Needle decompositions and Ricci curvature

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The Poincaré inequality

**Theorem (Poincaré, 1890 and 1894)**

Let $K \subseteq \mathbb{R}^3$ be convex and open. Let $f : K \rightarrow \mathbb{R}$ be $C^1$-smooth, with $\int_K f = 0$. Then,

$$\lambda_K \int_K f^2 \leq \int_K |\nabla f|^2$$

where $\lambda_K \geq (16/9) \cdot Diam^{-2}(K)$.

- In 2D, Poincaré got a better constant, $24/7$.
- Related to Wirtinger’s inequality on periodic functions in one dimension (sharp constant, roughly a decade later).
- The largest possible $\lambda_K$ is the **Poincaré constant** of $K$.
- Proof: Estimate $\int_{K \times K} |f(x) - f(y)|^2 dx dy$ via segments.
Motivation: The heat equation

- Suppose $K \subseteq \mathbb{R}^3$ with $\partial K$ an ‘insulator’, i.e., heat is not escaping/entering $K$.
- Write $u_t(x)$ for the temperature at the point $x \in K$ at time $t \geq 0$.

Heat equation (Neumann boundary conditions)

\[
\begin{cases}
\dot{u}_t = \Delta u_t & \text{in } K \\
\frac{\partial u_t}{\partial n} = 0 & \text{on } \partial K
\end{cases}
\]

Fourier’s law: Heat flux is proportional to the temp. gradient.

Rate of convergence to equilibrium

\[
\frac{1}{|K|} \int_K u_0 = 1 \quad \Rightarrow \quad \|u_t - 1\|_{L^2(K)} \leq e^{-t\lambda_K} \|u_0 - 1\|_{L^2(K)}
\]
Higher dimensions

The Poincaré inequality was generalized to all dimensions:

**Theorem (Payne-Weinberger, 1960)**

Let $K \subseteq \mathbb{R}^n$ be convex and open, let $\mu$ be the Lebesgue measure on $K$. If $f : K \rightarrow \mathbb{R}$ is $C^1$-smooth with $\int_K f d\mu = 0$, then,

$$\frac{\pi^2}{\text{Diam}^2(K)} \int_K f^2 d\mu \leq \int_K |\nabla f|^2 d\mu.$$  

- The constant $\pi^2$ is best possible in every dimension $n$.
  E.g.,
  $$K = (-\pi/2, \pi/2), \quad f(x) = \sin(x).$$

- In contrast, Poincaré’s proof would lead to an exponential dependence on the dimension.

- Not only the Lebesgue measure on $K$, we may consider any log-concave measure.
The Poincaré inequality was generalized to all dimensions:

**Theorem (Payne-Weinberger, 1960)**

Let $K \subseteq \mathbb{R}^n$ be convex and open, let $\mu$ be any log-concave measure on $K$. If $f : K \to \mathbb{R}$ is $C^1$-smooth with $\int_K f\,d\mu = 0$, then,

$$\frac{\pi^2}{\text{Diam}^2(K)} \int_K f^2\,d\mu \leq \int_K |\nabla f|^2\,d\mu.$$  

- The constant $\pi^2$ is best possible in every dimension $n$. E.g.,
  $$K = [-\pi/2, \pi/2], \quad f(x) = \sin(x).$$  
- In contrast, Poincaré’s proof would lead to an exponential dependence on the dimension. 
- A log-concave measure $\mu$ on $K$ is a measure with density of the form $e^{-H}$, where the function $H$ is convex.
The role of convexity / log-concavity

- For $\Omega \subseteq \mathbb{R}^n$, the Poincaré coefficient $\lambda_\Omega$ measures the connectivity or conductance of $\Omega$.

**Convexity is a strong form of connectedness**

Without convexity/log-concavity assumptions:

*long time to reach equilibrium, regardless of the diameter*
Many other ways to measure connectivity

The isoperimetric constant

For an open set $K \subset \mathbb{R}^n$ define

$$ h_K = \inf_{A \subset K} \frac{|\partial A \cap K|}{\min\{|A|, |K \setminus A|\}} $$

- If $K$ is convex, the infimum is attained when $|A| = |K|/2$ (Sternberg-Zumburn, 1999).

Theorem (Cheeger ’70, Buser ’82, Ledoux ’04)

For any open, convex set $K \subset \mathbb{R}^n$,

$$ \frac{h_K^2}{4} \leq \lambda_K \leq 9h_K^2. $$

- Mixing time of Markov chains, algorithms for estimating volumes of convex bodies (Dyer-Freeze-Kannan ’89, ...)

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Needle decompositions and Ricci curvature
How to prove dimension-free bounds for convex sets?

- Payne-Weinberger approach: **Hyperplane bisections.**
  (developed by Gromov-Milman ’87, Lovász-Simonovits ’93)

- Need to prove, for $K \subset \mathbb{R}^n$, $f : K \to \mathbb{R}$ and $\mu$ log-concave:

$$\int_K f \, d\mu = 0 \implies \int_K f^2 \, d\mu \leq \frac{\text{Diam}^2(K)}{\pi^2} \int_K |\nabla f|^2 \, d\mu.$$

Find a hyperplane $H \subset \mathbb{R}^n$ through barycenter of $K$ such that

$$\int_{K \cap H^+} f \, d\mu = \int_{K \cap H^-} f \, d\mu = 0,$$

where $H^-$, $H^+$ are the two half-spaces determined by $H$.

- It suffices to prove, given $\int_{K \cap H^\pm} f \, d\mu = 0$, that

$$\int_{K \cap H^\pm} f^2 \, d\mu \leq \frac{\text{Diam}^2(K \cap H^\pm)}{\pi^2} \int_{K \cap H^\pm} |\nabla f|^2 \, d\mu.$$
BISECTING AGAIN AND AGAIN

- Repeat bisecting recursively. After \( \ell \) steps, obtain a partition of \( K \) into \( 2^\ell \) convex bodies \( K_1, \ldots, K_{2^\ell} \) with

\[
\int_{K_i} f \, d\mu = 0 \quad \text{for } i = 1, \ldots, 2^\ell.
\]

THE LIMIT OBJECT (AFTER INDUCTION ON DIMENSION):

1. A partition \( \{K_\omega\}_{\omega \in \Omega} \) of \( K \) into segments (a.k.a “needles”).
2. A disintegration of measure: prob. measures \( \{\mu_\omega\}_{\omega \in \Omega} \)
on \( K \), and \( \nu \) on \( \Omega \), with

\[
\mu = \int_{\Omega} \mu_\omega \, d\nu(\omega)
\]

3. \( \nu \)-Each \( \mu_\omega \) is supported on \( K_\omega \) with \( \int_{K_\omega} f \, d\mu_\omega = 0 \).
4. \( \nu \)-Each \( \mu_\omega \) is log-concave, by Brunn-Minkowski!
Examples

1. Take $K = [0, 1]^2 \subseteq \mathbb{R}^2$ and $f(x, y) = f(x)$. Assume $\int_K f = 0$.

Here the needles $\mu_\omega$ are just Lebesgue measures,

$$d\mu_\omega(x) = dx.$$

2. Take $K = B(0, 1) \subseteq \mathbb{R}^2$ and $f(x, y) = f(\sqrt{x^2 + y^2})$ with $\int_K f = 0$. Here the needles $\mu_\omega$ satisfy

$$d\mu_\omega(r) = rdr.$$

(which is log-concave)
Reduction to one dimension

The Payne-Weinberger inequality is reduced to a 1D statement:

\[
\int_{K_\omega} f d\mu_\omega = 0 \implies \int_{K_\omega} f^2 d\mu_\omega \leq \frac{\text{Diam}^2(K_\omega)}{\pi^2} \int_{K_\omega} |\nabla f|^2 d\mu_\omega.
\]

This is because

1. \(\mu = \int_\Omega \mu_\omega d\nu(\omega)\),

2. All \(\mu_\omega\) are log-concave with \(\int_{K_\omega} f d\mu_\omega = 0\).

Usually, 1D inequalities for log-concave measures aren’t hard:

**Lemma**

*Let \(\mu\) be a log-concave measure, \(\text{Supp}(\mu) \subseteq [-D, D]\). Then,

\[
\int_{-D}^{D} f d\mu = 0 \implies \int_{-D}^{D} f^2 d\mu \leq \frac{4D^2}{\pi^2} \int_{-D}^{D} |f'|^2 d\mu.
\]*
These needle decompositions have many applications, such as:

**Theorem ("reverse Hölder inequality", Bourgain ’91, Bobkov ’00, ...)**

Let $K \subseteq \mathbb{R}^n$ be convex, $\mu$ a log-concave prob. measure on $K$. Let $p$ be any polynomial of degree $d$ in $n$ variables. Then,

$$\|p\|_{L^2(\mu)} \leq C_d \|p\|_{L^1(\mu)}$$

where $C_d > 0$ depends only on $d$ (and not the dimension).

**Theorem ("waist of the sphere", Gromov ’03)**

Let $f : S^n \to \mathbb{R}^k$ be continuous, $k \leq n$. Then for some $x \in \mathbb{R}^k$,

$$|f^{-1}(x) + \varepsilon| \geq |S^{n-k} + \varepsilon| \quad \text{for all } \varepsilon > 0,$$

where $A + \varepsilon = \{x \in S^n ; d(x, A) < \varepsilon\}$ and $S^{n-k} \subseteq S^n$. 
Bisections work only in symmetric spaces...

What is the analog of the needle decompositions in an abstract Riemannian manifold $\mathcal{M}$?

- Bisections are no longer possible.
- Are there other ways to construct partitions into segments?

Monge, 1781

A transportation problem induces a partition into segments.

Let $\mu$ and $\nu$ be smooth prob. measures in $\mathbb{R}^n$, disjoint supports. A transportation is a map $T: \mathbb{R}^n \to \mathbb{R}^n$ with

$$T_\ast \mu = \nu.$$

- There is a transportation such that the segments $\{(x, T(x))\}_{x \in \text{Supp}(\mu)}$ do not intersect (unless overlap).
Monge’s heuristics

Let $\mu$ and $\nu$ be smooth measures in $\mathbb{R}^n$, same total mass. Consider a transportation $T : \mathbb{R}^n \to \mathbb{R}^n$ that minimizes the cost

$$\int_{\mathbb{R}^n} |Tx - x| d\mu(x) = \inf_{S^*=\nu} \int_{\mathbb{R}^n} |Sx - x| d\mu(x).$$

Use the triangle inequality: Assume by contradiction that

$$(x, Tx) \cap (y, Ty) = \{z\}.$$
The Monge-Kantorovich transportation problem

1. Suppose that $\mathcal{M}$ is an $n$-dimensional Riemannian manifold. Either complete, or at least **geodesically convex**.

2. A measure $\mu$ on $\mathcal{M}$ with a smooth density. (maybe the Riemannian volume measure.)

3. A measurable function $f : \mathcal{M} \to \mathbb{R}$ with $\int_{\mathcal{M}} f d\mu = 0$ (and some mild integrability assumption).

Consider the transportation problem between the two measures

\[ d\nu_1 = f^+ d\mu \quad \text{and} \quad d\nu_2 = f^- d\mu. \]

We study a transportation $T_*=\nu_1 = \nu_2$ of minimal cost

\[ c(T) = \int_{\mathcal{M}} d(x, Tx) d\nu_1(x). \]
Structure of the optimal transportation

- Recall that $\int_{\mathcal{M}} f d\mu = 0$. Then the optimal transportation $T$ exists and it induces the following structure:

**Theorem ("Resolution of the Monge-Kantorovich problem")**

There exists a partition $\{I_\omega\}_{\omega \in \Omega}$ of $\mathcal{M}$ into minimizing geodesics and measures $\nu$ on $\Omega$, and $\{\mu_\omega\}_{\omega \in \Omega}$ on $\mathcal{M}$ with

$$\mu = \int_{\Omega} \mu_\omega d\nu(\omega) \quad \text{(disintegration of measure)},$$

and for $\nu$-any $\omega \in \Omega$, the measure $\mu_\omega$ is supported on $I_\omega$ with

$$\int_{I_\omega} f d\mu_\omega = 0.$$

- A result of Evans and Gangbo ’99, Caffarelli, Feldman and McCann ’02, Ambrosio ’03, Feldman and McCann ’03.
- Like localization, but where is the log-concavity of needles?
Example - the sphere $S^n$

In this example:

- $\mathcal{M} = S^n$
- The measure $\mu$ is the Riemannian volume on $S^n \subseteq \mathbb{R}^{n+1}$.
- $f(x_0, \ldots, x_n) = x_n$, clearly $\int_{S^n} f d\mu = 0$.

1. We obtain a partition of $S^n$ into needles which are **meridians**.
2. The density on each needle is proportional to

   $$\rho(t) = \sin^{n-1} t \quad t \in (0, \pi)$$

   in arclength parametrization (“spherical polar coordinates”).
3. Note that $\left(\rho^{\frac{1}{n-1}}\right)'' + \rho^{\frac{1}{n-1}} = 0$. 

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Needle decompositions and Ricci curvature
Ricci curvature appears

Assume $\mu$ is the Riemannian volume on $\mathcal{M}$, and $\int f d\mu = 0$.

**Theorem ("Riemannian needle decomposition")**

There is a partition $\{\mathcal{I}_\omega\}_{\omega \in \Omega}$ of $\mathcal{M}$ and measures $\nu$ on $\Omega$, and $\{\mu_\omega\}_{\omega \in \Omega}$ on $\mathcal{M}$ with $\mu = \int_{\Omega} \mu_\omega d\nu(\omega)$ such that for any $\omega \in \Omega$,

1. The measure $\mu_\omega$ is supported on the minimizing geodesic

   \[ \mathcal{I}_\omega = \{\gamma_\omega(t)\}_{t \in (a_\omega, b_\omega)} \]  

   (arclength parametrization)

   with $C^\infty$-smooth, positive density $\rho = \rho_\omega : (a_\omega, b_\omega) \to \mathbb{R}$.

2. $\int_{\mathcal{I}_\omega} f d\mu_\omega = 0$.

3. Set $\kappa(t) = \text{Ricci}(\dot{\gamma}(t), \dot{\gamma}(t))$, $n = \text{dim}(\mathcal{M})$. Then we have

   \[ \left(\rho^{\frac{1}{n-1}}\right)^{\prime\prime} + \frac{\kappa}{n-1} \cdot \rho^{\frac{1}{n-1}} \leq 0. \]
Remarks on the theorem

- If $\mu$ is not the Riemannian measure, replace the dimension $n$ by $N \in (-\infty, 1] \cup [n, +\infty]$ and use the generalized Ricci tensor (Bakry-Émery, '85):

$$Ricci_{\mu,N} = Ricci_M + Hess\psi - \frac{\nabla\psi \otimes \nabla\psi}{N - n}$$

where $d\mu/d\lambda_M = \exp(-\Psi)$. Also set $Ricci_{\mu} = Ricci_{\mu,\infty}$.

**When $Ricci_M \geq 0$, the needle density $\rho$ satisfies**

$$\left(\rho^{\frac{1}{n-1}}\right)'' \leq \left(\rho^{\frac{1}{n-1}}\right)'' + \frac{\kappa}{n-1} \cdot \rho^{\frac{1}{n-1}} \leq 0.$$ 

Thus $\rho^{1/(n-1)}$ is concave and in particular $\rho$ is **log-concave**.

- This recovers the case of $\mathbb{R}^n$, without use of bisections.
- Already generalized to measure-metric spaces (Cavalletti and Mondino ’15) and to Finsler manifolds (Ohta ’15).
Suppose $\mathcal{M}$ is $n$-dimensional, geodesically-convex, and

$$Ricci_\mathcal{M} \geq n - 1 \ (= Ricci_{S^n}).$$

For a subset $A \subseteq \mathcal{M}$ denote

$$A + \varepsilon = \{ x \in \mathcal{M} ; d(x, A) < \varepsilon \},$$

the $\varepsilon$-neighborhood of $A$.

Let $\mu$ and $\sigma$ be Riemannian measures on $\mathcal{M}$ and $S^n$, respectively, normalized to be prob. measures.

**Theorem ("Lévy-Gromov isoperimetric inequality")**

For any $A \subseteq \mathcal{M}$ and a geodesic ball $B \subseteq S^n$,

$$\mu(A) = \sigma(B) \implies \forall \varepsilon > 0, \ \mu(A + \varepsilon) \geq \sigma(B + \varepsilon).$$
Proof of Lévy-Gromov’s isoperimetric inequality

- Given measurable \( A \subseteq M \) with \( \mu(A) = \lambda \in (0, 1) \), define
  \[
  f(x) = (1 - \lambda) \cdot 1_A(x) - \lambda \cdot 1_{M \setminus A}(x).
  \]

- Apply needle decomposition for \( f \) to obtain
  \[
  \mu = \int_\Omega \mu_\omega d\nu(\omega),
  \]
  where \( \nu \) and \( \{\mu_\omega\} \) are prob. measures.

Properties of the needle decomposition

1. Set \( A_\omega = A \cap I_\omega \), where \( I_\Omega = \text{Supp}(\mu_\omega) \) is a minimizing geodesic. Then,
  \[
  \mu_\omega(A_\omega) = \lambda \quad \forall \omega \in \Omega.
  \]

2. For any \( \varepsilon > 0 \),
  \[
  \mu(A + \varepsilon) = \int_\Omega \mu_\omega(A + \varepsilon) d\nu(\omega) \geq \int_\Omega \mu_\omega(A_\omega + \varepsilon) d\nu(\omega)
  \]
  with equality when \( M = S^n \) and \( A = B \) is a cap in \( S^n \).
Proof of Lévy-Gromov’s isoperimetric inequality

Our needle density $\rho$ is “more concave” than polar spherical coordinates, i.e., needles with density $\sin^{n-1} t$.

One-dimensional lemma
Let $\rho : (a, b) \rightarrow \mathbb{R}$ be smooth and positive with

$$\left(\rho^{\frac{1}{n-1}}\right)^{\prime\prime} + \rho^{\frac{1}{n-1}} \leq 0.$$  \hfill (1)

Let $A \subseteq (a, b)$ and $B = [0, t_0] \subseteq [0, \pi]$. Then for any $\varepsilon > 0$,

$$\frac{\int_A \rho}{\int_a^b \rho} = \frac{\int_B \sin^{n-1} t \, dt}{\int_0^\pi \sin^{n-1} t \, dt} \quad \Rightarrow \quad \frac{\int_{A+\varepsilon} \rho}{\int_a^b \rho} \geq \frac{\int_{B+\varepsilon} \sin^{n-1} t \, dt}{\int_0^\pi \sin^{n-1} t \, dt}.$$

In fact, from (1) the isoperimetric profile $I$ of $(\mathbb{R}, | \cdot |, \rho)$ satisfies

$$\left(I^{\frac{n}{n-1}}\right)^{\prime\prime} + n \cdot I^{\frac{1}{n-1} - 1} \leq 0.$$
More applications of needle decompositions

Assume that $\mathcal{M}$ is geodesically-convex with non-negative Ricci. Using Needle decompositions we can obtain:

1. **Poincaré constant** (Li-Yau ’80, Yang-Zhong ’84):
   \[
   \lambda_{\mathcal{M}} \geq \frac{\pi^2}{\text{Diam}^2(\mathcal{M})}
   \]

2. **Brunn-Minkowski type inequality**: For any measurable $A, B \subseteq \mathcal{M}$ and $0 < \lambda < 1$,
   \[
   \text{Vol}(\lambda A + (1 - \lambda)B) \geq \text{Vol}(A)^\lambda \text{Vol}(B)^{1-\lambda}
   \]
   where $\lambda A + (1 - \lambda)B$ consists of all points $\gamma(\lambda)$ where $\gamma$ is a geodesic with $\gamma(1) \in A, \gamma(0) \in B$. (Cordero-Erausquin, McCann, Schmuckenschlaeger ’01).

3. **Log-Sobolev inequalities** (Wang ’97), reverse Cheeger inequality $\lambda_{\mathcal{M}} \leq c \cdot h^2_{\mathcal{M}}$ (Buser ’84), spectral gap and Lipschitz functions (E. Milman ’09).
Another application: The 4 functions theorem

Assume $\mathcal{M}$ is geodesically-convex, $\mu$ a measure, $\text{Ricci}_\mu \geq 0$.

The four functions theorem (Riemannian version of KLS ’95)

Let $\alpha, \beta > 0$. Let $f_1, f_2, f_3, f_4 : \mathcal{M} \to [0, +\infty)$ be measurable functions. Assume that for any probability measure $\eta$ on $\mathcal{M}$ which is a log-concave needle,

$$
\left( \int_{\mathcal{M}} f_1 \, d\eta \right)^\alpha \left( \int_{\mathcal{M}} f_2 \, d\eta \right)^\beta \leq \left( \int_{\mathcal{M}} f_3 \, d\eta \right)^\alpha \left( \int_{\mathcal{M}} f_4 \, d\eta \right)^\beta,
$$

whenever $f_1, f_2, f_3, f_4$ are $\eta$-integrable. Then,

$$
\left( \int_{\mathcal{M}} f_1 \, d\mu \right)^\alpha \left( \int_{\mathcal{M}} f_2 \, d\mu \right)^\beta \leq \left( \int_{\mathcal{M}} f_3 \, d\mu \right)^\alpha \left( \int_{\mathcal{M}} f_4 \, d\mu \right)^\beta.
$$

- Recall: A log-concave needle is a measure, supported on a minimizing geodesic, with a log-concave density in arclength parameterization.
One last application: Dilation inequalities

**Definition (Nazarov, Sodin, Volberg ’03, Bobkov and Nazarov ’08, Fradelizi ’09)**

For $A \subseteq \mathcal{M}$ and $0 < \varepsilon < 1$, the set $\mathcal{N}_\varepsilon(A)$ contains all $x \in \mathcal{M}$ for which there exists a minimizing geodesic $\gamma : [a, b] \rightarrow \mathcal{M}$ with $\gamma(a) = x$ and

$$\lambda_1 (\{ t \in [a, b] ; \gamma(t) \in A \}) \geq (1 - \varepsilon) \cdot (b - a),$$

where $\lambda_1$ is the Lebesgue measure in the interval $[a, b] \subseteq \mathbb{R}$.

Thus $\mathcal{N}_\varepsilon(A)$ is a kind of an $\varepsilon$-dilation of the set $A$.

**Theorem (Riemannian version of Bobkov-Nazarov ’08)**

Assume $\mathcal{M}$ is $n$-dimensional, geodesically-convex, $\mu$ is prob., $\text{Ricci}_\mu \geq 0$. Let $A \subseteq \mathcal{M}$ be measurable with $\mu(A) > 0$. Then,

$$\mu(\mathcal{M} \setminus A)^{1/n} \geq (1 - \varepsilon) \cdot \mu(\mathcal{M} \setminus \mathcal{N}_\varepsilon(A))^{1/n} + \varepsilon.$$
Comparison with the quadratic cost

- Given probability measures \( \nu_1, \nu_2 \) on \( \mathcal{M} \), consider all transportations \( T_\star \nu_1 = \nu_2 \) with the \textbf{quadratic cost}

\[
C(T) = \int_{\mathcal{M}} d^2(x, Tx) d\nu_1(x).
\]

**Theorem (Brenier ’87)**

When \( \mathcal{M} = \mathbb{R}^n \), the map \( T \) of minimal quadratic cost has the form

\[
T = \nabla \Phi
\]

where \( \Phi \) is a convex function on \( \mathbb{R}^n \). (and vice versa)

- Generalization to Riemannian manifolds by McCann ’01:
  The optimal map \( T \) has the form \( T(x) = \exp_x(\nabla \Phi) \), where
  \(-\Phi \) is a \( d^2/2 \)-concave function.
- This yields some of the aforementioned applications.
Two open problems in isoperimetry

The “Cartan-Hadamard” conjecture

Suppose $\mathcal{M}$ is complete, $n$-dimensional, simply-connected, non-positive sectional curvature. Then for any $A \subseteq \mathcal{M}$,

$$Vol_{n-1}(\partial A) \geq n \cdot Vol_n(A)^{\frac{n-1}{n}} \cdot Vol_n(B_n^2)^{1/n}$$

where $B_n^2 = \{ x \in \mathbb{R}^n ; |x| \leq 1 \}$.

Known for $n = 2, 4, 3$, by Weil ’26, Croke ’84 and Kleiner ’92.

The Kannan-Lovász-Simonovits conjecture (1995)

Let $K \subseteq \mathbb{R}^n$ be convex, bounded and open. Then

$$\inf_{|A\cap K| = |K|/2} Vol_{n-1}(\partial A \cap K) \geq c \cdot \inf_{|H\cap K| = |K|/2} Vol_{n-1}(\partial H \cap K)$$

where $A$ ranges over all measurable sets and $H$ ranges over all half spaces in $\mathbb{R}^n$. Here, $c > 0$ is a universal constant.
Thank you!

One of the images (sphere with meridians) was taken from www2.rdrop.com/~half