Let $(a_n)_{n \geq 0}$ be a log-concave sequence. Then for every non-negative integers k and l we have*

$$\sum_{n \geq 0} \binom{n}{k} \binom{l-n}{k} a_n a_{l-n} \geq \sum_{n \geq 0} \binom{n}{k-1} \binom{l-n}{k+1} a_n a_{l-n}, \qquad (20)$$

where here we set $\binom{n}{k} = 0$ in case where $k > n$ or $k < 0$ or $n < 0$.

Proof. Inequality (20) holds trivially if $2k > \ell$. We may thus assume that $2k \leq \ell$. Let U be a random subset of cardinality $2k + 1$ of $\{1, \ldots, l+1\}$ chosen uniformly. Let $X_1, \ldots, X_{2k+1}$ be the elements of U in increasing order. Observe that the law of X_{k+1} is given by

$$\mathbb{P}(X_{k+1} = n + 1) = \frac{\binom{n}{k} \binom{l-n}{k}}{\binom{l+1}{2k+1}}, \qquad (n \geq 0).$$

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Therefore
$$\sum_n \binom{n}{k}\binom{l-n}{k}a_n a_{l-n} = \binom{l+1}{2k+1}\mathbb{E}[f(X_{k+1})]$$
where f is the function given by
$$f(n) = a_{n-1}a_{l+1-n}, \quad \forall n,$$
where we set $a_k = 0$ for $k < 0$. In a similar way
$$\sum_n \binom{n}{k-1}\binom{l-n}{k+1}a_n a_{l-n} = \binom{l+1}{2k+1}\mathbb{E}[f(X_k)].$$

Hence the desired inequality boils down to
$$\mathbb{E}[f(X_k)] \le \mathbb{E}[f(X_{k+1})].$$

By Fubini it suffices to prove that
$$\mathbb{P}(f(X_k) > t) \le \mathbb{P}(f(X_{k+1}) > t) \qquad \forall t \ge 0.$$

The function f satisfies $f(l + 2 - n) = f(n)$ for all n. The crucial observation is that because of the log–concavity of the sequence (a_n) the farther n from the midpoint $(l+2)/2$ the smaller $f(n)$. Therefore the level set $\{f > t\}$ is either empty or else an interval of the form $[\![n, l + 2 - n]\!]$ for some integer $n \le (l+2)/2$. Hence it is enough to prove that for every such n,
$$\mathbb{P}(X_k \in [\![n, l + 2 - n]\!]) \le \mathbb{P}(X_{k+1} \in [\![n, l + 2 - n]\!]). \tag{21}$$

Intuitively, since X_{k+1} is the middle element of U, it is more likely to be close to the center of the interval $[\![1, l + 1]\!]$ than any other element. More precisely, since $X_k \le X_{k+1}$,
$$\begin{aligned}
&\mathbb{P}(X_k \in [\![n, l + 2 - n]\!]) - \mathbb{P}(X_{k+1} \in [\![n, l + 2 - n]\!]) \\
&= \mathbb{P}(X_k \le l + 2 - n;\ X_{k+1} > l + 2 - n) - \mathbb{P}(X_k < n;\ X_{k+1} \ge n) \\
&= \frac{\binom{l+2-n}{k}\binom{n-1}{k+1}}{\binom{l+1}{2k+1}} - \frac{\binom{n-1}{k}\binom{l+2-n}{k+1}}{\binom{l+1}{2k+1}}.
\end{aligned}$$

In order to complete the proof we need to show that this expression is non-positive, assuming that $k \le \ell/2$ and $n \le (\ell+2)/2$. Note that
$$\frac{\binom{l+2-n}{k}\binom{n-1}{k+1}}{\binom{n-1}{k}\binom{l+2-n}{k+1}} = \frac{(n-1-k)!(\ell-n-k+1)!}{(\ell+2-n-k)!(n-k-2)!} = \frac{(n-1)-k}{(\ell+2-n)-k}. \tag{22}$$

We need to show that the expression in (22) is at most one. The denominator in (22) is positive, as
$$\ell + 2 - n - k = 1 + [(\ell+2)/2 - n] + (\ell/2 - k) \ge 1.$$

The numerator in (22) is smaller than the denominator, as $n - 1 \le \ell + 2 - n$. Hence the expression in (22) is at most one, completing the proof of the lemma. $\qquad\square$

Direct proof of Proposition 3.4: Let $k \geq 0$ be an integer. We need to prove that

$$\sum_{n,m} \binom{n}{k}\binom{m}{k} a_n a_m \geq \sum_{n,m} \binom{n}{k-1}\binom{m}{k+1} a_n a_m.$$

By grouping the terms according to the value of $n + m$ we see that it suffices to prove that for any $\ell, k \geq 0$,

$$\sum_n \binom{n}{k}\binom{l-n}{k} a_n a_{l-n} \geq \sum_n \binom{n}{k-1}\binom{l-n}{k+1} a_n a_{l-n}.$$

This is, however, exactly the statement of the previous lemma. $\qquad\square$

When μ is a finite, log-concave measure on $[0, \infty)$, it is well-known (e.g., [6]) that $\mu([t, \infty)) \leq A\mathrm{e}^{-Bt}$ for all $t > 0$, where $A, B > 0$ depend only on μ. It follows that the Laplace transform φ defined in (2) is analytic for $t \in \mathbb{C}$ with $\mathrm{Re}(t) > -C$ for some $C > 0$ depending on μ.

Proof of Theorem 1.5. By Theorem 1.1, the alternating Taylor coefficients sequence $(a_t(n))_{n\geq 0}$ is log-concave for any $t > 0$. From Corollary 3.2 we thus learn that

$$f(t) = c_{k,\ell,m,n}(t) = a_t(\ell)a_t(m) - a_t(k)a_t(n)$$

satisfies $(-1)^j f^{(j)}(t) \geq 0$ for any $t > 0$ and $j \geq 0$. The function f is real-analytic in a neighborhood of $[0, \infty)$ and in particular it is continuous in $[0, \infty)$. The function f is thus absolutely-monotone, and according to Bernstein theorem, there exists a finite, non-negative measure ν for which (4) holds true. $\qquad\square$

We may rewrite conclusion (4) of Theorem 1.5 as follows: For any $t > 0$,

$$\int_0^\infty \frac{x^\ell}{\ell!} e^{-tx} d\mu(x) \int_0^\infty \frac{x^m}{m!} e^{-tx} d\mu(x) - \int_0^\infty \frac{x^k}{k!} e^{-tx} d\mu(x) \int_0^\infty \frac{x^n}{n!} e^{-tx} d\mu(x) = \int_0^\infty e^{-tx} d\nu(x).$$

Let us now consider the Fourier transform

$$F_\mu(t) = \int_0^\infty e^{-itx} d\mu(x) \qquad\qquad (t \in \mathbb{R}).$$

By analytic continuation, Theorem 1.5 immediately implies the following:

Proposition 3.6. *Let μ be a finite, log-concave measure on $[0, \infty)$. Then for any non-negative integers $k \leq \ell \leq m \leq n$ with $k + n = \ell + m$ there exists a finite, non-negative measure $\nu = \nu_{k,\ell,m,n}$ on $[0, \infty)$, such that for any $t > 0$,*

$$\frac{F_\mu^{(\ell)}(t)}{\ell!} \frac{F_\mu^{(m)}(t)}{m!} - \frac{F_\mu^{(k)}(t)}{k!} \frac{F_\mu^{(n)}(t)}{n!} = (-i)^{\ell+m} F_\nu(t).$$

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Corollary 3.7. *Let μ, k, ℓ, m, n, ν be as in Theorem 1.5. Write $P_j(\mu)$ for the measure whose density with respect to μ is $x \mapsto x^j/j!$. Write $E_t(\mu)$ for the measure whose density with respect to μ is $x \mapsto \exp(-tx)$. Then,*

  (i) We have $\nu = P_\ell(\mu) * P_m(\mu) - P_k(\mu) * P_n(\mu)$ where $*$ stands for convolution.

  (ii) For any $t > 0$, the measure $E_t(\nu)$ is the Berwald-Borell transform of $E_t(\mu)$ with the same parameters (k, ℓ, m, n). The same holds for any $t \in \mathbb{R}$ for which $E_t(\mu)$ is a finite measure.

Proof. We have that $F_\mu^{(j)}/j! = (-i)^j \cdot F_{P_j(\mu)}$. Proposition 3.6 thus shows that

$$F_{P_\ell(\mu)} F_{P_m(\mu)} - F_{P_k(\mu)} F_{P_n(\mu)} = F_\nu. \tag{23}$$

The Fourier transform maps products to convolutions. Conclusion (i) therefore follows from (23). Conclusion (ii) follows immediately from the definitions. $\square$

4   Borell type formula for the Poisson measure

In [4], Borell gave a new proof of the Prékopa-Leindler inequality based on the following stochastic variational formula. Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$. Given a standard n-dimensional Brownian motion B and a bounded function $f \colon \mathbb{R}^n \to \mathbb{R}$ we have

$$\log\left(\int_{\mathbb{R}^n} e^f \, d\gamma_n \right) = \sup_u \left\{ \mathbb{E}\left[f\left(B_1 + \int_0^1 u_s \, ds \right) - \frac{1}{2} \int_0^1 |u_s|^2 \, ds \right] \right\}, \tag{24}$$

where the supremum is taken over all bounded stochastic processes u which are adapted to the Brownian filtration, i.e. u_t is measurable with respect to the σ-field generated by $\{B_s;\ s \leq t\}$ for all $t \in [0, 1]$.

In this section we give a discrete version of Borell's formula in which the Gaussian measure and the Brownian motion are replaced by the Poisson distribution and the Poisson process, respectively. In the following section we shall apply our formula in order to deduce a discrete version of the Prékopa-Leindler inequality. We start with some background on counting processes with stochastic intensities. Let $T > 0$ be a fixed number, denote $\mathbb{R}_+ = [0, \infty)$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which our random variables will be defined.

Throughout this section, we let N be a Poisson point process on $[0, T] \times \mathbb{R}_+ \subseteq \mathbb{R}^2$ with intensity measure equal to the Lebesgue measure $\mathcal{L}$. In particular $N(F)$ is a Poisson random variable with parameter $\mathcal{L}(F)$ for any Borel set $F \subseteq [0, T] \times \mathbb{R}_+$. For a Borel subset $E \subseteq [0, T] \times \mathbb{R}_+$ we write $\mathcal{F}_E$ for the σ-field generated by the random variables

$$\{N(F);\ F \text{ is a Borel set, } F \subseteq E\}.$$

For $t \in [0, T]$ we set $\mathcal{F}_t = \mathcal{F}_{[0,t] \times \mathbb{R}_+}$. This defines a filtration of Ω. Our next goal is to define, given a suitable stochastic process $(\lambda_t)_{0 \leq t \leq T}$, a counting process $(X_t^\lambda)_{0 \leq t \leq T}$ which satisfies

$$X_t^\lambda = N(\{(s, u) \in [0, T] \times \mathbb{R}^+ ;\ s \leq t,\ u \leq \lambda_s\}). \tag{25}$$

In other words X_t^λ is the number of atoms of N which lie below the curve $\{(s, \lambda_s) \colon s \in [0, t]\}$. Let us now explain the technical assumptions regarding the stochastic intensity process $(\lambda_t)_{t \leq T}$. Recall that a process is called *predictable* if, as a function of $t \in [0, T]$ and $\omega \in \Omega$, it is measurable with respect to the σ-field $\mathcal{P}$ generated by the sets

$$\{ (s, t] \times A \,; \, s \leq t \leq T, \, A \in \mathcal{F}_s \}.$$

This is slightly more restrictive than being adapted, i.e., when λ_t is measurable with respect to $\mathcal{F}_t$. We have the following standard fact: if a process is left-continuous and adapted, then it is predictable. From now on we will always assume that the stochastic intensity process λ is non-negative, bounded and predictable. The counting process X^λ defined via (25) is clearly adapted and non-decreasing. Note that given $M > 0$, with probability 1 the process N has only finitely many atoms in the box $[0, T] \times [0, M]$ and no two of those lie on the same vertical line $\{t\} \times [0, M]$. Thus, with probability 1 the process X^λ has finitely many jumps, all of size 1, and it is right-continuous. We sometimes refer to the jumps of X^λ as *atoms*.

Lemma 4.1. *For every non-negative predictable process* $(H_t)_{0 \leq t \leq T}$ *we have*

$$\mathbb{E}\left[\int_0^T H_t \, X^\lambda(dt)\right] = \mathbb{E}\left[\int_0^T H_t \lambda_t \, dt\right], \tag{26}$$

where the integral on the left-hand side is a Riemann-Stieljes integral, i.e., here it is a sum of the values of H_t *at the atoms of* X^λ.

The proof of the technical Lemma 4.1 is deferred to the appendix. Equation (26) is frequently taken as the definition of a counting process with stochastic intensity λ. The process

$$\widetilde{X}_t^\lambda = X_t^\lambda - \int_0^t \lambda_s ds \qquad (0 \leq t \leq T)$$

is called the *compensated* process. By Lemma 4.1 it has the property that for every bounded, predictable process (H_t), the process

$$\left(\int_0^t H_s \widetilde{X}^\lambda(ds)\right)_{0 \leq t \leq T}$$

is a martingale. We are now in a position to state the analogue of Borell's formula for the Poisson measure. In the following theorem π_T denotes the Poisson measure of parameter T, i.e.,

$$\pi_T(n) = \frac{T^n}{n!} e^{-T} \qquad \text{for } n \in \mathbb{N} = \{0, 1, 2, \ldots\}$$

where we abbreviate $\pi_T(n) = \pi_T(\{n\})$.

Theorem 4.2. *Let* $f \colon \mathbb{N} \to \mathbb{R}$ *be bounded and let* $T > 0$. *Then we have*

$$\log\left(\int_\mathbb{N} e^f \, d\pi_T\right) = \sup_\lambda \left\{ \mathbb{E}\left[f(X_T^\lambda) - \int_0^T (\lambda_t \log \lambda_t - \lambda_t + 1) \, dt\right]\right\}, \tag{27}$$

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where the supremum is taken over all bounded, non-negative, predictable processes $(\lambda_t)_{0 \leq t \leq T}$, *and* $(X_t^\lambda)_{0 \leq t \leq T}$ *is the associated counting process, defined by* (25). *Moreover the supremum is actually a maximum.*

Proof. Let $(P_t)_{t \geq 0}$ be the Poisson semigroup: For every $g \colon \mathbb{N} \to \mathbb{R}$

$$P_t g(x) = \sum_{n \in \mathbb{N}} g(x + n) \, \pi_t(n).$$

We shall show that for every predictable bounded process (λ_t) we have

$$\log P_T(e^f)(0) \geq \mathbb{E}\left[f(X_T^\lambda) - \int_0^T (\lambda_t \log \lambda_t - \lambda_t + 1) \, dt \right], \qquad (28)$$

with equality if λ is chosen appropriately. Let us start with the inequality. Note that for every $g : \mathbb{N} \to \mathbb{R}$ and $t \geq 0$,

$$\partial_t P_t g = \partial_x P_t g$$

where $\partial_x g(x) = g(x + 1) - g(x)$ denotes the discrete gradient. Letting

$$F(t, x) = \log P_{T-t}(e^f)(x)$$

we obtain

$$\partial_t F = -e^{\partial_x F} + 1.$$

Let λ be a predictable, non-negative, bounded process and let

$$M_t = F(t, X_t^\lambda) - \int_0^t (\lambda_s \log \lambda_s - \lambda_s + 1) \, ds.$$

Almost surely, the process $(M_t)_{0 \leq t \leq T}$ is a piecewise absolutely-continuous function in t. Hence the distributional derivative of the function $t \mapsto M_t$ is almost-surely the sum of an integrable function on $[0, T]$ and finitely many atoms. Namely, for any fixed $t \in [0, T]$,

$$M_t = M_0 + \int_0^t \partial_x F(s, X_{s-}^\lambda) \, X^\lambda(ds) - \int_0^t \left(e^{\partial_x F(s, X_s^\lambda)} + \lambda_s \log \lambda_s - \lambda_s \right) ds \qquad (29)$$

where X_{s-}^λ denotes the left limit at s of X^λ. Note that since X^λ has only finitely many atoms we can replace X_s^λ by X_{s-}^λ in the second term of the right hand side. Setting $\alpha_t = \partial_x F(t, X_{t-}^\lambda)$, we may rewrite (29) as follows:

$$M_t - M_0 = \int_0^t \alpha_s \, \widetilde{X}^\lambda(ds) - \int_0^t (e^{\alpha_s} + \lambda_s \log \lambda_s - \lambda_s - \alpha_s \lambda_s) \, ds. \qquad (30)$$

Recall that $\widetilde{X}_t^\lambda = X_t^\lambda - \int_0^t \lambda_s \, ds$ is the compensated measure. Note that $F(t, x)$ is continuous in t and that (X_{t-}^λ) is left continuous in t. Thus (α_t) is left continuous. Since it is also

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adapted, it is predictable. Moreover both (α_t) and (λ_t) are bounded. Consequently the first summand on the right-hand side of (30) is a martingale. Moreover, since

$$\mathrm{e}^x + y \log y - y - xy \geq 0 \qquad \forall x \in \mathbb{R},\, y \in \mathbb{R}_+ \tag{31}$$

the second integral on the right-hand side of (30) is non-negative. Therefore $(M_t)_{0 \leq t \leq T}$ is a supermartingale. In particular $M_0 \geq \mathbb{E}[M_T]$, which is the desired inequality (28).

There is equality in (31) if $\mathrm{e}^x = y$. Hence if λ is such that

$$\lambda_t = \mathrm{e}^{\partial_x F(t, X_{t-}^\lambda)}, \tag{32}$$

for almost every t and with probability one, then M is a martingale and we have equality in (28). Note that the function $\mathrm{e}^{\partial_x F(t,x)}$ is continuous in t and bounded. We prove in Lemma 4.3 below that under these conditions, a solution to (32) does indeed exist, which finishes the proof of the theorem. $\qquad\square$

Remarks.

1. It is also possible to prove Theorem 4.2 by using the Girsanov change of measure formula for counting processes. The argument presented here has the advantage of being self-contained.

2. Theorem 4.2 can probably be generalized in several ways. Firstly, up to some annoying technical details, the argument should work just the same for a function f that depends on the whole trajectory of the process rather than just the terminal point. On the left hand side, the Poisson distribution should then be replaced by the law of the Poisson process of intensity 1 on $[0, T]$. In the Gaussian case, this pathspace version of the formula is known as the Boué-Dupuis formula, see [5]. Then one can also replace the interval $[0, T]$ equipped with the Lebesgue measure by a more general measure space, leading to a Borell type formula for Poisson point processes. In a sense this was already carried out by X. Zhang in [13], but the result in [13] does not seem to recover our theorem.

3. A dual version version of Borell's formula involving relative entropy was proved by the second-named author in [8]. This can be done in the Poisson case too. The formula then reads: if μ is a probability measure on $\mathbb{N}$ whose density with respect to π_T is bounded away from 0 and $+\infty$, then the relative entropy of μ with respect to π_T satisfies

$$\mathrm{H}(\mu \mid \pi_T) = \inf_\lambda \left\{ \mathbb{E} \left[\int_0^T (\lambda_t \log \lambda_t - \lambda_t + 1)\, dt \right] \right\},$$

where the infimum runs over all non-negative, predictable processes λ such that X_T^λ has law μ. This follows from the representation formula (27) and the Gibbs variational principle, which is the fact that the functionals $\nu \mapsto H(\nu | \pi_T)$ and $f \mapsto \log \int e^f d\pi_T$ are Legendre-Fenchel conjugates with respect to the pairing $(f, \nu) \mapsto \int f d\nu$.

We now state and prove the technical lemma used in the proof of Theorem 4.2.

Lemma 4.3. *Let $G\colon [0,T] \times \mathbb{N} \to \mathbb{R}_+$ and assume that G is continuous in the first variable and bounded. Then there exists a predictable, bounded, non-negative, left-continuous process $(\lambda_t)_{0 \leq t \leq T}$ satisfying*

$$\lambda_t = G(t, X_{t-}^\lambda),$$

for almost every $t \leq T$ and with probability 1.

Proof. Consider the map

$$H\colon (\lambda_t)_{0 \leq t \leq T} \mapsto (G(t, X_{t-}^\lambda))_{0 \leq t \leq T}$$

from the set of adapted, left-continuous, non-negative, bounded processes to itself. We will show that H has a fixed point. Let λ and μ be two processes in the domain of H. Since G is bounded, there is a constant C such that

$$\mathbb{E}[|G(t, X_{t-}^\lambda) - G(t, X_{t-}^\mu)|] \leq C\, \mathbb{P}(X_{t-}^\lambda \neq X_{t-}^\mu).$$

The probability that the integer-valued random variable X_{t-}^λ differs from X_{t-}^μ is dominated by $\mathbb{E}[|X_{t-}^\lambda - X_{t-}^\mu|]$, which in turn is the average number of atoms of N between the graphs of λ and μ on $[0, t)$. Since λ and μ are predictable, it follows from Lemma 4.1 that

$$\mathbb{E}|X_{t-}^\lambda - X_{t-}^\mu| = \mathbb{E}\left[\int_0^t |\lambda_s - \mu_s|\, ds\right].$$

Therefore

$$\mathbb{E}\left[|G(t, X_{t-}^\lambda) - G(t, X_{t-}^\mu)|\right] \leq C\, \mathbb{E}\left[\int_0^t |\lambda_s - \mu_s|\, ds\right].$$

This easily implies that H is Lipschitz with constant $1/2$ for the distance

$$d(\lambda, \mu) = \int_0^T \mathrm{e}^{-2Ct}\, \mathbb{E}[|\lambda_t - \mu_t|]\, dt.$$

Thus, being a contraction, the map H has a fixed point, which is the desired result. $\qquad\square$

5 A discrete Prékopa-Leindler inequality

Following Borell we derive in this section a Prékopa-Leindler type inequality from the representation formula (27). Recall that if x is a real number we denote its lower integer part by $\lfloor x \rfloor$ and its upper integer part by $\lceil x \rceil$. For $a, b \in \mathbb{R}$ we denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Recall that π_T denotes the Poisson distribution of parameter T.

Proposition 5.1. *Let $f, g, h, k\colon \mathbb{N} \to \mathbb{R}$ satisfy*

$$f(x) + g(y) \leq h\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) + k\left(\left\lceil \frac{x+y}{2} \right\rceil\right), \quad \forall x, y \in \mathbb{N}.$$

Then,

$$\int_{\mathbb{N}} \mathrm{e}^f\, d\pi_T \int_{\mathbb{N}} \mathrm{e}^g\, d\pi_T \leq \int_{\mathbb{N}} \mathrm{e}^h\, d\pi_T \int_{\mathbb{N}} \mathrm{e}^k\, d\pi_T.$$

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Proof. By approximation we may assume that all four functions are bounded. Let α and β be two non-negative, bounded, predictable processes. We observe that $\lfloor (X^\alpha + X^\beta)/2 \rfloor$ coincides with the process X^λ where

$$\lambda = (\alpha \wedge \beta) \chi + (\alpha \vee \beta) (1 - \chi)$$

where χ_t is the indicator function of the event that $X^\alpha_{t-} + X^\beta_{t-}$ is even. Similarly $\lceil (X + Y)/2 \rceil = X^\mu$ where

$$\mu = (\alpha \wedge \beta) (1 - \chi) + (\alpha \vee \beta) \chi.$$

Note that for every $t \in [0, T]$ either $\mu_t = \alpha_t$ and $\lambda_t = \beta_t$ or the other way around. In particular, for every function $\varphi : [0, \infty) \to \mathbb{R}$,

$$\varphi(\alpha_t) + \varphi(\beta_t) = \varphi(\lambda_t) + \varphi(\mu_t), \quad \forall t \in [0, T]$$

Using the hypothesis made on f, g, h, k we get that for a continuous function φ,

$$f(X^\alpha_T) + g(X^\beta_T) - \int_0^T \varphi(\alpha_t)\, dt - \int_0^T \varphi(\beta_t)\, dt$$
$$\leq h(X^\lambda_T) + k(X^\mu_T) - \int_0^T \varphi(\lambda_t)\, dt - \int_0^T \varphi(\mu_t)\, dt.$$

Choosing $\varphi(x) = x \log x + x - 1$, taking expectation, and using the representation formula in Theorem 4.2 for h and k we get

$$\mathbb{E}\left[f(X^\alpha_T) - \int_0^T \varphi(\alpha_t)\, dt \right] + \mathbb{E}\left[g(X^\beta_T) - \int_0^T \varphi(\beta_t)\, dt \right]$$
$$\leq \log\left(\int_{\mathbb{N}} \mathrm{e}^h\, d\pi_T \right) + \log\left(\int_{\mathbb{N}} \mathrm{e}^k\, d\pi_T \right).$$

Taking the supremum in α and β and using the representation formula for f and g yields the result. $\qquad\square$

Rescaling appropriately, we obtain as a corollary a Prékopa-Leindler type inequality for the counting measure on $\mathbb{Z}$.

Proof of Theorem 1.4. Let Y_n be a random variable having the Poisson law of parameter n and let $X_n = Y_n - n$. Applying the previous proposition to the functions f, g, h, k (translated by $-n$) we get

$$\mathbb{E}\left[\mathrm{e}^{f(X_n)} \right] \mathbb{E}\left[\mathrm{e}^{g(X_n)} \right] \leq \mathbb{E}\left[\mathrm{e}^{h(X_n)} \right] \mathbb{E}\left[\mathrm{e}^{k(X_n)} \right]. \tag{33}$$

On the other hand, for any fixed $k \in \mathbb{Z}$, letting n tend to $+\infty$ and using Stirling formula we get

$$\mathbb{P}(X_n = k) = \frac{n^{n+k}}{(n+k)!} \mathrm{e}^{-n} = \frac{1}{\sqrt{2\pi n}}(1 + o(1)).$$

Hence by the dominated convergence theorem

$$\sqrt{2\pi n} \cdot \mathbb{E}\left[\mathrm{e}^{f(X_n)}\right] \overset{n\to\infty}{\longrightarrow} \sum_{x\in\mathbb{Z}} \mathrm{e}^{f(x)},$$

and similarly for g, h, k. Hence multiplying (33) by n and letting n tend to $+\infty$ yields the result. $\qquad\square$

6 Appendix: Proof of Lemma 4.1

We write $\mathcal{B}([0,T])$ for the Borel σ-field of $[0,T]$. Let μ^+ and μ^- be the measures on $[0,T] \times \mathbb{R}_+ \times \Omega$ equipped with the σ-field $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$ defined by

$$\mu^+(dt, du, d\omega) = N(\omega)(dt, du)\,\mathbb{P}(d\omega)$$
$$\mu^-(dt, du, d\omega) = \mathcal{L}(dt, du)\,\mathbb{P}(d\omega),$$

where $\mathcal{L}$ is the Lebesgue measure on the strip $[0,T] \times \mathbb{R}_+$ while $N(\omega)$ is the measure on this strip given by the Poisson process N. Let $\mathcal{I}$ be the σ-field on $[0,T] \times \mathbb{R}_+ \times \Omega$ generated by the class

$$\mathcal{J} = \{E \times A;\ E \in \mathcal{B}([0,T] \times \mathbb{R}_+),\ A \in \mathcal{F}_{E^c}\}$$

where E^c denotes the complement of E. We claim that μ^+ and μ^- coincide on $\mathcal{I}$. This is in fact the statement of Theorem 1 in [10], we recall here the short proof for completeness. For $E \times A \in \mathcal{J}$, the random variable $N(E)$ is independent of the set A, hence

$$\mu^+(E \times A) = \mathbb{E}[N(E)\mathbb{1}_A] = \mathbb{E}[N(E)]\mathbb{P}(A) = \mathcal{L}(E)\mathbb{P}(A) = \mu^-(E \times A).$$

Since the class $\mathcal{J}$ is stable by finite intersections and generates the σ-field $\mathcal{I}$ the claim follows from the monotone class theorem.

Next recall the definition of the predictable σ-field $\mathcal{P}$ and observe that

$$\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{P} \subseteq \mathcal{I}.$$

As a result, since (H_t) and (λ_t) are predictable, as a function of (t, u, ω),

$$H_t\,\mathbb{1}_{\{u\le\lambda_t\}}$$

is measurable with respect to $\mathcal{I}$. We may therefore integrate $H_t\,\mathbb{1}_{\{u\le\lambda_t\}}$ with respect to μ^+ or μ^- and obtain the same outcome. In other words,

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_+} H_t\,\mathbb{1}_{\{u\le\lambda_t\}}\,N(dt, du)\right] = \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_+} H_t\,\mathbb{1}_{\{u\le\lambda_t\}}\,dtdu\right]. \tag{34}$$

From (34) we obtain that

$$\mathbb{E}\left[\int_0^T H_t X^\lambda(dt)\right] = \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_+} H_t \mathbb{1}_{\{u\leq\lambda_t\}} N(dt,du)\right]$$

$$= \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_+} H_t \mathbb{1}_{\{u\leq\lambda_t\}} dt du\right]$$

$$= \mathbb{E}\left[\int_0^T H_t \lambda_t\, dt\right],$$

completing the proof of lemma 4.1.

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