Marginals of Geometric Inequalities

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Summary. This note consists of three parts. In the first, we observe that a surprisingly rich family of functional inequalities may be proven from the Brunn–Minkowski inequality using a simple geometric technique. In the second part, we discuss consequences of a functional version of Santaló's inequality, and in the third part we consider functional counterparts of mixed volumes and of Alexandrov–Fenchel inequalities.

1 Introduction

In this note we review a simple, folklore, method for obtaining a functional inequality – an inequality about functions – from a geometric inequality, which here means an inequality about shapes and bodies. Given a compact set $K \subset \mathbb{R}^n$ and a k-dimensional subspace $E \subset \mathbb{R}^n$, the marginal of K on the subspace E is the function $f_{K,E} : E \to [0, \infty)$ defined as

$$f_{K,E}(x) = \mathrm{Vol}_{n-k}\big(K \cap [x + E^{\perp}]\big)$$

where $E^{\perp}$ is the orthogonal complement to E in $\mathbb{R}^n$, and Vol_{n-k} is the induced Lebesgue measure on the affine subspace $x + E^{\perp}$. A trivial observation is that an inequality of the form $\mathrm{Vol}_n(A) \geq \mathrm{Vol}_n(B)$ implies the inequality $\int_E f_{A,E} \geq \int_E f_{B,E}$. Thus geometric inequalities give rise to certain functional inequalities in a lower dimension.

The idea of recovering functional inequalities from different types of inequalities in higher dimension is not new, and neither is the use of marginals as explained above (see, e.g., [Bo, Er] or [KLS, page 548]). In this note we observe that this obvious method, when applied to some classical geometric inequalities, entails non-trivial functional inequalities. In particular, this method yields conceptually simple proofs of logarithmic Sobolev inequalities, Prékopa–Leindler and other inequalities: All follow as marginals of the Brunn–Minkowski inequality. Marginals of the Brunn–Minkowski inequality are the

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subject of the first part of this paper, that consists of Section 2 and Section 3. Although no new mathematical statements are presented in this part of the note, we hope that some readers will benefit from the clear geometric flavor added to the known proofs of these inequalities, in particular the approach of Bobkov and Ledoux to the gaussian log-Sobolev inequality [BoL]. We would also like to acknowledge the great influence of K. Ball's work [B2] and F. Barthe's work [Ba1] on our understanding of the interplay between log-concave functions and convex sets.

An application of the "marginals of geometric inequalities" approach to Santaló's inequality was carried out in [ArtKM]. By appropriately taking marginals of both sides of Santaló's inequality, the following new inequality was established: For any integrable function $g : \mathbb{R}^n \to [0, \infty)$ with a positive integral, there exists $x_0 \in \mathbb{R}^n$ such that $\tilde{g}(x) = g(x - x_0)$ satisfies

$$\int_{\mathbb{R}^n} \tilde{g} \int_{\mathbb{R}^n} \tilde{g}^\circ \leq (2\pi)^n \tag{1}$$

where $f^\circ(x) = \inf_{y \in \mathbb{R}^n} \left[e^{-\langle x,y \rangle} / f(y) \right]$ for any $f : \mathbb{R}^n \to [0, \infty)$. In the case where g is assumed to be an even function, the inequality (1) was proven by K. Ball [B1]. If $\int x g^\circ(x) dx = 0$, then we can take $x_0 = 0$ in (1). In that case, equality in (1) holds if and only if g is a gaussian function. Additionally, the left hand side of (1) is always bounded from below by c^n, for a universal constant $c > 0$ (see [KlM]).

Santalò's inequality, once translated into its functional form (1), attains power of its own. For example, it was shown in [ArtKM] following ideas of Maurey [M], that the inequality (1) implies a sharp concentration inequality for Lipshitz functions of gaussian variables. The second part of this paper describes further applications of the functional Santaló inequality (1). For example, with the aid of the transportation of measure technique, we derive the following corollary:

Corollary 1.1. *Let $K, T \subset \mathbb{R}^n$ be centrally-symmetric, convex bodies, and denote by $D \subset \mathbb{R}^n$ the standard Euclidean unit ball in $\mathbb{R}^n$. Then,*

$$\mathrm{Vol}_n(K \cap_2 T) \mathrm{Vol}_n(K^\circ \cap_2 T) \leq \mathrm{Vol}_n(D \cap_2 T)^2 \tag{2}$$

where $K^\circ = \{x \in \mathbb{R}^n ; \forall y \in K, \ \langle x, y \rangle \leq 1\}$ is the polar body, and $A \cap_2 B$ is defined as follows: If A is the unit ball of the norm $\| \cdot \|_A$ and B is the unit ball of the norm $\| \cdot \|_B$, then $A \cap_2 B$ is defined as the unit ball of the norm $\|x\|_{A \cap_2 B} = \sqrt{\|x\|_A^2 + \|x\|_B^2}$.

Here, a convex body is a compact, convex set with a non-empty interior. Note that $A \cap B \subset A \cap_2 B \subset \sqrt{2}(A \cap B)$ for any centrally-symmetric convex sets $A, B \subset \mathbb{R}^n$. Thus, Corollary 1.1 immediately implies that

$$\mathrm{Vol}_n(K \cap T) \mathrm{Vol}_n(K^\circ \cap T) \leq 2^n \mathrm{Vol}_n(D \cap T)^2 \tag{3}$$

for any centrally-symmetric convex bodies $K, T \subset \mathbb{R}^n$. Inequality (3) is probably not sharp; The constant 2^n on the right hand side seems unnecessary. The validity of (3), without the 2^n factor, was conjectured by Cordero–Erausquin in [C-E]. Cordero–Erausquin proved this conjecture for the case where $K, T \subset \mathbb{R}^{2n}$ are unit balls of complex Banach norms, and T is invariant under complex conjugation [C-E]. Another case in which a sharp version of (3) is known to hold, without the 2^n factor, is the case where T is an unconditional convex body. This follows from the methods in [C-EFM], and was also observed independently by Barthe and Cordero–Erausquin [Ba2]. Corollary 1.1 is derived from more general principles in Section 5, and so is the following corollary.

Corollary 1.2. *Let $\psi : \mathbb{R}^n \to (-\infty, \infty]$ be a convex, even function, and let $\alpha > 0$ be a parameter. Let μ be a measure on $\mathbb{R}^n$ whose density $F = \frac{d\mu}{dx}$ is*

$$F(x) = \int_0^\infty t^{n+1} e^{-\alpha t^2} e^{-\psi(tx)} dt. \tag{4}$$

Then, for any centrally-symmetric, convex body $K \subset \mathbb{R}^n$,

$$\mu(K)\mu(K^\circ) \le \mu(D)^2. \tag{5}$$

What types of measures arise in Corollary 1.2? By plugging in (4), e.g., $\alpha = 1, \psi(x) = \|x\|^2$ for some norm $\|\cdot\|$ on $\mathbb{R}^n$, we deduce that a measure μ whose density is $1/(1 + \|x\|^2)^{n+2}$ satisfies (5). Observe that these measures are not log-concave (see Section 4 for definition).

The third part of this note focuses on the Alexandrov–Fenchel inequalities for mixed volumes. Let $f : \mathbb{R}^n \to [0, \infty)$ be a function that is concave on its support. We define the Legendre transform of f to be

$$\mathcal{L}' f(x) = \sup_{y; f(y) > 0} \left[f(y) - \langle x, y \rangle \right].$$

Note that $\mathcal{L}'f$ is convex. We use the notation $\mathcal{L}'$ and not $\mathcal{L}$, since our transform is slightly different from the standard Legendre transform $\mathcal{L}$ of convex functions (see, e.g., [Ar] or (42) below). The transforms $\mathcal{L}$ and $\mathcal{L}'$ differ mainly by a trivial minus sign.

Theorem 1.3. *Let $f_0, ..., f_n : \mathbb{R}^n \to [0, \infty)$ be compactly-supported, continuous functions, that are concave on their support. Assume also that $\mathcal{L}' f_0, ..., \mathcal{L}' f_n$ posses continuous second derivatives. Denote*

$$V(f_0, ..., f_n) = \int_{\mathbb{R}^n} [\mathcal{L}' f_0](x) D\big(\mathrm{Hess}[\mathcal{L}' f_1](x), ..., \mathrm{Hess}[\mathcal{L}' f_n](x)\big) dx \tag{6}$$

where D stands for mixed discriminant (see, e.g., the Appendix below) and Hess *stands for the Hessian of a function. Then:*

1. *The multilinear form $V(f_0,...,f_n)$ may be extended to be defined for all compactly-supported non-negative functions that are concave on their support (without any smoothness or even continuity assumptions). The quantity $V(f_0,...,f_n)$ is finite also in this extended domain of definition.*
2. *The multilinear form V is continuous with respect to pointwise convergence of functions, in the space of compactly-supported non-negative functions that are concave on their support.*
3. *The multilinear form V is fully symmetric, i.e. for any permutation $\sigma \in S_{n+1}$,*
$$V(f_0,...,f_n) = V(f_{\sigma(0)},...,f_{\sigma(n)}),$$
whenever $f_0,...,f_n : \mathbb{R}^n \to [0,\infty)$ are compactly-supported functions that are concave on their support.
4. *Let $f_0,...,f_n,g_0,...,g_n : \mathbb{R}^n \to [0,\infty)$ be compactly-supported functions that are concave on their support. If $f_0 \geq g_0,...,f_n \geq g_n$, then*
$$V(f_0,...,f_n) \geq V(g_0,...,g_n) \geq 0.$$
5. *Let $f_0,...,f_n : \mathbb{R}^n \to [0,\infty)$ be compactly-supported functions that are concave on their support. The following "hyperbolic-type" inequality holds:*
$$V(f_0,f_1,...,f_n)^2 \geq V(f_0,f_0,f_2,...,f_n)V(f_1,f_1,f_2,...,f_n). \tag{7}$$

The analogy with mixed volumes of convex bodies is clear (see Section 5). Note that a function $g : \mathbb{R}^n \to \mathbb{R}$ is of the form $g = \mathcal{L}'f$ for some compactly-supported function $f : \mathbb{R}^n \to [0,\infty)$ that is concave on its support, if and only if g is convex and
$$\forall x \in \mathbb{R}^n, \quad 0 \leq g(x) - h_T(x) \leq C \tag{8}$$
for some $C > 0$ and a compact, convex set $T \subset \mathbb{R}^n$, where $h_T(x) = \sup_{y\in T}\langle x,y\rangle$ is the supporting functional of T. Thus, we could have reformulated Theorem 1.3 in terms of convex functions satisfying condition (8), rather than in terms of Legendre transform of concave functions.

Theorem 1.4. *Let $K \subset \mathbb{R}^n$ be a compact, convex set, and let $f_0,...,f_n : K \to [0,\infty)$ be concave functions that vanish on ∂K. Assume further that the functions have continuous second derivatives in the interior of K, and that the second derivatives are bounded. Denote*
$$I(f_0,...,f_n) = \int_K f_0(x)D\big(-\mathrm{Hess}f_1(x),...,-\mathrm{Hess}f_n(x)\big)dx, \tag{9}$$
where, as before, D stands for the mixed discriminant and Hess *stands for the Hessian of a function. Then:*

1. *The multilinear form $I(f_0, ..., f_n)$ is finite, and continuous with respect to pointwise convergence of functions (yet, trying to extend the multilinear form I to non-smooth functions, we may encounter situations where $I = \infty$. We thus choose to formally confine the domain of definition of I to smooth functions).*
2. *The multilinear form I is fully symmetric, i.e. for any permutation $\sigma \in S_{n+1}$,*
$$I(f_0, ..., f_n) = I(f_{\sigma(0)}, ..., f_{\sigma(n)}),$$
whenever $f_0, ..., f_n : K \to [0, \infty)$ are concave functions that vanish on ∂K and have continuous, bounded, second derivatives in the interior of K.
3. *Let $f_0, ..., f_n, g_0, ..., g_n : K \to [0, \infty)$ be concave functions that vanish on ∂K and have continuous, bounded, second derivatives in the interior of K. If $f_0 \geq g_0, ..., f_n \geq g_n$ then*
$$I(f_0, ..., f_n) \geq I(g_0, ..., g_n) \geq 0.$$
4. *Let $f_0, ..., f_n : K \to [0, \infty)$ be concave functions that vanish on ∂K and have continuous, bounded, second derivatives in the interior of K. The following "elliptic-type" inequality holds:*
$$I(f_0, f_1, ..., f_n)^2 \leq I(f_0, f_0, f_2, ..., f_n) I(f_1, f_1, f_2, ..., f_n). \tag{10}$$

The only significant difference between V from Theorem 1.3 and I from Theorem 1.4, is the fact that the Legendre transform is applied to the functions in Theorem 1.3 (compare the definition (9) with the definition (6)). The "elliptic" inequality (10) is transformed into the "hyperbolic" inequality (7) after an application of the Legendre transform. It would be desirable to have a deeper understanding of this fact. In particular, our proofs of (7) and of (10) are completely different; We would like to see a unifying scheme for both inequalities. Such a unifying approach might possibly shed new light on the highly non-trivial Alexandrov–Fenchel inequalities. The proofs of Theorem 1.3 and Theorem 1.4 appear in Section 5.1 and Section 5.2, respectively. Section 5 constitutes the third part of this note.

For the convenience of the reader, we also include a short appendix regarding some standard properties of mixed discriminants. Here, the letter D denotes both the unit Euclidean ball and the mixed discriminant, but the context will always distinguish between the two meanings.

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2 The Basic Setting

We work in Euclidean spaces $\mathbb{R}^m$, for various $m > 0$, and we denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the usual norm and scalar product in $\mathbb{R}^m$. Let $n, s > 0$ be integers, and let $f : \mathbb{R}^n \to [0, \infty)$ be a function. The support of f, denoted by $\mathrm{Supp}(f)$, is the closure of $\{x \in \mathbb{R}^n; f(x) > 0\}$. We say that f is s-concave if $\mathrm{Supp}(f)$ is compact, convex and $f^{\frac{1}{s}}$ is concave on $\mathrm{Supp}(f)$. An s-concave function is continuous in the interior of its support (see e.g., [Ro]). With any function $f : \mathbb{R}^n \to [0, \infty)$ we associate a set

$$\mathcal{K}_f = \left\{ (x, y) \in \mathbb{R}^{n+s} = \mathbb{R}^n \times \mathbb{R}^s; x \in \mathrm{Supp}(f), |y| \leq f^{\frac{1}{s}}(x) \right\} \tag{11}$$

where, for given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$, (x, y) are coordinates in $\mathbb{R}^{n+s}$. If the function f is measurable, so is the set $\mathcal{K}_f$. Additionally, the set $\mathcal{K}_f$ is convex if and only if f is s-concave. We also note that

$$\mathrm{Vol}(\mathcal{K}_f) = \int_{\mathrm{Supp}(f)} \kappa_s \cdot \left(f^{\frac{1}{s}}(x) \right)^s dx = \kappa_s \int f \tag{12}$$

where $\kappa_s = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$ is the volume of the s-dimensional Euclidean unit ball. For $\lambda > 0$ and $f : \mathbb{R}^n \to [0, \infty)$, we define the function $\lambda \times_s f : \mathbb{R}^n \to [0, \infty)$ to be

$$\left[\lambda \times_s f \right](x) = \lambda^s f\left(\frac{x}{\lambda} \right). \tag{13}$$

Note that $\mathcal{K}_{\lambda \times_s f} = \lambda \mathcal{K}_f = \{\lambda y; y \in \mathcal{K}_f\}$, and hence we view $\lambda \times_s f$ as a functional analog to homothety of bodies. If f is an s-concave function, so is $\lambda \times_s f$. Recall that for two sets $A, B \subset \mathbb{R}^n$, their Minkowski sum is defined by $A + B = \{a + b; a \in A, b \in B\}$. For two functions $f, g : \mathbb{R}^n \to [0, \infty)$, we define their "$s$-Minkowski sum" as

$$\left[f \oplus_s g \right](x) = \left(\sup_{\substack{y \in \mathrm{Supp}(f), z \in \mathrm{Supp}(g) \\ x = y + z}} f(y)^{\frac{1}{s}} + g(z)^{\frac{1}{s}} \right)^s \tag{14}$$

whenever $x \in \mathrm{Supp}(f) + \mathrm{Supp}(g)$. If $x \notin \mathrm{Supp}(f) + \mathrm{Supp}(g)$, we set $\left[f \oplus_s g \right](x) = 0$. Our definition is motivated by the fact that

$$\mathcal{K}_{f \oplus_s g} = \mathcal{K}_f + \mathcal{K}_g.$$

Note that whenever f, g are s-concave, the function $f \oplus_s g$ is also s-concave. The $\oplus_s$ and $\times_s$ operations induce a convex cone structure on the class of s-concave functions.

Arguably one of the most useful geometric inequalities in the theory of convex bodies is the Brunn–Minkowski inequality. This inequality states that for any non-empty compact sets $A, B \subset \mathbb{R}^m$,

$$\mathrm{Vol}(A + B)^{\frac{1}{m}} \geq \mathrm{Vol}(A)^{\frac{1}{m}} + \mathrm{Vol}(B)^{\frac{1}{m}}. \tag{15}$$

There are at least a handful of completely different proofs of (15), see, e.g., [BonF], [Bru1, Bru2], [GiM], [Gr], [GrM], [HO], [KnS], [Mc2]. For instance, along the lines of Blaschke's proof, we may use the easily verified fact that for any hyperplane $H \subset \mathbb{R}^m$,

$$S_H(A+B) \supset S_H(A) + S_H(B),$$

where S_H is the Steiner symmetrization with respect to the hyperplane H (see, e.g., [BonF]). We now derive (15) by applying a suitable sequence of Steiner symmetrizations, such that $S_{H_1}...S_{H_k}(A+B)$, $S_{H_1}...S_{H_k}(A)$ and $S_{H_1}...S_{H_k}(B)$ converge to Euclidean balls when $k \to \infty$.

The Brunn–Minkowski inequality (15) for $(n+s)$-dimensional sets implies that for any $\lambda, \mu > 0$ and measurable functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\mathrm{Vol}^*_{n+s}\left(\mathcal{K}_{[\lambda \times_s f] \oplus_s [\mu \times_s g]}\right)^{\frac{1}{n+s}} \geq \lambda \mathrm{Vol}_{n+s}(\mathcal{K}_f)^{\frac{1}{n+s}} + \mu \mathrm{Vol}_{n+s}(\mathcal{K}_g)^{\frac{1}{n+s}} \tag{16}$$

where Vol^*_{n+s} stands for outer Lebesgue measure (the set $\mathcal{K}_{[\lambda \times_s f] \oplus_s [\mu \times_s g]}$ may be non-measurable). We immediately conclude that (16) translates, using (12), to the following inequality: For all $\lambda, \mu > 0$, an integer $s > 0$ and measurable functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\left(\int^*_{\mathbb{R}^n} [\lambda \times_s f] \oplus_s [\mu \times_s g] \right)^{\frac{1}{n+s}} \geq \lambda \left(\int_{\mathbb{R}^n} f \right)^{\frac{1}{n+s}} + \mu \left(\int_{\mathbb{R}^n} g \right)^{\frac{1}{n+s}} \tag{17}$$

where $\int^*$ is the outer integral. We summarize this discussion with the following theorem.

Theorem 2.1. *Let $f, g, h : \mathbb{R}^n \to [0, \infty)$ be three integrable functions, and $s, \lambda, \mu > 0$ be real numbers. Assume that for any $x, y \in \mathbb{R}^n$,*

$$h(\lambda x + \mu y) \geq \left(\lambda f(x)^{\frac{1}{s}} + \mu g(y)^{\frac{1}{s}} \right)^s. \tag{18}$$

Then,

$$\left(\int h \right)^{\frac{1}{n+s}} \geq \lambda \left(\int f \right)^{\frac{1}{n+s}} + \mu \left(\int g \right)^{\frac{1}{n+s}}.$$

Proof. Assume first that s is an integer. In this case, the theorem follows from (17), as $h \geq [\lambda \times_s f] \oplus_s [\mu \times_s g]$ pointwise, and $\int h = \int^* h$. The case of an integer s suffices for all the applications we present below. Next, assume that $s = p/q$ is a rational number, and $p, q > 0$ are integers. Note that by Hölder's inequality, for any $x_1, ..., x_q, y_1, ..., y_q \in \mathbb{R}^n$,

$$\begin{aligned} \lambda \prod_{i=1}^q f(x_i)^{\frac{1}{qs}} + \mu \prod_{i=1}^q g(y_i)^{\frac{1}{qs}} &\leq \left(\prod_{i=1}^q \left(\lambda f(x_i)^{\frac{1}{s}} + \mu g(y_i)^{\frac{1}{s}} \right) \right)^{\frac{1}{q}} \\ &\leq \prod_{i=1}^q h(\lambda x_i + \mu y_i)^{\frac{1}{qs}} \end{aligned} \tag{19}$$

where the second inequality follows from (18). Our derivation of (19) is inspired by [GrM]. For a function $r : \mathbb{R}^n \to [0,\infty)$ we define ad-hoc $\tilde{r} : \mathbb{R}^{nq} \to [0,\infty)$ by $\tilde{r}(x) = \tilde{r}(x_1,...,x_q) = \prod_{i=1}^q r(x_i)$ where $x = (x_1,...,x_q) \in (\mathbb{R}^n)^q$ are coordinates in $\mathbb{R}^{nq}$. Thus, (19) implies that for any $x,y \in \mathbb{R}^{nq}$,

$$\tilde{h}(\lambda x + \mu y) \geq \left(\lambda \tilde{f}(x)^{\frac{1}{qs}} + \mu \tilde{g}(y)^{\frac{1}{qs}}\right)^{qs}. \tag{20}$$

Note that $qs = p$ is an integer, which is the case we already dealt with. Hence

$$\begin{aligned}\left(\int h\right)^{\frac{1}{n+s}} &= \left(\int \tilde{h}\right)^{\frac{1}{q(n+s)}} \geq \lambda\left(\int \tilde{f}\right)^{\frac{1}{q(n+s)}} + \mu\left(\int \tilde{g}\right)^{\frac{1}{q(n+s)}} \\ &= \lambda\left(\int f\right)^{\frac{1}{n+s}} + \mu\left(\int g\right)^{\frac{1}{n+s}}.\end{aligned}$$

and the theorem is proven for the case of a rational $s > 0$. The case of a real $s > 0$ follows by a standard approximation argument. □

Theorem 2.1 was first proven, for the case $n = 1$, by Henstock and Macbeath [HeM]. Later, it was proven for all $n \geq 1$ by Dinghas [D], by Borell [Bor] and by Brascamp–Lieb [BrL] independently. The notation in [Bor, BrL] is different from ours, and it covers only the case where $\lambda + \mu = 1$ in Theorem 2.1 (yet the general case follows easily). However, the framework in [Bor, BrL] also covers the case where $s \leq -n$, which does not seem to fit well into our discussion.

When $\lambda + \mu = 1$, letting s tend to infinity in Theorem 2.1, we recover the Prékopa–Leindler inequality [Le, Pr1, Pr2] as follows:

Corollary 2.2. *Let $f,g,h : \mathbb{R}^n \to [0,\infty)$ be three integrable functions and $0 < \lambda < 1$. Assume that for any $x,y \in \mathbb{R}^n$,*

$$h\left(\lambda x + (1-\lambda)y\right) \geq f(x)^\lambda g(y)^{1-\lambda}.$$

Then,

$$\int h \geq \left(\int f\right)^\lambda \left(\int g\right)^{1-\lambda}. \tag{21}$$

Proof. The argument is standard. Fix $M > 1$. The basic observation is that,

$$\left(\lambda x^{\frac{1}{s}} + (1-\lambda) y^{\frac{1}{s}}\right)^s \stackrel{s\to\infty}{\longrightarrow} x^\lambda y^{1-\lambda} \tag{22}$$

uniformly for $(x,y) \in \left(\frac{1}{M}, M\right) \times \left(\frac{1}{M}, M\right)$. Therefore for any $\varepsilon > 0$ there is $s_0(\varepsilon, M) > 0$, such that whenever $s > s_0(\varepsilon, M)$ and $\frac{1}{M} < f(x), g(y) < M$,

$$h\big(\lambda x + (1-\lambda)y\big) + \varepsilon \geq \left(\lambda f(x)^{\frac{1}{s}} + (1-\lambda) g(y)^{\frac{1}{s}}\right)^s.$$

Denote $K_f^M = \{x \in \mathbb{R}^n; \frac{1}{M} < f(x) < M\}$ and $K_g^M = \{x \in \mathbb{R}^n; \frac{1}{M} < g(x) < M\}$. Theorem 2.1 implies that for $\varepsilon > 0, s > s_0(\varepsilon, M)$,

$$\int_{\lambda K_f^M + (1-\lambda)K_g^M} \left[h(x) + \varepsilon\right] dx \geq \left[\lambda \Big(\int_{K_f^M} f \Big)^{\frac{1}{n+s}} + (1-\lambda) \Big(\int_{K_g^M} g \Big)^{\frac{1}{n+s}} \right]^{n+s}$$
$$\geq \Big(\int_{K_f^M} f \Big)^{\lambda} \Big(\int_{K_g^M} g \Big)^{1-\lambda}.$$

Since f, g are integrable, the sets $K_f^M, K_g^M \subset \mathbb{R}^n$ are bounded, and so is $\lambda K_f^M + (1-\lambda)K_g^M$. Letting first ε tend to zero, and then M tend to infinity, we conclude (21). $\square$

Note that in the proof of Corollary 2.2, we could confine s to be an integer, and use the simpler inequality (17) rather than Theorem 2.1. The proof of Corollary 2.2 is a prototype for the results we will obtain in the next section. The idea is to consider a geometric inequality in dimension $n + s$, to use the marginal of both sides of the inequality, and then let the extra dimension s tend to infinity. Thus our inequalities are traces of higher dimensional geometric inequalities, when the dimension tends to infinity.

3 Minkowski's Inequality

Suppose $K \subset \mathbb{R}^n$ is a convex set with the origin in its interior. For $x \in \mathbb{R}^n$ we define

$$\|x\|_K = \inf \left\{ \lambda > 0; \frac{x}{\lambda} \in K \right\}.$$

Then $\|\cdot\|_K$ is the (perhaps non-symmetric) norm whose unit ball is K. The dual norm, which again may be non-symmetric, is $\|x\|_* = \sup_{y \in K} \langle x, y \rangle$. In this section we will prove the following theorem:

Theorem 3.1. *Let $K \subset \mathbb{R}^n$ be a convex set with the origin in its interior. Let $\|\cdot\|$ be the norm that K is its unit ball (it may be a non-symmetric norm). Let $1 \leq p < \infty$, and let $F : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with $\int |F|^p, \int |\nabla F|^p < \infty$. Then,*

$$\int_{\mathbb{R}^n} F^p(x) \log \frac{cF^p(x)}{\int F^p(y) dy} dx \leq \int_{\mathbb{R}^n} \|\nabla F(x)\|^p dx \tag{23}$$

where $c = \mathrm{Vol}_n(K^\circ) e^n (\frac{q}{p})^{\frac{n}{q}} \Gamma(\frac{n}{q} + 1)$, and $q \geq 1$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$ (for $p = 1$ the value of c is $\mathrm{Vol}_n(K^\circ) e^n$, as interpreted by continuity). If $p > 1$, then equality in (23) holds for $F(x) = \alpha e^{-\|x\|_*^q / q}$, where $\|\cdot\|_*$ is the dual norm and $\alpha > 0$ is an arbitrary real number. The constant c is also optimal in the case $p = 1$.

Theorem 3.1 is equivalent, by a quick scaling argument produced below, to a family of inequalities which were explicitly stated and proven by Gentil [G] and independently by Agueh, Ghoussoub and Kang [AGK] (see also Remark (2) on Page 320 in [C-ENV]). The proof in [AGK] uses the mass-transportation method developed by Cordero–Erausquin, Nazareth and Villani [C-ENV] for the study of Sobolev and Gagliardo–Nirenberg inequalities. The proof in [G] relies on the Prékopa–Leindler inequality, and is related to the proof of the gaussian logarithmic Sobolev inequality by Bobkov and Ledoux [BoL]. Our approach is closer to that of Gentil, as we use Brunn–Minkowski, and our main contribution here is the clear geometric framework.

The case $p = 2$ and $\|\cdot\|$ being the Euclidean norm in (23) is particularly interesting; In this case (23) is simply equivalent to Stam's inequality from information theory [St]. Setting $F(x) = G(\sqrt{2}x)$ in (23) we may rewrite inequality (23) for $p = 2, \|\cdot\| = |\cdot|$ as follows:

$$\int_{\mathbb{R}^n} G^2(x) \log \frac{(e\sqrt{2\pi})^n G^2(x)}{\int G^2(y)dy} dx \leq 2 \int_{\mathbb{R}^n} |\nabla G(x)|^2 dx, \tag{24}$$

for any function G such that the right-hand side is finite. Furthermore, substituting $G(x) = \frac{e^{-\frac{|x|^2}{4}}}{(2\pi)^{\frac{n}{4}}} f(x)$ in (24), we obtain after integration by parts that

$$\int f^2(x) \log \frac{f^2(x)}{\int f^2(y) d\gamma_n(y)} d\gamma_n(x) \leq 2 \int |\nabla f(x)|^2 d\gamma_n(x) \tag{25}$$

where $\frac{d\gamma_n}{dx} = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$ is the density of the standard gaussian measure on $\mathbb{R}^n$. Inequality (25) is the logarithmic Sobolev inequality for the gaussian measure, first explicitly stated by Gross [Gro]. Inequality (25) is fundamental in the study of concentration inequalities in Gauss space, see [L]. We learned the fact that (25) and Stam's inequality are easily equivalent from [Be1, Be2]. In [Be2] it is also shown how (25) directly implies Nash's inequality.

For two sets $K, T \subset \mathbb{R}^m$ we denote the "T-surface area of K" by

$$\tilde{S}(K; T) = \frac{1}{m} \lim_{\varepsilon \to 0^+} \frac{\mathrm{Vol}_m(K + \varepsilon T) - \mathrm{Vol}_m(K)}{\varepsilon}$$

if the limit exists. The Brunn–Minkowski inequality implies that

$$\begin{aligned} \mathrm{Vol}_m(K + \varepsilon T) &\geq \left(\mathrm{Vol}_m(K)^{\frac{1}{m}} + \varepsilon \mathrm{Vol}_m(T)^{\frac{1}{m}} \right)^m \\ &\geq \mathrm{Vol}_m(K) + m\varepsilon \mathrm{Vol}_m(K)^{\frac{m-1}{m}} \mathrm{Vol}_m(T)^{\frac{1}{m}}. \end{aligned}$$

Consequently, whenever $\tilde{S}(K; T)$ exists,

$$\tilde{S}(K; T) \geq \mathrm{Vol}_m(K)^{\frac{m-1}{m}} \mathrm{Vol}_m(T)^{\frac{1}{m}}. \tag{26}$$

Inequality (26) is known as the Minkowski inequality (see e.g. [S, Theorem 3.2.1]). Note that K, T might be non convex in (26). Following our interest in marginals of Minkowski's inequality (26), we define, for any functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\tilde{\mathcal{S}}_s(f; g) = \frac{1}{n+s} \lim_{\varepsilon \to 0^+} \frac{\int [f \oplus_s (\varepsilon \times_s g)] - \int f}{\varepsilon} \tag{27}$$

whenever the integrals are defined and the limit exists. We interpret the Minkowski inequality (26) as follows:

Proposition 3.2. *Fix $s > 0$. Let $f, g : \mathbb{R}^n \to [0, \infty)$ be integrable functions such that $\tilde{\mathcal{S}}_s(f; g)$ exists. Then,*

$$\tilde{\mathcal{S}}_s(f; g) \geq \Big(\int f \Big)^{1 - \frac{1}{n+s}} \Big(\int g \Big)^{\frac{1}{n+s}}. \tag{28}$$

If $f = \lambda \times_s g$ and g is s-concave, then equality holds in (28).

Proof. By Theorem 2.1, whenever the functions are integrable,

$$\begin{aligned} \int \big[f \oplus_s (\varepsilon \times_s g) \big] &\geq \bigg(\Big(\int f \Big)^{\frac{1}{n+s}} + \varepsilon \Big(\int g \Big)^{\frac{1}{n+s}} \bigg)^{n+s} \\ &\geq \Big(\int f \Big) + \varepsilon (n+s) \Big(\int f \Big)^{1 - \frac{1}{n+s}} \Big(\int g \Big)^{\frac{1}{n+s}}. \end{aligned}$$

We assume that $\tilde{\mathcal{S}}(f; g)$ exists, hence the definition (27) implies the desired inequality. It is easy to verify that equality holds when $f = \lambda \times_s g$ is s-concave. $\square$

Recall from Section 1 that for a 1-concave function $f : \mathbb{R}^n \to [0, \infty)$, its Legendre transform is

$$\mathcal{L}' f(x) = \sup_{y \in \mathrm{Supp}(f)} \big[-\langle x, y \rangle + f(y) \big]. \tag{29}$$

The function $\mathcal{L}' f : \mathbb{R}^n \to \mathbb{R}$ is always convex. Additionally, for any numbers $\lambda, \mu > 0$ and functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\mathcal{L}' \Big\{ \big[(\lambda \times_s f) \oplus_s (\mu \times_s g) \big]^{\frac{1}{s}} \Big\} = \lambda \mathcal{L}' \big(f^{\frac{1}{s}} \big) + \mu \mathcal{L}' \big(g^{\frac{1}{s}} \big), \tag{30}$$

as the reader may easily verify. The inverse transform is

$$\mathcal{L}'^{-1} f(x) = \inf_{y \in \mathbb{R}^n} \big[\langle x, y \rangle + f(y) \big].$$

If f is 1-concave, then $\mathcal{L}'^{-1} \mathcal{L}' f = f$ on $\mathrm{Supp}(f)$. In this case, if $x \notin \mathrm{Supp}(f)$ then $\mathcal{L}'^{-1} \mathcal{L}' f(x) = -\infty$. Moreover, note that when f is concave, and is also differentiable and strictly concave in some neighborhood of a point x, then

$$y = \nabla f(x) \quad \Leftrightarrow \quad x = -\nabla \mathcal{L}' f(y).$$

Lemma 3.3. *Let $s > 0$ be an integer, and let $f, g : \mathbb{R}^n \to [0, \infty)$. Assume that f is continuous and that g is compactly-supported. Assume further that f is continuously differentiable in the interior of* $\mathrm{Supp}(f)$*. Then, for $x \in \mathbb{R}^n$ with $f(x) > 0$,*

$$\frac{d}{d\varepsilon}\big[f \oplus_s (\varepsilon \times_s g)\big](x)\Big|_{\varepsilon=0} = s f^{\frac{s-1}{s}}(x)\mathcal{L}'\big[g^{\frac{1}{s}}\big]\left(\nabla f^{\frac{1}{s}}(x)\right).$$

Moreover,

$$\frac{[f \oplus_s (\varepsilon \times_s g)](x) - f(x)}{\varepsilon} \overset{\varepsilon \to 0}{\longrightarrow} s f^{\frac{s-1}{s}}(x)\mathcal{L}'\left[g^{\frac{1}{s}}\right]\left(\nabla f^{\frac{1}{s}}(x)\right)$$

locally uniformly in x in the interior of $\mathrm{Supp}(f)$*.*

Proof. Begin with the definitions (13) and (14). For sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\big[f \oplus_s (\varepsilon \times_s g)\big](x) &= \sup_{\substack{y \in \mathrm{Supp}(f), z \in \mathrm{Supp}(g) \\ x = y + \varepsilon z}} \left(f^{\frac{1}{s}}(y) + \varepsilon g^{\frac{1}{s}}(z)\right)^s \\ &= \sup_{z \in \mathrm{Supp}(g)} \left(f^{\frac{1}{s}}(x - \varepsilon z) + \varepsilon g^{\frac{1}{s}}(z)\right)^s\end{aligned}$$

(all we need is that $x - \varepsilon z \in \mathrm{Supp}(f)$ for all $z \in \mathrm{Supp}(g)$; Recall that $Supp(g)$ is a bounded set). Since f is smooth and $f(x) > 0$, then $f^{\frac{1}{s}}$ is also continuously differentiable in a neighborhood of x, and

$$f^{\frac{1}{s}}(x - \varepsilon z) = f^{\frac{1}{s}}(x) - \varepsilon\langle \nabla f^{\frac{1}{s}}(x), z\rangle + |\varepsilon z|\alpha_x(\varepsilon z),$$

where $\alpha_x(y) \to 0$ as $y \to 0$, locally uniformly in x. Therefore,

$$\big[f \oplus_s (\varepsilon \times_s g)\big]^{\frac{1}{s}}(x) = f^{\frac{1}{s}}(x) + \varepsilon \sup_{z \in \mathrm{Supp}(g)} \left[-\langle \nabla f^{\frac{1}{s}}(x), z\rangle + g^{\frac{1}{s}}(z) + |z|\alpha_x(\varepsilon z)\right].$$

Denote $\alpha'_x(\varepsilon) = \sup_{z \in \mathrm{Supp}(g)} |z||\alpha_x(\varepsilon z)|$. Since $Supp(g)$ is compact, then $\alpha'_x(\varepsilon) \to 0$ as $\varepsilon \to 0$ locally uniformly in x, and

$$\left|[f \oplus_s (\varepsilon \times_s g)]^{\frac{1}{s}}(x) - f^{\frac{1}{s}}(x) - \varepsilon \sup_{z \in \mathrm{Supp}(g)} \left[-\langle \nabla f^{\frac{1}{s}}(x), z\rangle + g^{\frac{1}{s}}(z)\right]\right| \leq \varepsilon \alpha'_x(\varepsilon).$$

By (29) we conclude that

$$\frac{d}{d\varepsilon}[f \oplus_s (\varepsilon \times_s g)]^{\frac{1}{s}}(x)\Big|_{\varepsilon=0} = \mathcal{L}'\left[g^{\frac{1}{s}}\right]\left(\nabla f^{\frac{1}{s}}(x)\right)$$

and that the ε-derivative converges locally uniformly in x. This in turn implies that

$$\frac{d}{d\varepsilon}[f \oplus_s (\varepsilon \times_s g)](x)\Big|_{\varepsilon=0} = s f^{\frac{s-1}{s}}(x)\mathcal{L}'\left[g^{\frac{1}{s}}\right]\left(\nabla f^{\frac{1}{s}}(x)\right)$$

where the derivative converges locally uniformly in x. $\square$

Lemma 3.4. *Let $s > 1$, and let $f, g : \mathbb{R}^n \to [0, \infty)$ be compactly-supported bounded functions. Assume that the function f is of the form $f(x) = (A - G(x))_+^p$ for some $A, p > 0$ and for a continuous function $G : \mathbb{R}^n \to \mathbb{R}$, continuously differentiable in a neighborhood of $\{x; G(x) \leq A\}$. Assume that ∇G does not vanish on $\{x; G(x) = A\}$. Then,*

$$\tilde{\mathcal{S}}_s(f; g) = \frac{s}{n+s} \int_{\mathrm{Supp}(f)} f^{\frac{s-1}{s}}(x) \mathcal{L}' \left[g^{\frac{1}{s}} \right] \left(\nabla f^{\frac{1}{s}}(x) \right) dx < \infty. \tag{31}$$

Proof. Our task is basically to justify differentiation under the integral sign (see Lemma 5.2 for a less technical argument of the same spirit). For $\varepsilon > 0$, denote $F(\varepsilon, x) = \frac{[f \oplus_s (\varepsilon \times_s g)](x) - f(x)}{\varepsilon}$. According to (27),

$$\tilde{S}(f, g) = \frac{1}{n+s} \lim_{\varepsilon \to 0^+} \int F(\varepsilon, x) dx. \tag{32}$$

Let K be a compact set contained in the interior of $\mathrm{Supp}(f)$. By Lemma 3.3,

$$F(\varepsilon, x) \quad \overset{\varepsilon \to 0}{\longrightarrow} \quad s f^{\frac{s-1}{s}}(x) \mathcal{L}' \left[g^{\frac{1}{s}} \right] \left(\nabla f^{\frac{1}{s}}(x) \right)$$

uniformly on K. We conclude that

$$\int_K F(\varepsilon, x) dx \quad \overset{\varepsilon \to 0}{\longrightarrow} \quad s \int_K f^{\frac{s-1}{s}}(x) \mathcal{L}' \left[g^{\frac{1}{s}} \right] \left(\nabla f^{\frac{1}{s}}(x) \right) dx \tag{33}$$

for any compact set K contained in the interior of $\mathrm{Supp}(f)$. For $\delta > 0$, let K^δ be a compact set, contained in the interior of $\mathrm{Supp}(f)$, such that $\mathrm{Supp}(f) \setminus K^\delta$ is contained in a δ-neighborhood of $\partial \mathrm{Supp}(f)$. We will show that

$$\lim_{\delta \to 0^+} \limsup_{\varepsilon \to 0^+} \left| \int_{\mathbb{R}^n \setminus K^\delta} F(\varepsilon, x) dx \right| = 0. \tag{34}$$

It is straightforward to obtain (31) from (32), (33) and (34). Hence we focus our attention on proving (34). Denote $R = \max\{|x|; x \in \mathrm{Supp}(g)\}, m = \sup g^{1/s}$. Then for any $0 < \varepsilon < \frac{\delta}{R}$,

$$\int_{\mathbb{R}^n \setminus K^\delta} \left[f \oplus_s (\varepsilon \times_s g) \right](x) dx \leq \int_{(\partial \mathrm{Supp}(f))_\delta} \left(\sup_{|z| \leq R} \left[f(x - \varepsilon z)^{\frac{1}{s}} + \varepsilon m \right] \right)^s dx$$

where $T_\delta = \{x \in \mathbb{R}^n; \exists y \in T, |y - x| < \delta\}$ for any $T \subset \mathbb{R}^n$. Recall that $f(x) = (A - G(x))_+^p$ and denote $G_\varepsilon(x) = \inf_{|z-x|<\varepsilon} G(z)$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n \setminus K^\delta} F(\varepsilon, x) &\leq \int_{(\partial \mathrm{Supp}(f))_\delta} \frac{\left[\left(A - G_{\varepsilon R}(x) \right)_+^{\frac{p}{s}} + \varepsilon m \right]^s - \left(A - G(x) \right)_+^p}{\varepsilon} dx \\ &\leq \int_{(\partial \mathrm{Supp}(f))_\delta} C + \frac{\left(A - G_{\varepsilon R}(x) \right)_+^p - \left(A - G(x) \right)_+^p}{\varepsilon} dx \end{aligned}$$

for a small enough $\delta, \varepsilon > 0$, where in this proof we denote by c, C, c' etc. positive numbers independent of ε and δ. Therefore,

$$\Big| \int_{\mathbb{R}^n \setminus K^\delta} F(\varepsilon, x) \Big| \leq C \mathrm{Vol}_n((\partial \mathrm{Supp}(f))_\delta)$$

$$+ \int_{\mathrm{Supp}(f) \cap (\partial \mathrm{Supp}(f)_\delta} \hat{C} \left[(A - G(x))^{p-1} + 1 \right] \frac{G(x) - G_{\varepsilon R}(x)}{\varepsilon} dx \quad (35)$$

$$+ \int_{(\partial \mathrm{Supp}(f))_\delta \setminus \mathrm{Supp}(f)} \frac{(A - G_{\varepsilon R}(x))_+^p}{\varepsilon} dx. \quad (36)$$

Clearly $\mathrm{Vol}_n((\partial \mathrm{Supp}(f))_\delta) \to 0$ as $\delta \to 0$. Next, we will bound (35). Since G is continuously differentiable, we have that $\frac{G(x) - G_{R\varepsilon}(x)}{\varepsilon} < C$ on $\mathrm{Supp}(f)_\delta$. As the gradient of G does not vanish on the compact set $\partial \mathrm{Supp}(f)$, and since the vector $\nabla g(x)$ is normal to $\partial \mathrm{Supp}(f)$ for $x \in \partial \mathrm{Supp}(f)$, we conclude that for $x \in \mathrm{Supp}(f)$,

$$G(x) < A - c \cdot d\big(x, \partial \mathrm{Supp}(f)\big)$$

whenever $d(x, \partial \mathrm{Supp}(f)) < \tilde{c}$, where $d(x, A)$ stands for the distance between x and A. Therefore, (35) is smaller than

$$\int_{\mathrm{Supp}(f) \cap (\partial \mathrm{Supp}(f))_\delta} \tilde{C} \left[d\big(x, \partial \mathrm{Supp}(f)\big)^{p-1} + 1 \right] dx.$$

The latter integral actually converges even when we replace the domain of integration with the entire $\mathrm{Supp}(f)$, because $p > 0$. Hence (35) tends to zero as $\delta \to 0$, regardless of ε. All that remains is to bound (36). The integrand of (36) is non-zero only on $(\partial \mathrm{Supp}(f))_{\varepsilon R} \setminus \mathrm{Supp}(f)$. The volume of this set is bounded by $\tilde{C}\varepsilon$, and thus (36) is smaller than

$$\tilde{C} \sup_{x \in (\partial \mathrm{Supp}(f))_{\varepsilon R} \setminus \mathrm{Supp}(f)} \big(A - G_{\varepsilon R}(x)\big)_+^p \stackrel{\varepsilon \to 0^+}{\longrightarrow} 0$$

independently of δ. This establishes (34) and the lemma is proven. □

Next, we will prove Theorem 3.1. Aside from some technicalities, Theorem 3.1 follows simply by letting s tend to ∞ in Minkowski's inequality, in the form of Proposition 3.2.

Proof of Theorem 3.1. First, assume that $p > 1$. Let $s > 1$ and denote

$$g^{\frac{1}{s}}(x) = (1 - \|x\|_*^q)_+^{\frac{1}{q}}.$$

Then g is concave and compactly-supported. Hölder's inequality implies that

$$\big[\mathcal{L}' g^{\frac{1}{s}}\big](x) = (1 + \|x\|^p)^{\frac{1}{p}}.$$

Next, let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $h(x) \to \infty$ when $|x| \to \infty$. Assume that h is a continuously differentiable function whose gradient is non-zero for $x \neq 0$. Assume also that

$$\int_{\mathbb{R}^n} e^{-\frac{h(x)}{q}} \left(h^{10}(x) + |\nabla h(x)|^p\right) dx < \infty. \tag{37}$$

Introduce

$$f^{\frac{1}{s}}(x) = \left(s - h(x)\right)_+^{\frac{1}{q}}.$$

Then f is compactly-supported, and by Lemma 3.4,

$$\tilde{\mathcal{S}}(f;g) = \frac{s}{n+s} \int_{\mathbb{R}^n} \left(s - h(x)\right)_+^{\frac{s-1}{q}} \left(1 + \frac{\|\nabla h(x)\|^p}{q^p (s - h(x))_+}\right)^{\frac{1}{p}} dx.$$

Set $t = \frac{1}{s}$. Proposition 3.2 along with some simple manipulations yields that for any $t > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(1 - th(x)\right)_+^{\frac{1}{q}\left(\frac{1}{t}-1\right)} \left(1 + \frac{t\|\nabla h(x)\|^p}{q^p \left(1 - th(x)\right)_+}\right)^{\frac{1}{p}} dx \\ &\geq (1+nt) \left(\int_{\mathbb{R}^n} (1 - th(x))_+^{\frac{1}{qt}}\, dx\right)^{1-\frac{t}{nt+1}} \left(\frac{1}{t^{\frac{n}{q}}} \int_{\mathbb{R}^n} (1 - \|x\|_*^q)_+^{\frac{1}{qt}}\, dx\right)^{\frac{t}{nt+1}}. \end{aligned} \tag{38}$$

Note that by Proposition 3.2, equality in (38) holds for $h(x) = \|x\|_*^q$. Denote by $A(t)$ and by $B(t)$ the left and right hand sides of inequality (38), respectively. Then $A(t), B(t) \to \int e^{-h(x)/q}$ as $t \to 0$, and hence we set $A(0) = B(0) = \int e^{-h(x)/q}$. Our integrability assumptions on h allow us to differentiate $A(t)$ under the integral sign (see, e.g. [AlB], Theorem 20.4). We obtain

$$A'(0) = \int e^{-\frac{h(x)}{q}} \left(-\frac{h^2(x)}{2q} + \frac{h(x)}{q} + \frac{\|\nabla h(x)\|^p}{pq^p}\right) dx. \tag{39}$$

Regarding differentiation of the right hand side, recall that K° is the unit ball of $\|\cdot\|_*$. Note that

$$\begin{aligned} \frac{1}{t^{\frac{n}{q}}} \int_{\mathbb{R}^n} (1 - \|x\|_*^q)_+^{\frac{1}{qt}}\, dx &= \mathrm{Vol}(K^\circ) \frac{1}{t^{\frac{n}{q}+1}} \int_0^1 (1 - s^q)^{\frac{1}{qt}-1} s^{q-1} s^n ds \\ &= \frac{\mathrm{Vol}(K^\circ)}{t^{\frac{n}{q}+1} q} \int_0^1 s^{\frac{n}{q}} (1-s)^{\frac{1}{qt}-1} ds \\ &= \frac{\mathrm{Vol}(K^\circ) q^{\frac{n}{q}} \Gamma\left(\frac{n}{q}+1\right)}{tn+1} \cdot \frac{\left(\frac{1}{qt}\right)^{\frac{n}{q}} \Gamma\left(\frac{1}{qt}\right)}{\Gamma\left(\frac{1}{qt} + \frac{n}{q}\right)} \end{aligned}$$

which tends to $\mathrm{Vol}(K^\circ) c'_{n,q} = \mathrm{Vol}(K^\circ) q^{\frac{n}{q}} \Gamma(n/q+1)$ as $t \to 0$. Next, we will compute the derivative of $B(t)$ (again, using differentiation under the integral sign, justified by [AlB], Theorem 20.4). We derive

$$B'(0) = \int_{\mathbb{R}^n} e^{-\frac{h(x)}{q}} \left[-\frac{h^2(x)}{2q} + n - \log \int e^{-\frac{h(y)}{q}} dy + \log\left(\mathrm{Vol}(K^\circ) c'_{n,q}\right)\right] dx.$$

Since $A(0) = B(0)$ and $A(t) \geq B(t)$ for all t, we conclude that $A'(0) \geq B'(0)$. Thus,

$$\int_{\mathbb{R}^n} e^{-\frac{h(x)}{q}} \left(\frac{h(x)}{q} + \frac{\|\nabla h(x)\|^p}{pq^p} \right) dx \geq \int_{\mathbb{R}^n} e^{-\frac{h(x)}{q}} \log \frac{\tilde{c}_{n,q} \mathrm{Vol}(K^\circ)}{\int e^{-\frac{h(y)}{q}} dy} dx$$

where $\tilde{c}_{n,q} = e^n c'_{n,q} = (eq^{1/q})^n \Gamma(n/q+1)$, with equality for $h(x) = \|x\|_*^q$. Now, introduce $\tilde{F}(x) = e^{-\frac{h(x)}{pq}}$. Then we get

$$p^{p-1} \int \|\nabla \tilde{F}(x)\|^p dx \geq \int \tilde{F}^p(x) \log \frac{\tilde{c}_{n,q} \mathrm{Vol}(K^\circ) \tilde{F}^p(x)}{\int \tilde{F}^p(y) dy} dx, \tag{40}$$

and equality holds for $\tilde{F}(x) = e^{-\frac{\|x\|_*^q}{pq}}$. Our final manipulation is setting $\tilde{F}(x) = F(x/p^{1/q})$. We obtain, after a simple change of variables,

$$\int \|\nabla F(x)\|^p dx \geq \int F^p(x) \log \frac{c_{n,q} \mathrm{Vol}(K^\circ) F^p(x)}{\int F^p(y) dy} dx$$

where $c_{n,q} = p^{-n/q} \tilde{c}_{n,q} = (e(q/p)^{1/q})^n \Gamma(1+n/q)$. Equality holds for $F(x) = e^{-\|x\|_*^q/q}$. Note that if F is smooth, decays fast enough at infinity, and the gradient of F does not vanish for $x \neq 0$, then the integrability assumption (37) on $h(x) = -c_1 \log F(c_2 x)$ automatically holds. This implies inequality (23) for a class of functions F that is dense in $W^{1,p}(\mathbb{R}^n)$. A standard approximation argument entails the conclusion of the theorem for any function F with $\int |F|^p, \int |\nabla F|^p < \infty$. This ends the case $p > 1$. The case $p = 1$ of inequality (23) is obtained by continuity, with the sharp constant $e^n = \lim_{q \to \infty} (e(\frac{q}{p})^{1/q})^n \Gamma(1 + \frac{n}{q})$. This concludes the proof. □

Next we present the equivalence of Theorem 3.1 and the inequalities proven by Gentil [G] and by Agueh, Ghoussoub and Kang [AGK]. Note that our formulation is indeed equivalent to that in [G, AGK], since a convex function that is homogenous of degree p, is necessarily $\|x\|^p$ for some norm $\|\cdot\|$, which is not necessarily a symmetric norm.

Corollary 3.5. *Let $K \subset \mathbb{R}^n$ be a convex set with the origin in its interior. Let $\|\cdot\|$ be the (possibly non-symmetric) norm for which K is its unit ball. Let $1 \leq p < \infty$, and let $F : \mathbb{R}^n \to [0, \infty)$ be a smooth function with $\int F^p(x) dx = 1$. Then,*

$$\int F^p(x) \log F^p(x) dx + \log \left[c_{n,p} \mathrm{Vol}(K^\circ) \right] \leq n \log \left(\int \|\nabla F(x)\|^p dx \right)^{\frac{1}{p}}$$

where $c_{n,p} = [(eq)^{\frac{n}{q}} n^{\frac{n}{p}} \Gamma(\frac{n}{q}+1)]/p^n$ and $\frac{1}{p} + \frac{1}{q} = 1$ (the constant $c_{n,1} = n^n$ is interpreted by continuity). If $p > 1$, equality holds for $F(x) = \alpha e^{-\beta \|x\|_*^q}$, where $\|\cdot\|_*$ is the dual norm, and $\alpha, \beta > 0$ are such that $\int F^p(x) dx = 1$. The constant is also optimal for $p = 1$.

Proof. The argument is standard. For any $t > 0$, let $G_t(x) = F(tx)$. Applying Theorem 3.1 for the function G_t, we obtain

$$t^p \int \|\nabla F(x)\|^p dx \geq \int F^p(x) \log \frac{cF^p(x)}{\int F^p(y)dy} dx + (n \log t) \int F^p(x) dx.$$

Optimizing over t, we set $t = (\frac{n \int F^p(x)dx}{p \int \|\nabla F(x)\|^p dx})^{\frac{1}{p}}$. Thus,

$$\begin{aligned} &\frac{n}{p} \int F^p(x)dx \\ &\quad \geq \int F^p(x) \log \frac{cF^p(x)}{\int F^p(y)dy} dx + \frac{n}{p} \int F^p(x)dx \cdot \log \frac{n \int F^p(x)dx}{p \int \|\nabla F(x)\|^p dx}. \end{aligned}$$

Recall that $\int F^p(x) = 1$. We conclude that

$$\frac{n}{p} \log \int \|\nabla F(x)\|^p dx \geq \int F^p(x) \log F^p(x) dx + \log c + \frac{n}{p}\Big(\log \frac{n}{p} - 1\Big). \quad \square$$

4 Santaló's Inequality

Let $K \subset \mathbb{R}^n$ be a compact set. Santaló's inequality (see, e.g. [MeP]) states that for some $x_0 \in \mathbb{R}^n$, and $\tilde{K} = K - x_0$ we have

$$\mathrm{Vol}_n(\tilde{K})\mathrm{Vol}_n(\tilde{K}^\circ) \leq \mathrm{Vol}_n(D)^2 \tag{41}$$

where, as before, $\tilde{K}^\circ = \{x \in \mathbb{R}^n; \forall y \in \tilde{K},\ \langle x, y \rangle \leq 1\}$ is the polar body and $D \subset \mathbb{R}^n$ is the Euclidean unit ball. Inequality (1), which is a functional version of Santaló's inequality, was proven in [ArtKM] by taking marginals of both sides in (41). See also [B1, FM]. Here, for simplicity, we focus attention on the case where the functions involved are even, as in [B1]. Recall that the standard Legendre transform of a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is defined by (e.g. [Ar])

$$\mathcal{L}\varphi(x) = \sup_{y \in \mathbb{R}^n} \big[\langle x, y \rangle - \varphi(y)\big]. \tag{42}$$

For a convex continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ we have $\mathcal{L}\mathcal{L}\varphi = \varphi$. Note that the only function for which $\mathcal{L}\varphi = \varphi$ is $\varphi(x) = |x|^2/2$. In the case of even functions, inequality (1) reads as follows:

Proposition 4.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an even, measurable function such that $0 < \int e^{-\varphi} < \infty$. Then,*

$$\int_{\mathbb{R}^n} e^{-\varphi} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\varphi} dx \leq \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} dx \right)^2$$

with equality iff φ is a.e. a positive definite quadratic form.

The inequality in Proposition 4.1 is due to K. Ball [B1], and the equality case was settled in [ArtKM]. One advantage of switching from geometric inequalities to analytic ones, is the availability of a new arsenal of analytic techniques. This will be demonstrated in this section, where we apply the results of Brenier, McCann and Caffarelli to Proposition 4.1.

We begin with standard definitions. A measure μ on $\mathbb{R}^n$ is a logarithmically concave measure (log-concave for short) if for any compact sets $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$,

$$\mu\big(\lambda A + (1-\lambda)B\big) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}. \tag{43}$$

The Lebesgue measure on $\mathbb{R}^n$ is a log-concave measure, as follows from the Brunn–Minkowski inequality (15). Given a function $f : \mathbb{R}^n \to [0,\infty)$, we say that f is a log-concave function if

$$f\big(\lambda x + (1-\lambda)y\big) \geq f(x)^\lambda f(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n, 0 < \lambda < 1$. The notions of a log-concave function and a log-concave measure are closely related. Borell showed in [Bor] that if μ is a measure on $\mathbb{R}^n$ whose support is not contained in any affine hyperplane, then μ is a log-concave measure if and only if μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, and the density of μ is a log-concave function. In this section we will prove the following:

Theorem 4.2. *Let $f : \mathbb{R}^n \to (-\infty,\infty]$ be an even measurable function, let $\alpha > 0$, and assume that μ is an even log-concave measure on $\mathbb{R}^n$. Then,*

$$\int_{\mathbb{R}^n} e^{-\alpha f} d\mu \int_{\mathbb{R}^n} e^{-\alpha \mathcal{L} f} d\mu \leq \left(\int_{\mathbb{R}^n} e^{-\alpha\frac{|x|^2}{2}} d\mu\right)^2 \tag{44}$$

whenever at least one of the integrals on the left-hand side is both finite and non-zero.

We recently learned that Theorem 4.2 was also proven independently, using the same method as ours, by Barthe and Cordero–Erausquin [Ba2]. Given two Borel probability measures μ_1, μ_2 on $\mathbb{R}^n$ and a Borel map $T : \mathbb{R}^n \to \mathbb{R}^n$ we say that T transports μ_1 to μ_2 (or pushes forward μ_1 to μ_2) if for any Borel set $A \subset \mathbb{R}^n$,

$$\mu_2(A) = \mu_1\big(T^{-1}(A)\big).$$

Equivalently, for any compactly-supported, bounded, measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \varphi(x) d\mu_2(x) = \int_{\mathbb{R}^n} \varphi(Tx) d\mu_1(x).$$

Brenier's theorem [Bre], as refined by McCann [Mc1], is the following:

Theorem 4.3. *Let μ_1 and μ_2 be two probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to the standard Lebesgue measure. Then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ such that $T = \nabla F$ exists μ_1-almost everywhere, and T transports μ_1 to μ_2. Moreover, the map T, called "Brenier map", is uniquely determined μ_1-almost everywhere.*

A corollary of the uniqueness part in Theorem 4.3 is that if both μ_1 and μ_2 are invariant under a linear map $L \in GL_n(\mathbb{R})$, then T is also invariant under L. Recall that we denote by γ_n the standard gaussian probability measure on $\mathbb{R}^n$, i.e. $\frac{d\gamma_n}{dx} = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$. For the case where $\mu_1 = \gamma_n$ and $\frac{d\mu_2}{d\gamma_n}$ is a log-concave function, the following useful result was proven by Caffarelli [C]:

Theorem 4.4. *Let μ be a probability measure on $\mathbb{R}^n$ such that $\psi = \frac{d\mu}{d\gamma_n}$ exists, and is a log-concave function. Let T be the Brenier map that transports γ_n to μ. Then T is a non-expansive map, i.e. $|Tx - Ty| \le |x - y|$ for all $x, y \in \mathbb{R}^n$.*

The following simple lemma demonstrates a certain relation between Legendre transform and non-expansive maps.

Lemma 4.5. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a function. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a non-expansive map. Denote*

$$a(x) = f(Tx) + \frac{|x|^2 - |Tx|^2}{2}, \quad b(x) = \mathcal{L}f(Tx) + \frac{|x|^2 - |Tx|^2}{2}.$$

Then, for any $x \in \mathbb{R}^n$,

$$b(x) \ge \mathcal{L}a(x).$$

Proof. By (42), for any $x, y \in \mathbb{R}^n$,

$$f(Tx) + \mathcal{L}f(Ty) \ge \langle Tx, Ty \rangle.$$

Hence, for all $x, y \in \mathbb{R}^n$,

$$f(Tx) - \frac{|Tx|^2}{2} + \mathcal{L}f(Ty) - \frac{|Ty|^2}{2} \ge -\frac{|Tx - Ty|^2}{2} \ge -\frac{|x - y|^2}{2}$$

as T is a non-expansive map. We conclude that

$$a(x) + b(y) = f(Tx) + \frac{|x|^2 - |Tx|^2}{2} + \mathcal{L}f(Ty) + \frac{|y|^2 - |Ty|^2}{2} \ge \langle x, y \rangle.$$

The definition (42) implies that $b \ge \mathcal{L}a$ (and also that $a \ge \mathcal{L}b$). $\square$

Proof of Theorem 4.2. First consider the case $\alpha = 1$. We may clearly assume that the support of μ is n-dimensional (otherwise, we may pass to a subspace of a lower dimension). By Borell's theorem, $\psi := \frac{d\mu}{dx}$ exists and is a log-concave function. Let $d\nu(x) = \frac{1}{\kappa}\psi(x)e^{-|x|^2/2}dx$ where $\kappa = \int \psi(x)e^{-|x|^2/2}dx$. Then,

$$\int_{\mathbb{R}^n} e^{-f} d\mu \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu = \kappa^2 \int_{\mathbb{R}^n} e^{\frac{|x|^2}{2} - f(x)} d\nu(x) \int_{\mathbb{R}^n} e^{\frac{|x|^2}{2} - \mathcal{L}f(x)} d\nu(x).$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the Brenier map that transports the probability measure γ_n to the probability measure ν. Then,

$$\int_{\mathbb{R}^n} e^{-f} d\mu \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu = \kappa^2 \int_{\mathbb{R}^n} e^{\frac{|Tx|^2}{2} - f(Tx)} d\gamma_n(x) \int_{\mathbb{R}^n} e^{\frac{|Tx|^2}{2} - \mathcal{L}f(Tx)} d\gamma_n(x)$$
$$= \frac{\kappa^2}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\Big(\frac{|Tx|^2 - |x|^2}{2} - f(Tx)\Big) dx$$
$$\cdot \int_{\mathbb{R}^n} \exp\Big(\frac{|Tx|^2 - |x|^2}{2} - \mathcal{L}f(Tx)\Big) dx.$$

Denote $g(x) = f(Tx) + \frac{|x|^2 - |Tx|^2}{2}$ and $h(x) = \mathcal{L}f(Tx) + \frac{|x|^2 - |Tx|^2}{2}$. Note that from Theorem 4.4, we know that T is a non-expansive map. Lemma 4.5 implies that $h \geq \mathcal{L}g$. Furthermore, since ψ is even, by the uniqueness of the Brenier map (Theorem 4.3) we also know that T is an even map. Hence h and g are even functions. Assume that $0 < \int_{\mathbb{R}^n} e^{-f} d\mu = \frac{\kappa}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-g} < \infty$. Proposition 4.1 implies that

$$\frac{\kappa^2}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-g} \int_{\mathbb{R}^n} e^{-h} \leq \frac{\kappa^2}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-g} \int_{\mathbb{R}^n} e^{-\mathcal{L}g} \leq \kappa^2$$

and the theorem follows for $\alpha = 1$. If $0 < \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu = \frac{\kappa}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-h} < \infty$ we repeat the last argument with h in place of g (note that $g \geq \mathcal{L}h$). This ends the case $\alpha = 1$.

For the general case, let μ_α be the measure defined by $\mu_\alpha(A) = \mu(\alpha^{-\frac{1}{2}} A)$. Note that

$$\int \varphi(x) d\mu_\alpha(x) = \int \varphi(\sqrt{\alpha} x) d\mu(x) \tag{45}$$

for any test function φ. Let $g : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary even function, and set $f(x) = \alpha g(x/\sqrt{\alpha})$. It is readily verified that $\mathcal{L}f(x) = \alpha \mathcal{L}g(x/\sqrt{\alpha})$. The measure μ_α is log-concave and even. Since f is also an even function, we conclude, from the case treated above, that

$$\int_{\mathbb{R}^n} e^{-f} d\mu_\alpha \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu_\alpha \leq \Big(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} d\mu_\alpha\Big)^2$$

whenever the integrals converge. This translates, with the help of (45), into

$$\int_{\mathbb{R}^n} e^{-f(\sqrt{\alpha}x)} d\mu \int_{\mathbb{R}^n} e^{-\mathcal{L}f(\sqrt{\alpha}x)} d\mu \leq \Big(\int_{\mathbb{R}^n} e^{-\frac{\alpha|x|^2}{2}} d\mu\Big)^2.$$

According to the definition of f we get that

$$\int_{\mathbb{R}^n} e^{-\alpha g} d\mu \int_{\mathbb{R}^n} e^{-\alpha \mathcal{L}g} d\mu \leq \Big(\int_{\mathbb{R}^n} e^{-\frac{\alpha|x|^2}{2}} d\mu(x)\Big)^2$$

whenever $0 < \int e^{-\alpha g} d\mu < \infty$ or $0 < \int e^{-\alpha \mathcal{L}g} d\mu < \infty$. □

Remark 4.6. For $n = 1$, the equality case in Theorem 4.2 is easily characterized: If μ is not a multiple of the Lebesgue measure on $\mathbb{R}$, then equality holds if and only if $f(x) = |x|^2/2$. If μ is a multiple of the Lebesgue measure on $\mathbb{R}$, then equality holds if and only if $f(x) = cx^2$ for some $c > 0$.

Theorem 4.2 has some interesting consequences, two of which were formulated in Section 1.

Proof of Corollary 1.1. For a centrally-symmetric convex set $A \subset \mathbb{R}^n$, we denote by $\|\cdot\|_A$ the norm whose unit ball is A. Let $d\mu = e^{-\|x\|_T^2/2}dx$, and consider the function $f(x) = \|x\|_K^2/2$. Then

$$\mathcal{L}f(x) = \frac{\|x\|_{K^\circ}^2}{2}.$$

Note that for any centrally-symmetric convex set $A \subset \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} e^{-\frac{\|x\|_A^2}{2}} dx = 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right) \mathrm{Vol}_n(A)$$

(see e.g., [P], page 11). In particular,

$$\int_{\mathbb{R}^n} e^{-\frac{\|x\|_K^2}{2}} d\mu = \int_{\mathbb{R}^n} e^{-\frac{\|x\|_K^2+\|x\|_T^2}{2}} dx = 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right) \mathrm{Vol}_n(K \cap_2 T)$$

and similar identities hold for $K^\circ \cap_2 T$ and $D \cap_2 T$. By Theorem 4.2,

$$\int e^{-\frac{\|x\|_K^2}{2}} d\mu \int e^{-\frac{\|x\|_{K^\circ}^2}{2}} d\mu \le \left(\int e^{-\frac{|x|^2}{2}} d\mu\right)^2.$$

We conclude that

$$\mathrm{Vol}_n(K \cap_2 T)\mathrm{Vol}_n(K^\circ \cap_2 T) \le \mathrm{Vol}_n(D \cap_2 T)^2$$

and (2) is proven. □

Proof of Corollary 1.2. Introduce $d\nu = e^{-\psi}dx$ and note that ν is even and log-concave. Then, for an arbitrary centrally-symmetric convex set $K \subset \mathbb{R}^n$,

$$\begin{aligned}\int e^{-\alpha\frac{\|x\|_K^2}{2}} d\nu &= \int_0^\infty \alpha t e^{-\frac{\alpha t^2}{2}} \nu(tK)dt \\ &= \alpha \int_0^\infty t e^{-\frac{\alpha t^2}{2}} \int_{tK} e^{-\psi(x)} dx dt \\ &= \alpha \int_0^\infty \int_K t^{n+1} e^{-\frac{\alpha t^2}{2}} e^{-\psi(tx)} dx dt = \alpha\mu(K)\end{aligned}$$

(everything is positive, so we may interchange the order of integration). Therefore, the inequality

$$\int_{\mathbb{R}^n} e^{-\frac{\alpha\|x\|_K^2}{2}} d\nu \int_{\mathbb{R}^n} e^{-\frac{\alpha\|x\|_{K^\circ}^2}{2}} d\nu \le \left(\int_{\mathbb{R}^n} e^{-\frac{\alpha|x|^2}{2}} d\nu\right)^2$$

of Theorem 4.2 translates to

$$\alpha^2\mu(K)\mu(K^\circ) \le \alpha^2\mu(D)^2.$$

This concludes the proof. □

Remarks.

1. Assume that μ is an even, log-concave measure, whose density is $F(x)$. Assume further that $F(x) = F(x_1, ..., x_n)$ actually depends only on $x_1, ..., x_{\lfloor \varepsilon n \rfloor}$, for some $0 < \varepsilon < 1$. By using techniques similar to those in [ArtKM], it is possible to show that for any centrally-symmetric convex set $K \subset \mathbb{R}^n$,
$$\mu(K)\mu(K^\circ) \leq \big(1 + c(\varepsilon)\big)\mu(D)^2$$
for some function $c(\varepsilon)$ that tends to zero as $\varepsilon \to 0$. The important feature is that $c(\varepsilon)$ depends solely on ε (and not on the dimension n).
2. What is the class of measures μ that satisfy (44), for all even measurable functions f and $\alpha > 0$? This class contains all even, log-concave measures, according to Theorem 4.2. If F is a density of a measure satisfying (44) and $\beta > 0$, then also the measure whose density is the function
$$x \mapsto \int_0^\infty t^{n+1} e^{-\beta t^2} F(tx) dt \tag{46}$$
satisfies (44), for all even functions f and $\alpha > 0$. This follows by combining the one-dimensional Prékopa–Leindler inequality with the proof of Corollary 1.2, similarly to the argument in [B1] (see also [ArtKM, Theorem 2.1]). We omit the details. We conclude that the class of densities of measures μ that satisfy (44) is closed under the transform (46).

5 Mixed Volumes

5.1 The V Functional

As observed by Minkowski (see, e.g., [S]), for any compact, convex sets $K_1, ..., K_N \subset \mathbb{R}^n$, the function
$$(\lambda_1, ..., \lambda_N) \mapsto Vol_n\left(\sum_{i=1}^N \lambda_i K_i\right),$$
defined for $\lambda_1, ..., \lambda_N > 0$, is a homogeneous polynomial of degree $n+1$ in the variables $\lambda_1, ..., \lambda_N$. Minkowski concluded (see, e.g., the Appendix here) that there exists a unique symmetric multilinear n-form V defined on the space of compact, convex sets in $\mathbb{R}^n$ such that
$$\mathrm{Vol}(K) = V(K, ..., K)$$
for any compact, convex set $K \subset \mathbb{R}^n$. The symmetry and multilinearity mean that

1. For any compact, convex sets $A, B, K_2, ..., K_n \subset \mathbb{R}^n$ and $\lambda, \mu > 0$,
$$V(\lambda A + \mu B, K_2, ..., K_n) = \lambda V(A, K_2, ..., K_n) + \mu V(B, K_2, ...K_n).$$

2. For any compact, convex sets $K_1, ..., K_n \subset \mathbb{R}^n$ and a permutation $\sigma \in S_n$,

$$V(K_1, ...K_n) = V(K_{\sigma(1)}, ..., K_{\sigma(n)}).$$

We say that $V(K_1, ..., K_n)$ is the mixed volume of $K_1, ..., K_n$. The mixed volume $V(K_1, ..., K_n)$ depends continuously on the convex sets $K_1, ..., K_n$, with respect to the Hausdorff metric on the space of convex sets. Two fundamental properties of mixed volumes of convex bodies are:

1. $K_1 \subset T_1, ..., K_n \subset T_n$ imply that $0 \le V(K_1, ..., K_n) \le V(T_1, ..., T_n)$.
2. Alexandrov–Fenchel inequalities:

$$V(C, T, K_1, ..., K_{n-2})^2 \ge V(C, C, K_1, ..., K_{n-2}) V(T, T, K_1, ..., K_{n-2})$$

for any compact, convex sets $C, T, K_1, ..., K_{n-2} \subset \mathbb{R}^n$.

Functional analogs of mixed volumes of convex bodies will be considered here. We will restrict ourselves to 1-concave functions, as the formulae are simpler in this case. Part of our discussion generalizes directly to the s-concave case, with an integer s. For any 1-concave functions $f_1, ..., f_N : \mathbb{R}^n \to \mathbb{R}$, the function

$$(\lambda_1, ..., \lambda_N) \mapsto \int \left[(\lambda_1 \times_1 f_1) \oplus_1 ... \oplus_1 (\lambda_N \times_1 f_N) \right] = \mathrm{Vol}_{n+1}\left(\sum_{i=1}^N \lambda_i \mathcal{K}_{f_i} \right),$$

defined for $\lambda_1, ..., \lambda_N > 0$, is a homogeneous polynomial of degree n in the variables $\lambda_1, ..., \lambda_N$. This follows from Minkowski's theorem (recall that 1-concave functions have compact support, hence the integral is always finite). Therefore there exists a unique symmetric multilinear $(n+1)$-form V defined on the space of 1-concave functions on $\mathbb{R}^n$ that satisfies the following:

1. For any 1-concave functions $f_0, ..., f_n : \mathbb{R}^n \to [0, \infty)$, and any permutation $\sigma \in S_{n+1}$,
$$V(f_0, ..., f_n) = V(f_{\sigma(0)}, ..., f_{\sigma(n)}).$$
2. For any 1-concave functions $f, g, h_1, ..., h_n : \mathbb{R}^n \to [0, \infty)$ and $\lambda, \mu > 0$,
$$V((\lambda \times_1 f) \oplus_1 (\mu \times_1 g), h_1, ..., h_n) = \lambda V(f, h_1, ..., h_n) + \mu V(g, h_1, ..., h_n).$$
3. For any 1-concave function $f : \mathbb{R}^n \to [0, \infty)$,
$$V(f, ..., f) = \int_{\mathbb{R}^n} f(x) dx.$$
4. If $f_0 \le g_0, ..., f_n \le g_n$ are all 1-concave functions, then
$$0 \le V(f_0, ..., f_n) \le V(g_0, ..., g_n).$$

5. For any 1-concave functions $f, g, h_2, ..., h_n$,

$$V(f, g, h_2, ..., h_n)^2 \geq V(f, f, h_2, ..., h_n) V(g, g, h_2, ..., h_n).$$

The proof of these five properties is a direct application of the known properties of Minkowski's mixed volumes and our definitions (11), (12), (13) and (14). We will see below that the multilinear form V satisfies the conclusions of Theorem 1.3 (and agrees with the definition given in the formulation of Theorem 1.3).

Mixed volumes of convex bodies are continuous with respect to the Hausdorff metric. We thus conclude that $V(f_0, ..., f_n)$ is continuous with respect to uniform convergence in the functions $f_0, ..., f_n$. Indeed, if $f, f^1, f^2, ... : \mathbb{R}^n \to [0, \infty)$ are 1-concave functions such that $f^m \to f$ uniformly in $\mathbb{R}^n$, then $\mathcal{K}_{f^m} \to \mathcal{K}_f$ in the Hausdorff metric. Actually, arguing as in Theorem 10.8 from [Ro], it is not very difficult to see that if $f^m \to f$ pointwise in $\mathbb{R}^n$, then $\mathcal{K}_{f^m} \to \mathcal{K}_f$ in the Hausdorff metric. We thus conclude that V satisfies property 1 from Theorem 1.3. The next lemma is standard in convex analysis, and follows e.g. from Theorem 1.1 in [CoH1]. We omit the details.

Lemma 5.1. *Let $f, f_1, f_2, ... : \mathbb{R}^n \to [0, \infty)$ be continuous, 1-concave functions. Assume that $f_k \to f$ uniformly in $\mathbb{R}^n$ when $k \to \infty$. Then, for any continuous, non-negative function $\varphi : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} \varphi\big(\nabla f_k(x)\big) dx \stackrel{k \to \infty}{\longrightarrow} \int_{\mathbb{R}^n} \varphi\big(\nabla f(x)\big) dx$$

(since the functions are concave on their support, the gradient exists a.e. and so the integrals are well-defined).

The next lemma is a minor modification of Lemma 3.4 (for the case $s = 1$).

Lemma 5.2. *Let $f, g : \mathbb{R}^n \to [0, \infty)$ be continuous, 1-concave functions. Then,*

$$V(f, ..., f, g) = \frac{1}{n+1} \int_{\mathrm{Supp}(f)} \mathcal{L}' g\big(\nabla f(x)\big) dx. \tag{47}$$

Proof. Since $\mathcal{L}'g$ is a non-negative continuous function, both sides in (47) are continuous in f with respect to uniform convergence, according to Lemma 5.1. By approximation, we may assume that f equals, on its support, to the minimum of finitely many affine functionals; Indeed, the set of functions of this form is dense among continuous 1-concave functions, in the topology of uniform convergence. Thus, we may suppose that

$$\mathrm{Supp}(f) = \bigcup_{i=1}^{N} A_i, \quad i \neq j \ \Rightarrow \ A_i \cap A_j = \emptyset,$$

for some convex sets $A_1, ..., A_N$, and that for $x \in A_i$ we have $f(x) = \langle x, \theta_i \rangle + c_i$. Let $R = \max_{x \in \mathrm{Supp}(g)} |x|$. If $x \in A_i$ and $d(x, A_j) > R\varepsilon$ for all $j \neq i$, then

$$\begin{aligned}\big[f \oplus_1 (\varepsilon \times_1 g)\big](x) &= \sup_{\substack{y\in \mathrm{Supp}(f), z\in \mathrm{Supp}(g)\\ y+\varepsilon z = x}} \big[f(y) + \varepsilon g(z)\big] \\ &= \sup_{z\in \mathrm{Supp}(g)} \big[\langle x-\varepsilon z, \theta_i\rangle + c_i + \varepsilon g(z)\big]\end{aligned}$$

as $z \in \mathrm{Supp}(g)$ implies that $y = x - \varepsilon z \in A_i \subset \mathrm{Supp}(f)$. Hence,

$$\big[f \oplus_1 (\varepsilon \times_1 g)\big](x) = f(x) + \varepsilon \sup_{z\in \mathrm{Supp}(g)} \big[g(z) - \langle z, \theta_i\rangle\big] = f(x) + \varepsilon \mathcal{L}' g(\theta_i).$$

Denote $B_\varepsilon = \{x \in \mathbb{R}^n; \exists i = 1, ..., N; d(x, \partial A_i) < R\varepsilon\}$. Then $\mathrm{Vol}_n(B_\varepsilon) \le C\varepsilon$, for some $C > 0$ independent of ε, and

$$\int_{\mathrm{Supp}(f)\setminus B_\varepsilon} \big[f \oplus_1 (\varepsilon \times_1 g)\big](x)dx = \int_{\mathrm{Supp}(f)\setminus B_\varepsilon} f(x) + \varepsilon \mathcal{L}' g\big(\nabla f(x)\big)dx.$$

Let $\omega(\delta)$ be the modulus of continuity of f, and let $M = \sup g$. Then,

$$\begin{aligned}&\left|\int_{B_\varepsilon} \big[f \oplus_1 (\varepsilon \times_1 g)\big](x)dx - \int_{B_\varepsilon} f(x)dx\right| \\ &= \left|\int_{B_\varepsilon} \sup_{z\in \mathrm{Supp}(g)} \big[f(x-\varepsilon z) - f(x) + \varepsilon g(z)\big]dx\right| \\ &\le \mathrm{Vol}_n(B_\varepsilon)\big(\omega(R\varepsilon) + \varepsilon M\big).\end{aligned}$$

Note that $\mathrm{Supp}(f \oplus_1 (\varepsilon \times_1 g)) \subset \mathrm{Supp}(f) \cup B_\varepsilon$. Consequently,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \big[f \oplus_1 (\varepsilon \times_1 g)\big](x)dx - \int_{\mathbb{R}^n} f(x) - \varepsilon \int_{\mathrm{Supp}(f)\setminus B_\varepsilon} \mathcal{L}' g\big(\nabla f(x)\big)dx\right| \\ < \mathrm{Vol}_n(B_\varepsilon)\big(\omega(R\varepsilon) + \varepsilon M\big).\end{aligned}$$

Since $\omega(R\varepsilon) \to 0$ as $\varepsilon \to 0$, we conclude that

$$\frac{1}{\varepsilon}\left[\int_{\mathbb{R}^n} f \oplus_1 (\varepsilon \times_1 g) - \int_{\mathbb{R}^n} f\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \int_{\mathrm{Supp}(f)} \mathcal{L}' g\big(\nabla f(x)\big). \tag{48}$$

By linearity and symmetry of V,

$$\begin{aligned}\int_{\mathbb{R}^n} f \oplus_1 (\varepsilon \times_1 g) &= V\big(f \oplus_1 (\varepsilon \times_1 g), ..., f \oplus_1 (\varepsilon \times_1 g)\big) \\ &= \left(\int_{\mathbb{R}^n} f\right) + (n+1)\varepsilon V(f, ..., f, g) + O(\varepsilon^2).\end{aligned} \tag{49}$$

The lemma follows from (48) and (49). □

We have proven almost all of the properties of V that were announced in Theorem 1.3. In fact, all that remains is to show that our definition of V agrees with the one given in Theorem 1.3.

Lemma 5.3. *Let $f_0, ..., f_n$ be continuous, 1-concave functions on $\mathbb{R}^n$. Assume that $\mathcal{L}' f_1, ..., \mathcal{L}' f_n$ have continuous second derivatives. Then,*

$$V(f_0, ..., f_n) = \frac{1}{n+1} \int_{\mathbb{R}^n} \mathcal{L}' f_0(y) D\big(\mathrm{Hess}\mathcal{L}' f_1(y), ..., \mathrm{Hess}\mathcal{L}' f_n(y)\big) dy$$

where D is the mixed discriminant.

Proof. By Lemma 5.2, for any continuous 1-concave functions f, g, we have

$$\begin{aligned} V(f, ..., f, g) &= \frac{1}{n+1} \int_{\mathrm{Supp}(f)} \mathcal{L}' g(\nabla f) \\ &= \frac{1}{n+1} \int_{Im(\nabla f)} \mathcal{L}' g(y) \det \mathrm{Hess}\big(\mathcal{L}' f(y)\big) dy \end{aligned} \tag{50}$$

where we have used the following standard change of variables: We set $y = \nabla f(x)$ and so $x = -\nabla \mathcal{L}' f(y)$. Note that

$$y \notin Im(\nabla f) = \big\{\nabla f(z); z \in \mathrm{Supp}(f)\big\} \quad \Rightarrow \quad \mathcal{L}' f(y) = \sup_{x \in \mathrm{Supp}(f)} \langle y, -x \rangle.$$

Hence $\mathcal{L}' f$ equals the support function of the convex set $-\mathrm{Supp}(f)$ on the complement of $\overline{Im(\nabla f)}$. We conclude that if $y \notin \overline{Im(\nabla f)}$, then $\mathcal{L}' f(ty) = t\mathcal{L}' f(y)$ for t close to 1, and hence $\det(\mathrm{Hess}\mathcal{L} f(y)) = 0$. Hence we may extend the integral in (50) and write,

$$V(f, ..., f, g) = \frac{1}{n+1} \int_{\mathbb{R}^n} \mathcal{L}' g(y) \det \mathrm{Hess}\big(\mathcal{L}' f(y)\big) dy.$$

By polarizing, we obtain

$$V(f_0, ..., f_n) = \frac{1}{n+1} \int_{\mathbb{R}^n} \mathcal{L}' f_0(y) D(\mathrm{Hess}\mathcal{L}' f_1, ..., \mathrm{Hess}\mathcal{L}' f_n) dy. \qquad \square$$

The proof of Theorem 1.3 is complete. We transfer our attention to the functional I.

5.2 The *I* Functional

The I functional was considered, using different terminology, in [CoH2] and in [TW3]. In the latter work, applications to partial differential equations were discussed. Let $K \subset \mathbb{R}^n$ be a compact, convex set. Recall that for $f_0, ..., f_n : K \to [0, \infty)$ smooth, concave functions that vanish on ∂K and that have bounded derivatives in the interior of K, we set

$$I(f_0, ..., f_n) = \int_K f_0(x) D\big(- \mathrm{Hess} f_1(x), ..., -\mathrm{Hess} f_n(x)\big) dx. \tag{51}$$

The multilinear form I is continuous with respect to pointwise convergence of the functions $f_0, ..., f_n$. This is essentially the content of Theorem 1.1 in [CoH1]. Unlike the multilinear form V from the previous section, the extension of I to general concave functions that vanish on ∂K, but that are not assumed to have bounded derivatives, may fail to be finite (e.g. $K = [-1, 1] \subset \mathbb{R}$ and $f_0(t) = f_1(t) = \sqrt{1-t^2}$). We therefore choose not to extend the definition of I to the class of general concave functions. Detailed explanations regarding such "Hessian measures" appear in [CoH1, CoH2, TW1, TW2, TW3].

Let us begin with establishing the symmetry of I. This symmetry is based on a certain relation between mixed discriminants and Hessians. Some readers might prefer to formulate this relation in the language of exterior forms, which is more suitable for applications of Stokes theorem (see, e.g., [Gr]). We stick to the more elementary mixed discriminants. Following the notation of [R], we define the Kronecker symbol $\delta^{i_1,...,i_k}_{j_1,...,j_k}$ to be 1 if $i_1, ..., i_k$ are distinct and are an even permutation of $j_1, ..., j_k$, to be -1 if $i_1, ..., i_k$ are distinct and are an odd permutation of $j_1, ..., j_k$, and to be zero otherwise. $[A]^i_j$ denotes the (i, j)-element of the matrix A. Then if $A_1, ..., A_n$ are $n \times n$ symmetric matrices,

$$D(A_1, ..., A_n) = \frac{1}{n!} \sum \delta^{i_1,...,i_n}_{j_1,...,j_n} [A_1]^{i_1}_{j_1} ... [A_n]^{i_n}_{j_n}$$

where the sum is over all $i_1, ..., j_n, j_1, ..., j_n \in \{1, ..., n\}$. For matrices A, B we write $\langle A, B\rangle = Tr(A^t B)$, for A^t being the transpose of A, and $Tr(A)$ standing for the trace of the matrix A. This is indeed a scalar product. We define $T(A_1, ..., A_{n-1})$ to be the unique matrix such that $\langle T(A_1, ..., A_{n-1}), B\rangle = D(A_1, ..., A_{n-1}, B)$ for any matrix B. In coordinates,

$$\big[T(A_1, ..., A_{n-1})\big]^i_j = \frac{1}{n!} \sum \delta^{i_1,...,i_{n-1},i}_{j_1,...,j_{n-1},j} [A_1]^{i_1}_{j_1} ... [A_{n-1}]^{i_{n-1}}_{j_{n-1}}$$

where the sum is over all $i_1, ..., j_{n-1}, j_1, ..., j_{n-1} \in \{1, ..., n\}$.

Given a symmetric matrix A, we denote by $[A]_i$ the i^{th} row or column of A. The next lemma was essentially noted in [R].

Lemma 5.4. *Let $f^1, ..., f^{n-1} : \mathbb{R}^n \to \mathbb{R}$ be functions with continuous third derivatives. Then for any $1 \le i \le n$,*

$$div \left[T\big(\mathrm{Hess}(f^1), ..., \mathrm{Hess}(f^{n-1})\big)\right]_i = 0,$$

or equivalently, for any fixed $u \in \mathbb{R}^n$, $div\big(T(\mathrm{Hess}(f^1), ..., \mathrm{Hess}(f^{n-1}))u\big) = 0$.

Proof. We need to prove that for any $1 \le i \le n$,

$$\sum_{j=1}^n \frac{\partial}{\partial j} \sum \delta^{i_1,...,i_{n-1},i}_{j_1,...,j_{n-1},j} f^1_{i_1,j_1} ... f^{n-1}_{i_{n-1},j_{n-1}} = 0.$$

We write f_j for the derivative with respect to the j^{th} variable. It is sufficient to prove that for any $1 \le i, k \le n$,

$$\sum \delta^{i_1,...,i_{n-1},i}_{j_1,...,j_{n-1},j} f^1_{i_1,j_1} ... f^{k-1}_{i_{k-1},j_{k-1}} f^k_{i_k,j_k,j} f^{k+1}_{i_{k+1},j_{k+1}} ... f^{n-1}_{i_{n-1},j_{n-1}} = 0 \tag{52}$$

where the sum is over $i_1, ..., i_{n-1}, j_1, ..., j_{n-1}, j$. Since $f_{i_k,j_k,j} = f_{i_k,j,j_k}$, then the left-hand side of (52) is equal to

$$\sum \delta^{i_1,...,i_{n-1},i}_{j_1,...,j_{n-1},j} f^1_{i_1,j_1} ... f^{k-1}_{i_{k-1},j_{k-1}} f^k_{i_k,j,j_k} f^{k+1}_{i_{k+1},j_{k+1}} ... f^{n-1}_{i_{n-1},j_{n-1}} \tag{53}$$

(j_k and j were switched). But since δ is reversed when we switch j and j_k, then (52) also equals the negative of the left-hand side of (53). We conclude that the sum is zero. $\square$

We would also like to use $I(f_0, ..., f_n)$ for non-concave functions. For any bounded, sufficiently smooth functions $f_0, ..., f_n : K \to [0, \infty)$ with bounded first and second derivatives, we use (51) as the definition of $I(f_0, ..., f_n)$.

Lemma 5.5. *Let $K \subset \mathbb{R}^n$ be a convex set, and let $f_0, ..., f_n : K \to [0, \infty)$ be bounded functions that vanish on ∂K. Assume that these functions have continuous third derivatives in the interior of K, and that the first and second derivatives are bounded in the interior of K. Then, for any permutation $\sigma \in S_{n+1}$*

$$I(f_0, ..., f_n) = I(f_{\sigma(0)}, ..., f_{\sigma(n)}).$$

Moreover,

$$I(f_0, ..., f_n) = \int_K D\big(-\mathrm{Hess}(f_2), ..., -\mathrm{Hess}(f_n), \nabla f_0 \otimes \nabla f_1 \big).$$

Proof. Since mixed discriminant is symmetric, clearly

$$I(f_0, f_1, ..., f_n) = I(f_0, f_{\sigma(1)}, ..., f_{\sigma(n)})$$

for any permutation σ of $\{1, ..., n\}$. Thus it suffices to show that

$$I(f, g, h_2, ..., h_n) = I(g, f, h_2, ..., h_n)$$

for any bounded functions $f, g, h_2, ..., h_n : K \to [0, \infty)$, that vanish on ∂K, have continuous third derivatives in the interior of K, and whose first and second derivatives are bounded in the interior of K. Abbreviate $T = T(-\mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n))$. Fix $1 \leq i \leq n$. By Stokes Theorem,

$$0 = \int_{\partial K} g f_i \langle [T]_i, \nu_x \rangle dx = \int_K \mathrm{div}\big(f_i g [T]_i \big) \tag{54}$$

where ν_x is the outer unit normal to ∂K at x. The use of Stokes theorem here is legitimate: To see this, take a sequence of domains $K_\delta \subset K$ with $K_\delta \to K$. In K_δ we may clearly apply Stokes theorem. By our assumptions,

$[T]_i, f_i$ are bounded on K, and hence $(gf_i[T]_i)(x) \to 0$ uniformly as $x \to \partial K$. This justifies (54). We conclude that

$$0 = \int_K f_i g \mathrm{div}([T]_i) + f_i \langle \nabla g, [T]_i \rangle + g \langle \nabla f_i, [T]_i \rangle.$$

By Lemma 5.4, $\mathrm{div}([T]_i) = 0$, and summing for all $1 \le i \le n$,

$$\int_K \sum_{i=1}^n f_i \langle \nabla g, [T]_i \rangle + \int_K \sum_{i=1}^n g \langle \nabla f_i, [T]_i \rangle = 0.$$

By the definitions of I and T,

$$I(g, f, h_2, ..., h_n) = \int_K \sum_{i=1}^n \langle [T]_i, -\nabla f_i \rangle \, g(y) dy = \int_K \sum_{i=1}^n f_i \langle \nabla g, [T]_i \rangle.$$

We conclude that

$$\begin{aligned} &I(g, f, h_2, ..., h_n) \\ &= \int_K \langle T, \nabla f \otimes \nabla g \rangle = \int_K D\big(- \mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n), \nabla f \otimes \nabla g \big). \end{aligned}$$

Since $\mathrm{Hess}(h_i)$ is a symmetric matrix for $i = 2, ..., n$ and $(\nabla f \otimes \nabla g)^t = \nabla g \otimes \nabla f$, by (56) from the Appendix, we conclude that

$$\begin{aligned} &D\big(- \mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n), \nabla f \otimes \nabla g \big) \\ &\qquad = D\big(- \mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n), \nabla g \otimes \nabla f \big) \end{aligned}$$

and hence I is symmetric in f and g. □

Proof of Theorem 1.4. The multilinear form I is finite, since it is the integral of a continuous function on a compact set. The continuity of I was discussed right after (51). Thus the first property in Theorem 1.4 is valid. According to Lemma 5.5, the functional $I(f_0, ..., f_n)$ is symmetric for functions $f_0, ..., f_n$ which are sufficiently smooth in the interior of K. By continuity, we obtain property 2 of Theorem 1.4. To obtain property 3, note that $-\mathrm{Hess}(f_0), ..., -\mathrm{Hess}(f_n)$ are non-negative definite matrices, and hence $D(-\mathrm{Hess}(f_0), ..., -\mathrm{Hess}(f_n)) \ge 0$. Therefore, if $f_0 \ge g_0, ..., f_n \ge g_n$, then

$$\begin{aligned} I(f_0, f_1, ..., f_n) &= \int_K f_0 D\big(- \mathrm{Hess}(f_1), ..., -\mathrm{Hess}(f_n) \big) \\ &\ge \int_K g_0 D\big(- \mathrm{Hess}(f_1), ..., -\mathrm{Hess}(f_n) \big) = I(g_0, f_1, f_2, ..., f_n) \\ &\ge I(g_0, g_1, f_2, ..., f_n) \ge ... \ge I(g_0, ..., g_n) \ge I(0, ..., 0) = 0. \end{aligned}$$

Property 3 is thus established. It remains to prove property 4. This proof is similar to the proof of the Cauchy-Schwartz inequality. It is enough to consider sufficiently smooth concave functions $f, g, h_2, ..., h_n : K \to [0, \infty)$. For $t \in \mathbb{R}$, the function $f + tg$ may fail to be concave. Nevertheless, we still have,

$$I(f+tg, f+tg, h_2, ..., h_n) \\ = \int_K (f+tg) D\big(- \mathrm{Hess}(f+tg), -\mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n)\big).$$

According to Lemma 5.5, for any $t \in \mathbb{R}$,

$$I(f+tg, f+tg, h_2, ..., h_n) \\ = \int_K D\big(- \mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n), \nabla(f+tg) \otimes \nabla(f+tg)\big).$$

Note that $\nabla(f+tg) \otimes \nabla(f+tg)$ is a non-negative definite matrix. Since $-\mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n)$ are also non-negative definite, we conclude that

$$I(f+tg, f+tg, h_2, ..., h_n) \\ = t^2 I(g, g, h_2, ..., h_n) + 2tI(f, g, h_2, ..., h_n) + I(f, f, h_2, ..., h_n) \geq 0$$

for all $t \in \mathbb{R}$. The fact that the quadratic function $I(f+tg, f+tg, h_2, ..., h_n)$ is always non-negative, entails that its discriminant is non-positive. This is exactly the content of Property 4. The proof is complete. □

6 Appendix: Mixed Discriminants

Given $p : \mathbb{R}^m \to \mathbb{R}$ a homogeneous polynomial of degree k, there exists a unique symmetric multilinear form $\tilde{p} : (\mathbb{R}^m)^k \to \mathbb{R}$ such that

$$p(x) = \tilde{p}(x, x, ..., x)$$

for any $x \in \mathbb{R}^m$. We say that $\tilde{p}$ is the polarization of p. This is proven e.g. in Appendix A in [H]. In particular, let A be an $n \times n$ matrix. Then $\det(A)$ is a homogeneous polynomial of degree n in the n^2 matrix elements. Hence, we may define the "mixed discriminant of the matrices $A_1, ..., A_n$" to be $D(A_1, ..., A_n)$, a multilinear symmetric form such that

$$\det(A) = D(A, ..., A)$$

for any matrix A. Note that by linearity,

$$\det\left(\sum_{i=1}^N \lambda_i A_i\right) = \sum_{i_1, ..., i_n \in \{1, ..., N\}} D(A_{i_1}, ..., A_{i_n}) \prod_{j=1}^n \lambda_{i_j}. \tag{55}$$

In fact, (55) is the essence of the proof of the existence of the polarization. Also, since $\det(A) = \det(A^t)$, then

$$D(A_1, ..., A_n) = D(A_1^t, ..., A_n^t). \tag{56}$$

The mixed discriminants satisfy various inequalities. We would like to mention only Alexandrov's inequality, from which it follows that the mixed discriminant of non-negative definite matrices is a non-negative number.

Lemma 6.1. *Let $A_1, ..., A_{n-2}, B, C$ be non-negative definite $n \times n$ symmetric matrices. Then,*

$$D(A_1, ..., A_{n-2}, B, C)^2 \geq D(A_1, ..., A_{n-2}, B, B) D(A_1, ..., A_{n-2}, C, C). \tag{57}$$

Sketch of Proof. (See [H], pages 63-65.) First, suppose that the matrices are positive definite. Let $p(A) = det(A) = D(A, ..., A)$. For any symmetric matrix A and a positive-definite matrix B, the polynomial in the variable t,

$$p(tB + A) = \det(B) \det\left(tId + \sqrt{B^{-1}} A \sqrt{B^{-1}}\right)$$

has only real roots, as $\sqrt{B^{-1}} A \sqrt{B^{-1}}$ is a real, symmetric matrix. By Rolle's Theorem,

$$\frac{d}{dt} p(tB + A) = nD(B, tB + A, ..., tB + A)$$

also has only real roots. The fact that $D(C, tB + A, ..., tB + A)$ has only real roots for any positive definite matrices B, C and any symmetric matrix A, follows from the general theory of hyperbolic polynomials (see e.g. Proposition 2.1.31 in [H]). We may now differentiate $D(C, tB + A, ..., tB + A)$ and so forth. By induction we conclude that

$$q(t) = D(A_1, ..., A_{n-2}, tB + C, tB + C) \tag{58}$$

has only real roots for any positive definite matrices $A_1, ..., A_{n-2}, B$ and any symmetric matrix C. Since q is a quadratic polynomial, its discriminant is non-negative, which is exactly the inequality (57). Thus (57) is proven, for the case of positive definite matrices. The inequality for non-negative definite matrices follows by continuity. □

Remark. The fact that $D(A_1, ..., A_{n-3}, tB + C, tB + C, tB + C)$ has only real roots implies the inequality

$$6a_0 a_1 a_2 a_3 - 4a_2^3 a_0 + 3a_2^2 a_1^2 - 4a_3 a_1^3 - a_3^2 a_0^2 \geq 0$$

which holds for any non-negative definite matrices, where

$$a_i = D(A_1, ..., A_{n-3}; B, i; C, 3 - i),$$

i.e. B appears i times, C appears $3 - i$ times. See also [Ros].

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