Remarks on Minkowski Symmetrizations

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Abstract. Here we extend a result by J. Bourgain, J. Lindenstrauss, V.D. Milman on the number of random Minkowski symmetrizations needed to obtain an approximated ball, if we start from an arbitrary convex body in $\mathbb{R}^n$. We also show that the number of "deterministic" symmetrizations needed to approximate an Euclidean ball may be significantly smaller than the number of "random" ones.

1 Background and Notation

Let K be a compact convex (symmetric) set in $\mathbb{R}^n$ and let u be any vector in $S^{n-1} = \{u; |u| = 1\}$ where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. We denote by $\pi_u \in O(n)$ the reflection with respect to the hyperplane through the origin orthogonal to u, i.e. $\pi_u x = x - 2\langle x, u\rangle u$.

The Minkowski symmetrization (sometimes called Blaschke symmetrization) of K with respect to u is defined to be the convex set $\frac{1}{2}(\pi_u K + K)$. Denote by $\|\cdot\|$ the norm whose unit ball is K, and by $M^*(K)$ the half mean width of K: $M^*(K) := \int_{S^{n-1}} \|x\|^* d\sigma(x)$, where σ is the normalized rotation invariant measure on S^{n-1}, and $\|\cdot\|^*$ is the dual norm.

It is easy to verify that $M^*(K) = M^*(\frac{1}{2}(\pi_u K + K))$, so the mean width is preserved under Minkowski symmetrizations. Since successive Minkowski symmetrizations make the body symmetric with respect to more and more hyperplanes, one might expect convergence to a ball. However, surprisingly, very few symmetrizations are needed for that convergence, as stated and proved in [1]:

Proposition 1.1 *If we start with arbitrary body K, and perform $cn\log n + c(\epsilon)n$ "random" Minkowski symmetrizations, with high probability we obtain a body $\hat{K}$ such that*

$$(1-\epsilon)M^*D \subset \hat{K} \subset (1+\epsilon)M^*D$$

where $D = \{u; |u| \le 1\}$.

"Random" means that the $N = cn\log n + c(\epsilon)n$ symmetrizations are performed with respect to $u_1, ..., u_N \in S^{n-1}$, and the u_i's are chosen independently and uniformly (i.e. according to the probability rotation invariant measure on the sphere).

* Supported by the Israel Science Foundation founded by the Academy of Sciences and Humanities.

Now we give a formal description of the body $\hat{K}$. Take the body K, symmetrize with respect to $u_1, ..., u_T$ (in that order). The body achieved is $\hat{K} = \frac{1}{2^T} \sum_{D \subset \{1,..,T\}} \prod_{i \in D} \pi_{u_i}(K)$.

The technique we use forces us to work in the dual space, so the next formula would be needed:

$$|||x||| = \frac{1}{2^T} \sum_{D \subset \{1,..,T\}} \Big\| \prod_{i \in D} \pi_{u_i} x \Big\|^*$$

where $\| \cdot \|^*$ is the dual norm to K, and $||| \cdot |||$ is the dual norm to $\hat{K}$.

Note that our notation doesn't specify the exact order of multiplications in expressions like $\prod_{i \in D} \pi_{u_i} x$. One may think that this might be a source for problems, since the reflections do not generally commute, and when passing to the dual, the order is reversed. However, we do not rely anywhere on the order of multiplications.

In the following sections, the number of random (and non random) Minkowski symmetrizations needed in order to obtain approximately a ball is investigated. Tight estimates (from below and above) for the random case are given, and actually there exists a simple formula to describe this quantity. The behavior in the deterministic case is not as clear to us as in the random case, but few results about that behavior are presented.

By c, C we define universal constants, which are not the same at different appearances.

2 Random Case

Denote by $a(K)$ the half-diameter of K. i.e. $a(K) = \sup_{x \in S^{n-1}} \|x\|^*$.

This section relies heavily on statements and proofs from [1]. One of the main lemmas from [1] needed is the following:

Lemma 2.1 *Assume* $a(K) = 1$, *and* $M^*(K) \leq \frac{1}{8}$, *then with* $N_1 < c_1 n$ *we have*

$$\forall x \in \mathbb{R}^n \qquad |||x||| = 2^{-N_1} \sum_D \Big\| \prod_{i \in D} \pi_{u_i} x \Big\|^* \leq \frac{1}{2} |x|$$

with probability at least $1 - \exp(-c_2 n)$.

This means that the diameter of the new body is at most half the diameter of the original body, while its mean width remains the same. This happens after random $c_1 n$ symmetrizations.

If the new body $\hat{K}$ satisfies $a(\hat{K})/M^*(\hat{K}) \geq 8$ we can repeat the same process, and reduce the diameter by an additional factor of 2. After $\log \frac{a(K)}{M^*(K)}$ iterations, each involving less than $c_1 n$ symmetrizations, we achieve a body that satisfies $a(\hat{K})/M^*(\hat{K}) \leq 8$, with exponential probability (exponential in the dimension n).

When we have reached this stage, we can use the next lemma from [1]:

Lemma 2.2 *Assume* $a(K) \leq 1$, *and* $M^*(K) \geq \frac{1}{8}$, *then for all* $\epsilon > 0$ *and* $n > n(\epsilon)$ *and with* $N_2 = c(\epsilon)n$

$$\forall x \in \mathbb{R}^n \qquad (1-\epsilon)M^*|x| \leq 2^{-N_2} \sum_D \Big\| \prod_{i\in D} \pi_{u_i} x \Big\|^* \leq (1+\epsilon)M^*|x|$$

with probability of at least $1 - \exp(-c_1(\epsilon)n)$.

In this paper, we are interested only in the "isomorphic" symmetrization procedure, meaning we want our body to be close to a ball up to a factor, say 4: $\frac{1}{2}D \subset K \subset 2D$ (as opposed to "almost-isometric" symmetrizations, where dependence in ϵ is added). Our analysis here leads us to the following formulation of Proposition 1.1:

Proposition 2.3 *K is a convex symmetric body. Perform* $c_1 n \log \frac{a}{M^*}$ *"random" Minkowski symmetrizations. With probability greater than* $1-\exp(-c_2 n)$ *we obtain a body* $\hat{K}$ *such that*

$$\frac{1}{2}M^*D \subset \hat{K} \subset 2M^*D$$

where c_1, c_2 *are numerical constants, and "random" means that the symmetrizations are chosen independently and uniformly on the sphere.*

Since for every body $K \subset \mathbb{R}^n$, the ratio $a(K)/M^*(K)$ is bounded by $\sqrt{n}$, the worst body we can find demands $cn \log n$ symmetrizations. There exist bodies - such as the n dimensional cube - that satisfy $a(K)/M^*(K) < Const$ independent of n. Those bodies become close to an Euclidean ball only after cn random symmetrizations. Therefore, Proposition 2.3 is slightly more informative than Proposition 1.1.

We will analyze now the process of performing random Minkowski symmetrizations, starting with a specific body. This specific body I is just a simple segment. Let $v \in S^{n-1}$, and $I = [-v, v]$.

The dual norm of the segment is $\|x\|^* = |\langle x, v\rangle|$. We will denote the dual norm after T symmetrizations with respect to the vectors $u_1, ...u_T$ by

$$\|x\|_T = \frac{1}{2^T} \sum_{D\subset\{1,..,T\}} \Big|\langle \prod_{i\in D} \pi_{u_i} x, v\rangle\Big|$$

We will denote by I_T the body after T symmetrizations, the body that $\|\cdot\|_T$ is its dual norm. Note that the diameter of I is 1, while its mean width is $\approx 1/\sqrt{n}$. Therefore, also $M^*(I_T) \approx 1/\sqrt{n}$, and if I_T is close to a ball, then we must have $a(I_T) \leq 2/\sqrt{n}$.

In the next few lines we will show a lower bound on the diameter of I_T, and we will use this result to get that Proposition 2.3 is tight for the case of a segment.

Now, a straightforward calculation (by induction on k) yields the following equality: for any $x \in \mathbb{R}^n$,

$$\int_{S^{n-1}} ... \int_{S^{n-1}} \langle x, \prod_{i=1}^{k} \pi_{u_i} x \rangle d\sigma(u_1)...d\sigma(u_k) = \Big(1 - \frac{2}{n}\Big)^k |x|^2$$

Roughly, this equality states that if we take a point in S^{n-1}, reflect it randomly k times, and calculate the scalar-product of the point we got with the original point - then the result has expected value of $(1-\frac{2}{n})^k$.

The expectation of $\|v\|_T$ (where the expectation is over all the possible symmetrizations) is

$$\mathbb{E}\|v\|_T = \mathbb{E}[\frac{1}{2^T} \sum_{D \subset \{1,..,T\}} |\langle \prod_{i \in D} \pi_{u_i} v, v \rangle| \geq \Big(1 - \frac{2}{n}\Big)^T$$

Since the diameter of I_T is $\sup_{x \in S^{n-1}} \|x\|_T$, the expectation of the diameter of I_T is surely greater than $\mathbb{E}\|v\|_T$, and we conclude that $\mathbb{E}[a(I_T)] > \exp(-\frac{2T}{n})$.

Now we can move to a general body $K \subset \mathbb{R}^n$. This body contains a maximal interval: There exists $v \in \mathbb{R}^n$ with $|v| = a(K)$ such that $I = [-v, v] \subset K$. If we apply the same set of symmetrizations to I and to K, then $I_T \subset K_T$, and $\mathbb{E}a(I_T) \leq \mathbb{E}a(K_T)$.

If the body K_T is close to a ball in exponential (in the dimension) probability, then with large probability $a(K_T) \leq 2M^*$, and clearly $\mathbb{E}a(K_T) \leq 2M^*$. But

$$\mathbb{E}a(I_T) \leq 2M^* \Longrightarrow a(K)\Big(1 - \frac{2}{n}\Big)^T \leq 2M^*$$

which means that

$$T > \frac{1}{2} n \log \frac{a}{2M^*}$$

In other words, it takes at least $\frac{1}{2} n \log \frac{a}{2M^*}$ random symmetrizations just to reduce the diameter of K to size of $2M^*$.

The above discussion was actually a proof of the following theorem. Fix a body K. For $u_1, .., u_T \in S^{n-1}$, define $\chi^K_{u_1,..,u_T}$ to be indicator of the following event:

$$\frac{1}{2} M^* D \subset \frac{1}{2^T} \sum_{D \subset \{1,..,T\}} \prod_{i \in D} \pi_{u_i}(K) \subset 2M^* D$$

i.e. $\chi^K_{u_1,..,u_T}$ equals 1 if the event occurs, and equals 0 otherwise.

Now, for a body K, define $T(K)$ as the minimal T such that

$$measure\{(u_1, ..u_T) \in (S^{n-1})^T : \chi^K_{u_1,..,u_T} = 1\} > 1 - e^{-n}$$

Theorem 2.4 *There exist numerical constants C_1, C_2 such that:*

$$C_1 n \log \frac{a}{M^*} \leq T(K) \leq C_2 n \log \frac{a}{M^*}$$

(or in a shorter form: $T(K) \approx n \log \frac{a}{M^*}$).

We know that after typical $T(K)$ symmetrizations, our body becomes close to a ball, and further symmetrizations would keep it in such a shape. If we apply less than $T(K)$ symmetrizations, with high probability we are far away from a ball.

There is a parameter of the body - namely, the diameter - that decays regularly during the process of symmetrizations. If we look at the change of the diameter during the process of symmetrizations, we observe a phase-transition: By Lemma 2.1 and the discussion above we can see that there are numerical constants such that for $T \leq cT(K)$:

$$\exp\Big(-C_1 \frac{T}{n}\Big) \leq \frac{E[a(K_T)]}{a(K)} \leq \exp\Big(-C_2 \frac{T}{n}\Big)$$

But for $T \geq CT(K)$, the diameter stabilized, and with high probability it is very close to $M^*(K)$.

3 Deterministic Examples

Until now, we were interested only in the question: what happen to a body going through "typical" or "random" symmetrizations. A question that naturally arises is whether there exists a special choice of symmetrizations such that an approximated ball is achieved much faster than in a "typical" choice of symmetrizations.

In the spirit of the results in [2], where symmetrizations of another kind are discussed, one might expect the answer to be "no". In the case of Milman and Schechtman investigating, the random and the non-random behavior essentially coincide. One cannot significantly improve the convergence by choosing specific symmetrizations. "Random" symmetrizations are almost as good as the best ones.

However, in our question it is not like that. We will see a few examples where $cn \log n$ random symmetrizations are needed, while only cn specific symmetrizations suffice.

3.1 Example 1

Again, a segment. $v = \frac{1}{\sqrt{n}}(1, 1, ..., 1) \in S^{n-1}$ and $I = [-v, v]$. According to formula 2.4 we need at least $cn \log n$ random symmetrizations.

Choose the directions to be $e_1, .., e_n$ - the standard unit vectors. Denote by I_n the interval I after going through symmetrizations with respect to $e_1, .., e_n$, and by $|||\cdot|||$ the dual norm to I_n. For every $x = (x_1, .., x_n) \in \mathbb{R}^n$:

$$|||x||| = \frac{1}{2^n} \sum_{D \subset \{1,..,n\}} \big|\langle \prod_{i \in D} \pi_{e_i} x, v\rangle\big| = \frac{1}{\sqrt{n}} Ave_{\epsilon \in \{-1,1\}^n} \big|\sum_i \epsilon_i x_i\big|$$

By Khinchine's inequality we obtain:

$$\frac{1}{\sqrt{2n}}|x| \leq |||x||| \leq \frac{1}{\sqrt{n}}|x|$$

or

$$\frac{1}{\sqrt{2n}} D \subset I_n \subset \frac{1}{\sqrt{n}} D$$

Also we proved that for the segment, there exists a choice of directions such that n symmetrizations are sufficient to achieve approximately a ball (actually $n-1$ suffice. The last symmetrization is unnecessary).

3.2 Example 2

Define the cube Q as $Q = \{x = (x_1, .., x_n) \in \mathbb{R}^n : \forall i \;\; -1 \leq x_i \leq 1\}$, the L_∞^n ball. Here we will prove the next lemma, which will be very useful in obtaining another example of bodies with an essentially different number of random and deterministic symmetrizations needed to approximate the Euclidean ball.

Lemma 3.1 *If $K \subset \alpha Q$, and $K = convS$, where $|S| = N$, then by symmetrizing with respect to the standard unit vectors $e_1, .., e_n$, a body $\hat{K}$ is obtained, and it satisfies:*

$$diam(\hat{K}) \leq c\alpha\sqrt{\log N}$$

Proof. Denote $S = \{v_1, .., v_N\}$. K is a convex hull of S, so the dual norm of K is $\|x\|^* = \max_i |\langle x, v_i\rangle|$. The dual norm of the symmetrized body will be $|||\cdot|||$. For every $x \in \mathbb{R}^n$:

$$|||x||| = \frac{1}{2^n} \sum_{D \subset \{1,..,n\}} \big\| \prod_{i \in D} \pi_{e_i} x\big\| = \frac{1}{2^n} \sum \max_j \big|\langle \prod_{i \in D} \pi_{e_i} x, v_j\rangle\big|$$

For $1 \leq j \leq N$ and $\epsilon \in \{\pm 1\}^n$ define

$$f_x^j(\epsilon) = \big|\langle \sum_i \epsilon_i x_i e_i, v_j\rangle\big|$$

Then:

$$|||x||| = \mathbb{E}_\epsilon\big[\max_j f_x^j(\epsilon)\big]$$

$|||x|||$ is the expectation of maximum of N random variables (expectation by variable $\epsilon \in \{\pm 1\}^n$). We would like to bound from above $|||x|||$, and for that purpose we will estimate the ψ_2 norm (see, for example, in [1]) of those variables. The coordinates of v_j are $v_j = (v_{j1}, .., v_{jn}) \in \mathbb{R}^n$.

For every $j \in \{1, .., N\}$ we have

$$\|f_x^j\|_p = \Big(Ave_{\epsilon\in\pm 1^n}\big|\sum_i v_{ji}x_i\epsilon_i\big|^p\Big)^{\frac{1}{p}} \le c\sqrt{p}\sqrt{\sum_i (x_i v_{ji})^2} \le c\sqrt{p}|x|\|v_j\|_\infty$$

The first inequality follows by Khinchine's inequality. Since $K \subset \alpha Q$, for every $x \in S^{n-1}$ we get that $\|f_x^j\|_p \le c\sqrt{p}\alpha$. Since the p^{th} moment of f_x^j is bounded by $c\alpha\sqrt{p}$, we get that $\|f_x^j\|_{\psi_2} \le c\alpha$.

Moreover since $|||x||| = \mathbb{E}_\epsilon[\max_j f_x^j(\epsilon)]$, we can use the well known estimate for the expectation of maximum of ψ_2 variables (e.g. [3]):

$$\forall x \in S^{n-1} \qquad |||x||| \le c\alpha\sqrt{\log N}$$

Thus the lemma is proved. □

Application of that lemma to the case of $B(l_1^n) = \{x \in \mathbb{R}^n : \sum_i |x_i| \le 1\}$ is easy. $B(l_1^n)$ has diameter 1, and $M^*B(l_1^n) \approx \sqrt{\log n/n}$. Again, $cn\log n$ random symmetrizations are needed to make this body close to an Euclidean ball.

For simplicity, assume $\exists k$ with $n = 2^k$ (otherwise, embed $B(l_1^n)$ in such a space). Let $w_1, .., w_n \in S^{n-1}$ be the normalized Walsh vectors. Since $w_1, .., w_n$ is an orthonormal system, we can write as well $K = B(l_1^n) = conv_{i=1,..n}\{\pm w_i\}$. But since $K \subset \frac{1}{\sqrt{n}}Q$ we are in a position to use Lemma 3.1.

Our process of symmetrizations here consists of 2 steps. In the first step, we symmetrize with respect to the standard unit vectors $e_1, .., e_n$. By Lemma 3.1 we obtain a body K_n which has diameter $diam(K_n) \le c\sqrt{\log n}/\sqrt{n}$ or $K_n \subset cM^*D$ (because $M^*(K_n) = M^*B(l_1^n) \approx \sqrt{\log n}/\sqrt{n}$). For the second step, we will choose an additional cn symmetrizations "randomly", and according to Proposition 2.3 we will get an approximated ball.

To summarize, we used specific n symmetrizations, and after that additional cn symmetrizations - and achieved a body close to a ball.

3.3 General Convex Body

Let $e_1, .., e_n$ be the standard orthonormal basis in $\mathbb{R}^n$. Call a body K a 1-unconditional body if it satisfies $\forall 1 \le i \le n, \pi_{e_i}K = K$. The norm that K is its unit ball is a 1-unconditional norm.

The following lemma is quite known:

Lemma 3.2 *Let $K \subset \mathbb{R}^n$ be a convex 1-unconditional body with $M^*(K) = 1$. Then there exist a numerical constant c such that*

$$K \subset c\sqrt{n}B(l_1^n)$$

Proof. Take $x = (x_1, .., x_n) \in K$. Since K is a 1-unconditional body, all the vectors of the form $(\pm x_1, .., \pm x_n)$ are also inside K, and so is their convex hull. Denote by Q_x the convex hull of all such vectors. Clearly, Q_x is a rectangular parallelepiped. Q_x is equal to a Minkowski sum of n segments - those are the segments $[-x_1e_1, x_1e_1], .., [-x_ne_n, x_ne_n]$. Since mean width is additive with respect to Minkowski sum, and the mean width of a unit segment is $\approx \frac{1}{\sqrt{n}}$, we can compute the mean width of Q_x, that is: $M^*(Q_x) \approx \frac{1}{\sqrt{n}} \sum_i |x_i|$. On the other hand, the mean width of Q_x is less than 1, since $M^*(K) = 1$ and $Q_x \subset K$. We conclude that there exist c such that $\frac{1}{\sqrt{n}} \sum_i |x_i| \leq c$, and $x \in c\sqrt{n}B(l_1^n)$. This is true for any $x \in K$, hence the lemma is proved. □

Assume $K \subset \mathbb{R}^n$ is a convex body, $M^*(K) = 1$. Apply n symmetrizations to K with respect to $e_1, .., e_n$. Then the resulting body is a 1-unconditional body with mean width 1, and therefore is contained in $c\sqrt{n}B(l_1^n)$. We have proved that using cn symmetrization, $\sqrt{n}B(l_1^n)$ can be transformed to a body which is very close to an Euclidean ball of radius $c\sqrt{\log n}$ (this is just a renormalization of Example 2). In particular, this body has a diameter of $c\sqrt{\log n}$ at most. Since our body was inscribed by $c\sqrt{n}B(l_1^n)$, if we apply to it the same set of cn symmetrizations, we achieve a body with diameter less than $c\sqrt{\log n}$.

Note that we start with a general body K, use only cn deterministic symmetrizations - and reduce the diameter of the body to be just $c\sqrt{\log n}$. At this stage of symmetrizations, the symmetrized body satisfies $\frac{a}{M^*} \leq c\sqrt{\log n}$. Now we can turn to "random" symmetrizations. By theorem 2.4, we need additional $cn \log \frac{a}{M^*} \leq cn \log\log n$ "random" symmetrizations to achieve approximately an Euclidean ball. Therefore we proved the following theorem:

Theorem 3.3 *Let K be a convex body in $\mathbb{R}^n$. For $N = cn\log\log n$, there exist N vectors in S^{n-1} such that if we symmetrize K with respect to those vectors, we obtain a body $\hat{K}$ such that*

$$\frac{1}{2}M^*D \subset \hat{K} \subset 2M^*D$$

Where c is a numerical constant, M^* is the mean width of K

Thus, for a wide class of bodies (bodies with large diameter), there is an essential difference between the number of "random" and "deterministic" Minkowski symmetrizations needed in order to approximate an Euclidean ball.

3.4 Remarks

1. Note that in the case of Example 1, it is clear that at least $n-1$ symmetrizations are needed just to get an n dimensional body, so one cannot expect convergence to a ball in less than $n-1$ symmetrizations.
2. We still don't know whether cn deterministic symmetrizations are sufficient for every convex body in $\mathbb{R}^n$.

I would like to thank Vitali Milman for presenting me to the subject, for helping me in fruitful discussions, and for encouraging me to write this paper.

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