Moment Measures

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A bijection

We present a correspondence between **convex functions** and **Borel measures** on $\mathbb{R}^n$.

The bijection looks natural to us. Still, the main question is:

What is this correspondence good for?

(we can use it to interpolate between measures or convex functions)

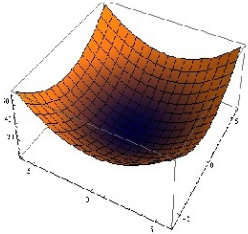
Three points of view for this bijection:

1. The classical Minkowski problem.
2. Toric Kähler manifolds.
3. PDE of Monge-Amperè type, transportation of measure.

Convex functions

We will consider convex functions $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

1. The convex set $\{\psi < +\infty\}$ has a non-empty interior.
2. We have $\lim_{x \to \infty} \psi(x) = +\infty$.



These two conditions imply that ψ grows at least linearly at ∞. They are equivalent to:

$$0 < \int_{\mathbb{R}^n} e^{-\psi} < +\infty.$$

Definition ("The moment measure associated with ψ")

The **moment measure** of ψ is the push-forward of the (log-concave) measure

$$e^{-\psi(x)} dx$$

under the map $x \mapsto \nabla\psi(x) \in \mathbb{R}^n$.

On the definition of moment measures

In short, the *moment measure* of ψ is the measure μ defined by

$$\mu = (\nabla \psi)_* \left(e^{-\psi(x)} dx \right).$$

- The convex function ψ is locally-Lipschitz, hence differentiable a.e., in the interior of $\{\psi < +\infty\}$.
- The map $x \mapsto \nabla \psi(x) \in \mathbb{R}^n$ is well-defined $e^{-\psi(x)} dx$-almost everywhere.
  (in fact, it's a map from a linear space to its dual).
- This "push-forward" means that for any test function f,

$$\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} f(\nabla \psi(x)) e^{-\psi(x)} dx.$$

- The measure μ is a finite, non-zero, Borel measure on $\mathbb{R}^n$.

Examples of moment measures

Example 1 – Discrete measures

Choose vectors $v_1, \dots, v_L \in \mathbb{R}^n$ and scalars $b_1, \dots, b_L$. Set

$$\psi(x) = \max_{i=1,\dots,L} [v_i \cdot x + b_i] \qquad (x \in \mathbb{R}^n).$$

The moment measure of ψ is a discrete measure, supported at the points $v_1, \dots, v_L \in \mathbb{R}^n$.

Example 2 – The uniform measure on the simplex

Let $v_0, \dots, v_n \in \mathbb{R}^n$ be vectors with zero sum that span $\mathbb{R}^n$. Set

$$\psi(x) = (n+1) \log \left[\sum_{i=0}^n \exp\left(\frac{x \cdot v_i}{n+1} \right) \right] \qquad (x \in \mathbb{R}^n).$$

Its moment measure is proportional to the uniform probability measure on the simplex whose vertices are $v_0, \dots, v_n$.

Explanation of the terminology

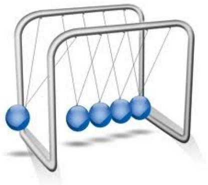
The simplex example is related to the Fubini-Study metric on the complex projective space. We will return to this point.

When $\psi : \mathbb{R}^n \to \mathbb{R}$ is smooth and convex, the set

$$\nabla\psi(\mathbb{R}^n) = \{\nabla\psi(x); x \in \mathbb{R}^n\}$$

is always convex.

- In certain cases, the latter set is a polytope, referred to as the **moment polytope**, and the the map $x \mapsto \nabla\psi(x)$ is essentially the **moment map** of a toric Kähler manifold.
- Our "moment measure", supported on this "moment polytope", perhaps fits in.
- The relation to momentum in Physics is very indirect, but we retain the name...



Restrictions on *moment measures*

Suppose that $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function whose moment measure is μ. Recall that for any test function f,

$$\int_{\mathbb{R}^n} f(y) d\mu(y) = \int_{\mathbb{R}^n} f(\nabla\psi(x)) e^{-\psi(x)} dx.$$

The barycenter of the moment measure

Suppose that ψ is **finite** and **smooth**. Then, for any i,

$$\int_{\mathbb{R}^n} y_i d\mu(y) = \int_{\mathbb{R}^n} \partial_i \psi(x) e^{-\psi(x)} dx = -\int_{\mathbb{R}^n} \partial_i \left(e^{-\psi}\right) = 0.$$

Thus, the barycenter of the moment measure μ is at the origin.

- "Bad Example": Select a convex body $K \subset \mathbb{R}^n$ and a vector $0 \neq \theta \in \mathbb{R}^n$. Consider the convex

$$\psi(x) = \begin{cases} x \cdot \theta & x \in K \\ +\infty & x \notin K \end{cases}$$

Essentially-continuous convex functions

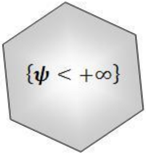
- Some regularity is needed in order to justify the "integration by parts" and to conclude that the barycenter is at zero.

Definition

A convex function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is **essentially-continuous** if

1. It is lower semi-continuous (i.e., the epigraph is a closed, convex set).
2. The set of points where ψ is discontinuous has zero $\mathcal{H}^{n-1}$-measure. Here, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

- Finite convex functions are continuous, hence essentially-continuous.
- This definition is concerned only with the boundary behavior of the function ψ near the set $\partial\{\psi < +\infty\}$. The function $e^{-\psi}$ should vanish a.e. at the boundary.



Essential-continuity seems to be the "right condition"

Suppose that μ is the moment measure of an **essentially continuous** convex function ψ with $0 < \int_{\mathbb{R}^n} e^{-\psi} < \infty$. Then,

1. $0 < \mu(\mathbb{R}^n) < +\infty$.
2. The barycenter of μ lies at the origin.
3. μ is not supported in a proper subspace $E \subsetneq \mathbb{R}^n$ (i.e., it is "truly n-dimensional").

Theorem (Cordero-Erausquin, K., '13)

Suppose that μ is a Borel measure on $\mathbb{R}^n$ satisfying the above three conditions.

Then μ is the moment measure of an essentially-continuous convex function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

This essentially-continuous, convex function ψ is uniquely determined, up to translations.

- No uniqueness without essential-continuity.

Why do the complex geometers care?

The smooth, convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ has moment measure μ with a smooth density ρ if and only if

$$\rho(\nabla\psi(x)) \det \nabla^2\psi(x) = e^{-\psi(x)} \qquad \text{for all } x \in \mathbb{R}^n.$$

Theorem (Wang-Zhu '04, Donaldson '08, Berman-Berndtsson '12)

Suppose $K \subset \mathbb{R}^n$ is a bounded, open, convex set. Suppose that μ is a measure on K, with C^∞-smooth density bounded from below and from above by positive numbers.

Assume that the barycenter of μ lies at the origin.

Then there exists a C^∞-smooth convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ whose moment measure is μ.

Moreover, ψ is uniquely determined, up to translations.

- Geometers probably don't need "generalized solutions"...

Where's "complex geometry"? so far everything is real!

Begin with a smooth, strictly-convex function $\psi : \mathbb{R}^n \to \mathbb{R}$.

1. Extend trivially the function ψ to $\mathbb{C}^n$ by setting

$$\psi(x + \sqrt{-1}y) = \psi(x) \qquad\qquad (x, y \in \mathbb{R}^n).$$

   Then ψ is pluri-sub-harmonic on $\mathbb{C}^n$ (and $\sqrt{-1}\mathbb{Z}^n$-periodic).

2. Consider the complex torus

$$\mathbb{T}^n_{\mathbb{C}} = \text{“}\mathbb{R}^n \times \sqrt{-1}\mathbb{T}^n_{\mathbb{R}}\text{”} = \mathbb{C}^n / (\sqrt{-1}\mathbb{Z}^n).$$

3. So we may view ψ as a p.s.h function on $\mathbb{T}^n_{\mathbb{C}}$. The induced Kähler manifold has a Riemannian tensor (in matrix form)

$$\begin{pmatrix} \nabla^2\psi & 0 \\ 0 & \nabla^2\psi \end{pmatrix}$$

   (this is a $2n \times 2n$ positive-definite matrix, by convexity of ψ).

Kähler-Einstein equations

- So, beginning with a smooth, convex function $\psi : \mathbb{R}^n \to \mathbb{R}$, we define a Kähler manifold X_ψ, with a free torus action.

The interest in "moment measures" stems from:

The Ricci tensor of X_ψ equals half of the metric tensor (i.e., X_ψ is an **Einstein manifold**) if and only if the moment measure of ψ is the uniform measure on a certain convex set.

- This construction is rather general:

Theorem (Atiyah, Guillemin-Sternberg '82; Lerman-Tolman '97)

Any compact, connected toric Kähler manifold (or orbifold) arises this way (i.e., it has a dense open subset of full measure isomorphic to X_ψ, for some convex ψ).

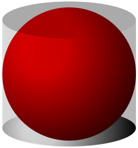
Revisiting the uniform measure on the simplex

This example is a reformulation of the classical fact:

The map

$$\mathbb{CP}^n \ni [Z_0 : \ldots : Z_n] \mapsto \frac{(|Z_0|^2, \ldots, |Z_n|^2)}{\sum_{i=0}^n |Z_i|^2} \in \mathbb{R}^{n+1}$$

pushes forward the uniform measure on $\mathbb{CP}^n$ to the uniform measure on the simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$.



1. Consider the dense, open set $X \subset \mathbb{CP}^n$ of all points $[Z_0 : \ldots : Z_n]$ with $Z_i \neq 0$ for all i.
2. We may reparameterize $X \cong \mathbb{R}^n \times \mathbb{T}^n_{\mathbb{R}}$, and the Fubini-Study metric becomes $g = \begin{pmatrix} \nabla^2\psi & 0 \\ 0 & \nabla^2\psi \end{pmatrix}$ for:

$$\psi(x) = (n+1)\log\left[\sum_{i=0}^n \exp\left(\frac{x \cdot v_i}{n+1}\right)\right] \qquad (x \in \mathbb{R}^n).$$

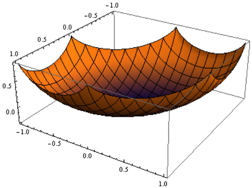
Other examples of moment measures

Given a measure μ, when can we explicitly find the convex function ψ whose moment measure is μ?

1. Radially-symmetric examples in $\mathbb{R}^n$:
   - The moment measure of $\psi(x) = |x|$ is the uniform measure on the sphere S^{n-1}.
   - The moment measure of $\psi(x) = |x|^2/2$ is the standard Gaussian on $\mathbb{R}^n$.
2. Cartesian products and linear images.
3. In the 1D case, there is an explicit inversion formula:

$$\left(\psi^{-1}\right)'\left(-\log\left|\int_x^{\infty} t d\mu(t)\right|\right) = \frac{1}{x}.$$

4. The case of the uniform measure on a **hexagon**: Numerical simulations by Doran, Headrick, Herzog, Kantor and Wiseman '08 and by Bunch and Donaldson '08.



The Legendre transform of ψ

Why do the convex geometers care?

Because of the **Minkowski Problem**.

Theorem (Minkowski, 1897)

Let $u_1, \dots, u_L \in S^{n-1}$ be distinct vectors that span $\mathbb{R}^n$, let $\lambda_1, \dots, \lambda_L > 0$ satisfy

$$\sum_{i=1}^{L} \lambda_i u_i = 0.$$

Then there exists a convex polytope $P \subset \mathbb{R}^n$ with exactly L facets of dimension $n-1$, denoted by $F_1, \dots, F_L$, such that

$$Vol_{n-1}(F_i) = \lambda_i, \qquad F_i \perp u_i. \tag{1}$$

This polytope P is uniquely determined by (1), up to translation.

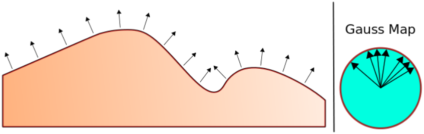
- Minkowski's proof is very much related to the **Brunn-Minkowski inequality**.

A more general formulation of Minkowski's theorem

- The boundary of a convex body $K \subset \mathbb{R}^n$ is a Lipschitz manifold, and the Gauss map (i.e., outer unit normal)

$$N : \partial K \to S^{n-1}$$

is defined for almost any $x \in \partial K$.



- The **surface area measure** of K is the Borel measure μ on the sphere S^{n-1} defined by

$$\mu(A) = \mathcal{H}^{n-1}(N^{-1}(A)).$$

i.e., it is the push-forward of the Lebesgue measure on ∂K under the Gauss map.

When $K \subset \mathbb{R}^n$ is a convex polytope, its surface area measure consists of finitely many atoms.

A more modern formulation (well, it's from the 1930s...)

Minkowski's 1897 theorem is the special case of **polytopes** of:

Theorem (Fenchel & Jessen '38, Aleksandrov '39)

Let μ be a finite, non-zero, Borel measure on S^{n-1}, which is not supported on a great subsphere. Assume that

$$\int_{S^{n-1}} x d\mu(x) = 0.$$

Then there exists a convex body $K \subset \mathbb{R}^n$, unique up to translations, whose surface area measure is μ.

- The case where μ has a continuous density on S^{n-1} was handled by Minkowski back in 1903.
- It is analogous to the moment measure theorem.
- One difference: The moment measure theorem requires only the structure of a **linear space**. Minkowski's problem is inherently Euclidean (not only because of Gauss map).

Moment measures and the Minkowski theorem

The "Moment measure theorem" is a variant of the Minkowski theorem.

- Can one deduce the moment measures thm from the Minkowski thm? how could **essential continuity** appear?

Other variants of the Minkowski problem:

1. The recent **logarithmic Minkowski problem** of Böröczky, Lutwak, Yang and Zhang '12: Push-forward the **cone measure** on ∂K under the Gauss map. [So far: Existence in the even case, no uniqueness].
2. Given a convex function ψ, push forward $s_1(\psi(x))dx$ under the map $s_2(\psi(x))\nabla\psi(x)$. Perhaps our method works for

$$s_1(t) = \left(1 - \frac{t}{k}\right)_+^k \approx e^{-t}, \qquad s_2(t) = t,$$

for various k, not only for the exponential $s_1(t) = \exp(-t)$.

On the proof of our main theorem

Theorem (Cordero-Erausquin, K., '13)

Suppose that μ is a finite Borel measure on $\mathbb{R}^n$ whose barycenter is at the origin, not supported in a subspace.

Then μ is the moment measure of an essentially-continuous convex function, uniquely determined up to translations.

- The proof of Minkowski's theorem used a variational problem based on the **Brunn-Minkowski inequality**:

$$\forall A, B \subset \mathbb{R}^n, \qquad Vol_n\left(\frac{A+B}{2}\right) \geq \sqrt{Vol_n(A) Vol_n(B)}.$$

- We will use a variational problem based on the **Prékopa-Leindler inequality** (an approach suggested already by Berman and Berndtsson, related work by Colesanti and Fragalrà).

Theorem (Prékopa-Leindler '70s. Variant of Brunn-Minkowski)

Suppose $f, g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *are measurable, and for any* $x, y \in \mathbb{R}^n$,

$$h\left(\frac{x+y}{2}\right) \leq \frac{f(x)+g(y)}{2}. \qquad (2)$$

Then,

$$\int_{\mathbb{R}^n} e^{-h} \geq \sqrt{\int_{\mathbb{R}^n} e^{-f} \int_{\mathbb{R}^n} e^{-g}}.$$

- The **Legendre transform** of $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is

$$u^*(y) = \sup_{\substack{x \in \mathbb{R}^n \\ u(x) < +\infty}} [x \cdot y - u(x)]$$

- Let $F, G : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Condition (2) holds with

$$f = F^*, g = G^*, \quad \text{and} \quad h = \left(\frac{F+G}{2}\right)^*.$$

Prékopa-Leindler revisited

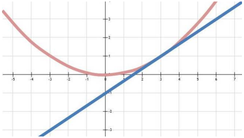
- For $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (finite near the origin) set
$$\mathcal{I}(u) = -\log \int_{\mathbb{R}^n} e^{-u^*}.$$

The Prékopa-Leindler inequality thus implies:

$$\mathcal{I}\left(\frac{u_1 + u_2}{2}\right) \leq \frac{\mathcal{I}(u_1) + \mathcal{I}(u_2)}{2}$$

for any functions $u_1, u_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, finite near the origin.

- The functional $\mathcal{I}$ is **convex**, hence it lies "above tangents".



Question: What is the tangent (i.e., first variation) of $\mathcal{I}$ at u_0?

Answer: It happens to be (minus) the moment measure of u_0^* if and only if the convex function u_0^* is **essentially-continuous**.

An "above-tangent" version of Prékopa

- Our goal: Formulating Prékopa in the form:

$$\mathcal{I}(\varphi_1) \geq \mathcal{I}(\varphi_0) + \left(\delta \mathcal{I}_{\varphi_0}\right)(\varphi_1 - \varphi_0)$$

Theorem (Prékopa revisited – Cordero-Erausquin, K., '13)

Let $\psi_0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex, essentially-continuous with $0 < \int e^{-\psi_0} < \infty$. Let μ be the probability measure proportional to its moment measure. Set $\varphi_0 = \psi_0^*$.

Then for any μ-integrable function φ_1, denoting $\psi_1 = \varphi_1^*$,

$$\log \int_{\mathbb{R}^n} e^{-\psi_0} - \log \int_{\mathbb{R}^n} e^{-\psi_1} \geq \int_{\mathbb{R}^n} (\varphi_0 - \varphi_1)\, d\mu. \qquad (3)$$

Conversely, if ψ_0 and μ are such that (3) holds true for any φ_1, then ψ_0 must be essentially-continuous.

Uniqueness part of the theorem

Suppose that ψ_0 and ψ_1 are convex functions, essentially-continuous, with the same moment measure μ.

- Then, by the "above-tangent" version of Prékopa:

$$\log \int_{\mathbb{R}^n} e^{-\psi_0} - \log \int_{\mathbb{R}^n} e^{-\psi_1} \geq \int_{\mathbb{R}^n} (\varphi_0 - \varphi_1)\, d\mu$$

and also

$$\log \int_{\mathbb{R}^n} e^{-\psi_1} - \log \int_{\mathbb{R}^n} e^{-\psi_0} \geq \int_{\mathbb{R}^n} (\varphi_1 - \varphi_0)\, d\mu.$$

So we are at the equality case of (our version of) Prékopa!

- The equality case in Prékopa is well-studied (since Dubuc '77), and it implies that there exists $x_0 \in \mathbb{R}^n$ with

$$\psi_0(x) = \psi_1(x - x_0) \qquad \text{for all } x \in \mathbb{R}^n.$$

Existence part of the theorem

- Given a probability measure μ on $\mathbb{R}^n$ with barycenter at the origin, we will maximize

$$\log \int_{\mathbb{R}^n} e^{-\varphi^*} - \int_{\mathbb{R}^n} \varphi d\mu$$

over all μ-integrable functions $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

- The (convex) maximizer φ would satisfy (set $\psi = \varphi^*$),

$$\log \int_{\mathbb{R}^n} e^{-\psi} - \log \int_{\mathbb{R}^n} e^{-\psi_1} \geq \int_{\mathbb{R}^n} (\varphi - \varphi_1)\, d\mu,$$

for any test function φ_1 (with $\psi_1 = \varphi_1^*$). Hence ψ must be essentially-continuous, with moment measure μ.

Basically, the supremum is finite thanks to the Santaló inequality.

- Only convex functions φ are relevant (because $\varphi^{**} \leq \varphi$ and $(\varphi^{**})^* = \varphi^*$).
- Compactness in the space of convex functions is relatively easy (even in local-Lipschitz topology).

Optimal transportation of measure

Given a measure μ on $\mathbb{R}^n$ with barycenter at the origin, we solved

$$\mu = (\nabla\psi)_* (\nu_\psi)$$

where $\nu_\psi = e^{-\psi(x)}dx$.

Theorem (Brenier '87, McCann '95)

Let μ, ν be probability measures on $\mathbb{R}^n$, with ν absolutely continuous. Then there exists a convex function ψ on $\mathbb{R}^n$ with

$$\mu = (\nabla\psi)_* \nu.$$

The map $x \mapsto \nabla\psi$ is uniquely defined ν-a.e., and it's **optimal***:*

$$\int_{\mathbb{R}^n} |x - \nabla\psi(x)|^2 d\nu(x) = \inf_T \int_{\mathbb{R}^n} |x - T(x)|^2 d\nu(x)$$

where the infimum runs over all maps that push-forward ν to μ.

Moment measures vs. Optimal transport

Optimal transport problem:

Given μ, ν, find ψ such that $\mu = (\nabla\psi)_* \nu$.

- This problem depends on the Euclidean structure. Stretching linearly both μ and ν in the same manner may completely alter the transporting map $T = \nabla\psi$.

  (In some sense, μ and ν live in dual spaces. If we stretch ν, then we are expected to shrink μ...).

Moment measure problem:

Given μ, find ψ such that $\mu = (\nabla\psi)_* \nu_\psi$, where $\nu_\psi = e^{-\psi(x)} dx$.

- Euclidean structure is irrelevant. We may use this to construct transport maps between any μ and any ν in a **linearly-invariant manner**! (assuming zero barycenters).

The end

Thank you!

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