On convex perturbations with a bounded isotropic constant

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Abstract

Let $K \subset \mathbb{R}^n$ be a convex body and $\varepsilon > 0$. We prove the existence of another convex body $K' \subset \mathbb{R}^n$, whose Banach-Mazur distance from K is bounded by $1 + \varepsilon$, such that the isotropic constant of K' is smaller than $\frac{c}{\sqrt{\varepsilon}}$, where $c > 0$ is a universal constant. As an application of our result, we present a slight improvement on the best general upper bound for the isotropic constant, due to Bourgain.

1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex body, i.e. a compact convex set with a non-empty interior. We say that K is isotropic or that K is in isotropic position, if $Vol(K) = 1$, the barycenter of K is at the origin, and

$$\int_K x_i x_j dx = L_K^2 \delta_{i,j}, \tag{1}$$

for some number $L_K > 0$, where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ are coordinates in $\mathbb{R}^n$, and $\delta_{i,j}$ is Kronecker's delta. When K is isotropic, we say that L_K as in (1) is the isotropic constant of K. It is well-known (e.g., [21]) that for any convex body $K \subset \mathbb{R}^n$, there exists an affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(K)$ is in isotropic position. This affine map T is unique, up to left multiplication by an orthogonal transformation (e.g., [21]). We define the isotropic constant of an arbitrary convex body $K \subset \mathbb{R}^n$ to be $L_K := L_{T(K)}$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is any affine map such that $T(K)$ is in isotropic position. The isotropic constant of K is well-defined, and is invariant under affine transformations. See below for a more direct definition of the isotropic constant of a non-isotropic convex body.

Among all convex bodies in $\mathbb{R}^n$, ellipsoids possess the minimal isotropic constant (this fact essentially goes back to Blaschke [3]. A proof appears, e.g., in [21]). It is straightforward to verify that c_n, the isotropic constant of an n-dimensional ellipsoid, satisfies $c_n \to \frac{1}{\sqrt{2\pi e}}$ when $n \to \infty$. Thus the minimal possible value of the isotropic constant of a convex body in $\mathbb{R}^n$ is well understood. In contrast, it is not even known what the order of

*The author is a Clay Research Fellow, and was also supported by NSF grant #DMS-0456590.

magnitude of the maximal isotropic constant is, among all convex bodies in $\mathbb{R}^n$. This is related to a basic open problem in asymptotic convex geometry, the validity of the "hyperplane conjecture" (e.g., [1, 4, 21]). The hyperplane conjecture suggests that for any convex body $K \subset \mathbb{R}^n$ of volume one, there exists an affine hyperplane $H \subset \mathbb{R}^n$ such that

$$Vol_{n-1}(K \cap H) > c$$

where $c > 0$ is a universal constant.

An equivalent formulation of the hyperplane conjecture reads as follows: For any dimension n, and any convex body $K \subset \mathbb{R}^n$, the isotropic constant L_K is bounded from above by some universal constant (see [21] for the aforementioned equivalence, and for additional equivalent, plausible, formulations of the hyperplane conjecture). Furthermore, any upper bound on the isotropic constant implies a lower bound on the volume of hyperplane sections, as follows: For any convex body $K \subset \mathbb{R}^n$ of volume one, there exists a hyperplane $H \subset \mathbb{R}^n$ with $Vol_{n-1}(K \cap H) > \frac{c}{L_K}$, where $c > 0$ is a universal constant (see e.g., [21]).

The hyperplane conjecture was verified for several large classes of convex sets: Unconditional convex bodies [4, 21], zonoids, duals to zonoids, [2] (see also [20]), bodies with a bounded outer volume ratio [21], random bodies [18], unit balls of Schatten norms [19], and others (e.g., [15]). A reduction of the problem to the case of bodies with a bounded volume ratio appears in [7, 8]. However, the best general bound known to date is Bourgain's estimate [5],

$$L_K < cn^{\frac{1}{4}} \log(n+1) \tag{2}$$

for any convex body $K \subset \mathbb{R}^n$. Bourgain's argument formally deals only with centrally-symmetric sets. See [23] for the non-symmetric case, or the last remark in [17] for a reduction of the general problem to the case of centrally-symmetric convex bodies. Additional proofs of the bound (2) were presented by Dar [10] and by Bourgain [6].

For two convex bodies $K_1, K_2 \subset \mathbb{R}^n$, we define their geometric distance as

$$d(K_1, K_2) = \inf \left\{ ab; a, b > 0, \exists x, y \in \mathbb{R}^n, \ \frac{1}{a}(K_1 + x) \subset K_2 + y \subset b(K_1 + x) \right\}.$$

Thus, the distance between K_1 and K_2 is small if, once we apply suitable translations, the body K_1 is close to a dilation of the body K_2. Clearly, $d(K_1, K_2)$ is not larger than the Banach-Mazur distance between K_1 and K_2 (see e.g., [13, page 767]). Our main result is the following theorem.

Theorem 1.1 *Let $K \subset \mathbb{R}^n$ be a convex body, and let $\varepsilon > 0$. Then there exists a convex body $T \subset \mathbb{R}^n$ such that*

1. $d(K, T) < 1 + \varepsilon$.
2. $L_T < \frac{c}{\sqrt{\varepsilon}}$.

Here, $c > 0$ is a universal constant.

A weaker version of Theorem 1.1, with a logarithmic factor, was obtained in [17]. A direct consequence of the recent Paouris theorem [25, 26], is that if $K, T \subset \mathbb{R}^n$ are convex bodies and $d(K,T) < 1 + \frac{1}{\sqrt{n}}$, then L_K and L_T have the same order of magnitude. Thus, the case $\varepsilon = \frac{1}{\sqrt{n}}$ in Theorem 1.1 entails the following slight improvement of (2).

Corollary 1.2 *Let $K \subset \mathbb{R}^n$ be a convex body. Then*

$$L_K < cn^{\frac{1}{4}},$$

where $c > 0$ is a universal constant.

The rest of the paper is organized as follows: In Section 2 we review some known results related to log-concave functions. Section 3 contains a description of our main tool, a certain transportation of measure. Theorem 1.1 and Corollary 1.2 are proven in Section 4.

Throughout this paper, the letters c, C, c_1, c' etc. denote positive universal constants, whose values are not necessarily the same in different appearances. We would like to emphasize that these constants are, in particular, independent of the dimension n. We use the notation $A \asymp B$ to abbreviate $c_1 A < B < c_2 A$, for $c_1, c_2 > 0$, universal constants.

2 Log-concave functions

In this section we summarize some facts, mostly standard, on log-concave functions. A function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if for any $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$,

$$f(\lambda x + (1-\lambda)y) \geq f^{\lambda}(x) f^{1-\lambda}(y),$$

(i.e., $\log f$ is concave). A log-concave function is always measurable. A log-concave function f with $0 < \int f < \infty$ has moments of all orders. In particular its barycenter

$$bar(f) = \frac{\int_{\mathbb{R}^n} x f(x) dx}{\int_{\mathbb{R}^n} f(x) dx} \in \mathbb{R}^n$$

is well-defined, as well as its inertia matrix $Cov(f) = (Cov(f)_{i,j})_{i,j=1,\ldots,n}$, whose entries are

$$Cov(f)_{i,j} = \frac{\int_{\mathbb{R}^n} x_i x_j f(x) dx}{\int_{\mathbb{R}^n} f(x) dx} - \frac{\int_{\mathbb{R}^n} x_i f(x) dx}{\int_{\mathbb{R}^n} f(x) dx} \frac{\int_{\mathbb{R}^n} x_j f(x) dx}{\int_{\mathbb{R}^n} f(x) dx}.$$

We also refer to $Cov(f)$ as the covariance matrix of f. For a log-concave function $f : \mathbb{R}^n \to [0, \infty)$ with $0 < \int f < \infty$, we define its isotropic constant as

$$L_f = \left(\frac{\sup_{x \in \mathbb{R}^n} f(x)}{\int_{\mathbb{R}^n} f(x) dx} \right)^{\frac{1}{n}} (\det Cov(f))^{\frac{1}{2n}}. \quad (3)$$

It is straightforward to verify that $L_f = L_{f \circ T}$ for any affine map $T : \mathbb{R}^n \to \mathbb{R}^n$, and also that $L_f = L_{af}$ for any $a > 0$. We say that f is in isotropic position if $\sup_{x\in\mathbb{R}^n} f(x) = \int f(x)dx = 1$ and $Cov(f)$ is a scalar matrix. In this case,

$$Cov(f) = L_f^2 Id,$$

where Id is the identity matrix.

We have already defined the isotropic constant of a convex body $K \subset \mathbb{R}^n$ in Section 1. This definition is consistent with (3) in the following sense: Denote by 1_K the characteristic function of K, a log-concave function. Then $L_{1_K} = L_K$.

Let us describe yet another characterization of the isotropic constant. We denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the standard Euclidean norm and scalar product in $\mathbb{R}^n$, respectively. We also write $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ for the unit sphere. Suppose that $f : \mathbb{R}^n \to [0,\infty)$ is a log-concave function with $0 < \int f < \infty$. Then, as is proven in [21],

$$nL_f^2 = \inf_{T:\mathbb{R}^n\to\mathbb{R}^n} \left(\frac{\sup_{x\in\mathbb{R}^n} f(x)}{\int_{\mathbb{R}^n} f(x)dx} \right)^{\frac{2}{n}} \int_{\mathbb{R}^n} |Tx|^2 f(x) \frac{dx}{\int f(y)dy}, \tag{4}$$

where the infimum runs over all volume-preserving affine maps $T : \mathbb{R}^n \to \mathbb{R}^n$.

The significance of log-concave functions stems mainly from the Brunn-Minkowski type inequalities. Suppose that $f : \mathbb{R}^n \to [0,\infty)$ is a log-concave function. Then, as follows from the Prékopa-Leindler inequality (e.g., first pages of [27]), for any compact sets $A, B \subset \mathbb{R}^n$

$$\int_{\frac{A+B}{2}} f(x)dx \geq \sqrt{\int_A f(x)dx \int_B f(x)dx}$$

where $\frac{A+B}{2} = \left\{ \frac{x+y}{2}; x \in A, y \in B \right\}$. Consequently, log-concave functions enjoy some concentration properties. For instance, Borell's lemma (e.g., [13, Page 717]) implies that for any $\theta \in \mathbb{R}^n$ and $p \geq 1$,

$$\int_{\mathbb{R}^n} |\langle x,\theta\rangle| f(x) \frac{dx}{\int f} \leq \left(\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^p f(x) \frac{dx}{\int f} \right)^{\frac{1}{p}} < cp \int_{\mathbb{R}^n} |\langle x,\theta\rangle| f(x) \frac{dx}{\int f}, \tag{5}$$

where $c > 0$ is a universal constant. Another immediate consequence of Borell's lemma reads as follows: Let $f : \mathbb{R}^n \to [0,\infty)$ be log-concave with $0 < \int f < \infty$, and denote by M the median of the Euclidean norm $|\cdot|$ with respect to f. That is, $\int_{|x|<M} f(x)dx = \frac{1}{2}\int_{\mathbb{R}^n} f(x)dx$. Then by Borell's lemma,

$$\int_{\mathbb{R}^n} |x|^2 f(x) \frac{dx}{\int f} \asymp M^2. \tag{6}$$

Next, we quote the results of K. Ball from [1]. The following lemma is precisely the content of (6), (7) in [1].

Lemma 2.1 *Suppose $g, h, m : [0,\infty) \to [0,\infty)$ are three measurable functions, such that for any $r, s > 0$,*

$$m\left(\frac{2}{\frac{1}{r}+\frac{1}{s}}\right) \geq g(r)^{\frac{s}{r+s}} h(s)^{\frac{r}{r+s}}. \tag{7}$$

Let $p \geq 1$, and denote

$$A = \int_0^\infty g(r) r^{p-1} dr, \quad B = \int_0^\infty h(r) r^{p-1} dr, \quad S = \int_0^\infty m(r) r^{p-1} dr.$$

Then,

$$S \geq \frac{2}{\frac{1}{A} + \frac{1}{B}}.$$

The next theorem is also due to K. Ball [1]. Since the theorem is proven in [1] only for even functions, for the reader's convenience we sketch the straightforward adaptation to the non-even case below.

Theorem 2.2 *Let $f : \mathbb{R}^n \to [0,\infty)$ be a log-concave function with $f(0) > 0$, and let $p \geq 1$. Then the set*

$$K_p(f) = \left\{ x \in \mathbb{R}^n; \int_0^\infty f(rx) r^{p-1} dr \geq \frac{f(0)}{p} \right\}$$

is convex.

Proof: Let $x, y \in K_p(f)$, and denote $g(r) = f(rx), h(r) = f(ry)$. Then,

$$A := \int_0^\infty g(r) r^{p-1} dr \geq \frac{f(0)}{p}, \quad B := \int_0^\infty h(r) r^{p-1} dr \geq \frac{f(0)}{p}.$$

We need to show that $\frac{x+y}{2} \in K_p(f)$. Equivalently, if $m(r) = f\left(r\frac{x+y}{2}\right)$, then it is sufficient to prove that

$$S := \int_0^\infty m\left(r\right) r^{p-1} dr \geq \frac{f(0)}{p}.$$

Let $r, s > 0$. Set $\lambda = \frac{s}{r+s}, u = rx, v = sy$, and use the log-concavity of f to obtain

$$m\left(\frac{2rs}{r+s}\right) = f(\lambda u + (1-\lambda)v) \geq f^\lambda(u) f^{1-\lambda}(v) = g(r)^{\frac{s}{r+s}} h(s)^{\frac{r}{r+s}}.$$

Thus g, h, m satisfy requirement (7) of Lemma 2.1. From the conclusion of that lemma, $S \geq \frac{f(0)}{p}$, and the theorem follows. □

The set

$$K_p(f) = \{x \in \mathbb{R}^n; p \int_0^\infty f(rx) r^{p-1} dr \geq f(0)\},$$

defined for any Borel measurable function $f : \mathbb{R}^n \to [0,\infty)$, will play an important rôle later on. Note that $0 \in K_p(f)$ for any $p \geq 1$, as $\int_0^\infty f(0) r^{p-1} dr = \infty \geq f(0)$. Recall that for a set $K \subset \mathbb{R}^n$ we denote by 1_K the characteristic function of K.

Lemma 2.3 *Let $K \subset \mathbb{R}^n$ be a convex body containing the origin. Let $p \geq 1$. Then,*

$$K_p(1_K) = K.$$

Proof: For any $x \in \mathbb{R}^n$ denote $r_x = sup\{r \geq 0; rx \in K\}$, and observe that

$$p \int_0^\infty 1_K(rx) r^{p-1} dr = \int_0^{r_x} p r^{p-1} dr = r_x^p.$$

Thus, $x \in K_p(1_K)$ if and only if $r_x \geq 1$, which holds if and only if $x \in K$. $\square$

Lemma 2.4 *Let $f, g : \mathbb{R}^n \to [0, \infty)$ be two measurable functions with $f(0) = g(0) > 0$, let $p \geq 1$, and denote $m = \sup_{g(x)>0} \frac{f(x)}{g(x)}$. Then,*

$$K_p(f) \subset m^{\frac{1}{p}} K_p(g). \tag{8}$$

Proof: Suppose that $x \in K_p(f)$. Then,

$$\int_0^\infty g\left(rm^{-\frac{1}{p}}x\right) r^{p-1} dr = \int_0^\infty m g(rx) r^{p-1} dr \geq \int_0^\infty f(rx) r^{p-1} dr \geq \frac{f(0)}{p} = \frac{g(0)}{p}.$$

Therefore $m^{-\frac{1}{p}} x \in K_p(g)$ and $x \in m^{\frac{1}{p}} K_p(g)$. This proves (8). $\square$

The next lemma is due to Fradelizi [12, Theorem 4].

Lemma 2.5 *Let $g : \mathbb{R}^n \to [0, \infty)$ be a log-concave function such that $0 < \int_{\mathbb{R}^n} g < \infty$. Let $x_0 = bar(g)$ be the barycenter of g. Then,*

$$\sup_{x \in \mathbb{R}^n} g(x) \leq e^n g(x_0).$$

The following lemma is a standard one-dimensional computation. It is almost identical, e.g., to [17, Lemma 2.4]. For completeness, we sketch its easy roof.

Lemma 2.6 *Let $n \geq 1$ be an integer, and let $g : [0, \infty) \to [0, \infty)$ be a log-concave function with $g(0) = 1, 0 < \int_0^\infty g(t) t^{n-1} dt < \infty$ and $\sup_x g(x) \leq e^n$. Then*

$$c_1 < \frac{n^{\frac{n+1}{n}}}{e(n+1)} \leq \frac{\int_0^\infty g(t) t^n dt}{\left(\int_0^\infty g(t) t^{n-1} dt\right)^{\frac{n+1}{n}}} \leq \frac{n!}{((n-1)!)^{\frac{n+1}{n}}} < c_2, \tag{9}$$

where $c_1, c_2 > 0$ are universal constants.

Proof: Set $A = \int_0^\infty g(t) t^{n-1} dt$, and let $r > 0$ be such that $\int_0^r e^n t^{n-1} dt = A$. Since $g(t) \leq e^n$ for any $t > 0$, then each $x > 0$ satisfies

$$\int_x^\infty g(t) t^{n-1} dt = A - \int_0^x g(t) t^{n-1} dt \geq A - \int_0^{\min\{r,x\}} e^n t^{n-1} dt = \int_{\min\{r,x\}}^r e^n t^{n-1} dt.$$

Consequently, by integrating by parts we obtain

$$\int_0^\infty g(t) t^n dt = \int_0^\infty \int_x^\infty g(t) t^{n-1} dt dx \geq \int_0^\infty \int_{\min\{r,x\}}^r e^n t^{n-1} dt dx = \int_0^r e^n t^n dt = \frac{(nA)^{\frac{n+1}{n}}}{e(n+1)}.$$

This proves the left hand side of (9). Next we focus our attention on the right hand side of (9). Select $a > 0$ such that

$$\int_0^\infty e^{-at} t^{n-1} dt = \int_0^\infty g(t) t^{n-1} dt = A. \tag{10}$$

By (10), it is impossible that always $g(t) < e^{-at}$ or always $g(t) > e^{-at}$. Hence necessarily $t_0 = \inf\{t > 0; e^{-at} \geq g(t)\}$ is finite. The function $-\log g$ is convex and vanishes at zero, therefore $\tilde{g}(t) = \frac{-\log g(t)}{t}$ is non-decreasing. Thus $\tilde{g}(t) \leq a$ for $t < t_0$, and $\tilde{g}(t) \geq a$ for $t > t_0$. Equivalently, $g(t) \geq e^{-at}$ for $t < t_0$ and $g(t) \leq e^{-at}$ for $t > t_0$. We conclude that for $x \geq t_0$,

$$\int_x^\infty g(t) t^{n-1} dt \leq \int_x^\infty e^{-at} e^{n-1} dt. \tag{11}$$

Using (10) we deduce that (11) holds also for $0 < x \leq t_0$. Thus (11) holds for all $x > 0$. By integrating by parts, as before, we conclude that

$$\int_0^\infty g(t) t^n dt = \int_0^\infty \int_x^\infty g(t) t^{n-1} dt dx \leq \int_0^\infty \int_x^\infty e^{-at} t^{n-1} dt dx = \int_0^\infty e^{-at} t^n dt.$$

To establish the right hand side of (9), observe that $\int_0^\infty t^n e^{-at} dt = \left(\frac{A}{(n-1)!}\right)^{\frac{n+1}{n}} (n+1)!$ and that $((n-1)!)^{1/n} \asymp n$. The proof is complete. $\square$

Next we compare, along the lines of [1] and [21], some volumetric characteristics of the function f and the body $K_{n+1}(f)$.

Lemma 2.7 *Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $0 < \int f < \infty$, and suppose that its barycenter lies at the origin, i.e. $\int x f(x) = 0$. Then also the barycenter of $K_{n+1}(f)$ lies at the origin. Furthermore,*

$$cL_f < L_{K_{n+1}(f)} < CL_f$$

where $c, C > 0$ are universal constants.

Proof: According to Lemma 2.5, necessarily $f(0) > 0$, since otherwise $f \equiv 0$. Both the assumptions and the conclusions of the lemma are invariant under replacement of f by af, for any $a > 0$. Thus we may assume that $f(0) = 1$. For $\theta \in S^{n-1}$ denote

$$r_\theta = \sup\{t > 0; t\theta \in K_{n+1}(f)\} = \sup\left\{t > 0; (n+1)\int_0^\infty f(rt\theta) r^n dr \geq 1\right\}. \tag{12}$$

From (12) we conclude that for any $\theta \in S^{n-1}$,

$$r_\theta = \left((n+1)\int_0^\infty f(r\theta) r^n dr\right)^{\frac{1}{n+1}}. \tag{13}$$

Integration in polar coordinates then yields

$$\begin{aligned}
&\int_{K_{n+1}(f)} \langle x, \theta \rangle dx \qquad (14)\\
&= \int_{S^{n-1}} \int_0^{r_\theta} \langle ry, \theta \rangle r^{n-1} dr dy = \frac{1}{n+1} \int_{S^{n-1}} \langle y, \theta \rangle r_y^{n+1} dy.\\
&= \int_0^\infty \int_{S^{n-1}} f(ry) \langle y, \theta \rangle r^n dr dy = \int_{\mathbb{R}^n} \langle x, \theta \rangle f(x) dx = 0,
\end{aligned}$$

since the barycenter of f lies at the origin. We deduce from (14) that the barycenter of $K_{n+1}(f)$ lies at the origin. Furthermore, by arguing as in (14), we conclude that for any $\theta \in S^{n-1}$,

$$\int_{K_{n+1}(f)} |\langle x, \theta \rangle| dx = \int_{\mathbb{R}^n} |\langle x, \theta \rangle| f(x) dx. \qquad (15)$$

We integrate by polar coordinates and use (13) to obtain

$$Vol(K_{n+1}(f)) = \frac{1}{n} \int_{S^{n-1}} r_\theta^n d\theta = \frac{(n+1)^{\frac{n}{n+1}}}{n} \int_{S^{n-1}} \left(\int_0^\infty f(r\theta) r^n dr \right)^{\frac{n}{n+1}} d\theta. \qquad (16)$$

According to Lemma 2.5, for any $x \in \mathbb{R}^n$,

$$f(x) \leq e^n. \qquad (17)$$

Based on (17), Lemma 2.6 implies that for any $\theta \in S^{n-1}$,

$$\left(\int_0^\infty f(r\theta) r^n dr \right)^{\frac{n}{n+1}} \asymp \int_0^\infty f(r\theta) r^{n-1} dr \qquad (18)$$

(note that the quantities in (18) are finite; Since $0 < \int f < \infty$, then any restriction of f to a straight line has a finite integral). Combining (16) and (18), we get

$$Vol(K_{n+1}(f)) \asymp \int_{S^{n-1}} \int_0^\infty f(r\theta) r^{n-1} dr d\theta = \int_{\mathbb{R}^n} f(x) dx. \qquad (19)$$

Next, (15) and (19) imply that for any $\theta \in S^{n-1}$,

$$\int_{K_{n+1}(f)} |\langle x, \theta \rangle| \frac{dx}{Vol(K_{n+1}(f))} \asymp \int_{\mathbb{R}^n} |\langle x, \theta \rangle| f(x) \frac{dx}{\int f}.$$

Using (5), we deduce that for any $\theta \in S^{n-1}$,

$$\int_{K_{n+1}(f)} \langle x, \theta \rangle^2 \frac{dx}{Vol(K_{n+1}(f))} \asymp \int_{\mathbb{R}^n} \langle x, \theta \rangle^2 f(x) \frac{dx}{\int f}. \qquad (20)$$

The estimate (20) entails that the inertia matrices $Cov(f)$ and $Cov(K_{n+1}(f)) := Cov(1_{K_{n+1}(f)})$ satisfy

$$c_1 Cov(f) < Cov(K_{n+1}(f)) < c_2 Cov(f) \qquad (21)$$

in the sense of positive definite matrices, for some universal constants $c_1, c_2 > 0$. According to (19), clearly $Vol(K_{n+1}(f))^{\frac{1}{n}} \asymp (\int f)^{\frac{1}{n}}$. Since $f(0) = 1$, we conclude by (3), (17) and (21) that

$$L_f \asymp L_{K_{n+1}(f)}.$$

□

This section's results are consolidated in the following lemma.

Lemma 2.8 *Let $K \subset \mathbb{R}^n$ be a convex body, and let $f : K \to (0, \infty)$ be a log-concave function. Suppose that $m > 1$ satisfies*

$$\sup_{x \in K} f(x) \leq m^n \inf_{x \in K} f(x).$$

Then there exist a convex set $T \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ such that

1. $\frac{1}{m}(T - x_0) \subset K - x_0 \subset m(T - x_0)$.
2. $c_1 L_f < L_T < c_2 L_f$ *where $c_1, c_2 > 0$ are universal constants.*

Proof: Suppose first that the barycenter of f lies at the origin. Multiplying f by a positive constant, if necessary, we may assume that $f(0) = 1$. Let $T = K_{n+1}(f) = \left\{x \in \mathbb{R}^n; (n+1)\int_0^\infty f(rx) r^n dr \geq f(0)\right\}$. The set T is convex according to Theorem 2.2. According to our assumptions,

$$\sup_{1_K(x) > 0} \frac{f(x)}{1_K(x)} \leq m^n \leq m^{n+1}, \quad \sup_{f(x) > 0} \frac{1_K(x)}{f(x)} \leq m^n \leq m^{n+1}. \tag{22}$$

Recall that $K_{n+1}(1_K) = K$ by Lemma 2.3. Lemma 2.4 and (22) entail that

$$\frac{1}{m} T \subset K \subset mT.$$

Moreover, according to Lemma 2.7, the barycenter of T lies at the origin and

$$L_T = L_{K_{n+1}(f)} \asymp L_f.$$

Thus the lemma is proven, with $x_0 = 0$, in the case where the barycenter of f is the origin. The general case is easily reduced to the case where the barycenter of f lies at the origin. Indeed, set $x_0 = bar(f) = \frac{\int x f(x) dx}{\int f(x) dx}$, and consider the log-concave function $\tilde{f}(x) = f(x + x_0)$, that is supported on $\tilde{K} = K - x_0$. Since the barycenter of $\tilde{f}$ lies at the origin, we know that $\tilde{T} = K_{n+1}(\tilde{f})$ satisfies $L_{\tilde{T}} \asymp L_{\tilde{f}} = L_f$, and also $\frac{1}{m}\tilde{T} \subset \tilde{K} \subset m\tilde{T}$. Therefore $T = \tilde{T} + x_0$ satisfies

$$\frac{1}{m}(T - x_0) \subset K - x_0 \subset m(T - x_0).$$

Since $L_T = L_{\tilde{T}} \asymp L_f$, the lemma is proven. □

3 Transportation map

Let $K \subset \mathbb{R}^n$ be a convex body. We consider the following function $F_K : \mathbb{R}^n \to \mathbb{R}$,

$$F_K(x) = \log \int_K e^{\langle x,y \rangle} \frac{dy}{Vol(K)}.$$

Our use of the function F_K is inspired by a remark by Gromov in [14]. The function F_K also resembles the partition functions of statistical mechanics. It might be useful to note that F_K is defined, in principle, on the dual space to $\mathbb{R}^n$, and that there is no need to fix a scalar product in $\mathbb{R}^n$ in order to define F_K. A few simple properties of F_K are established in the next lemma.

Lemma 3.1 *Suppose $K \subset \mathbb{R}^n$ is a convex body. Then F_K is C^2-smooth, strictly convex, and $Im(\nabla F_K) := \{\nabla F_K(x); x \in \mathbb{R}^n\}$ satisfies*

$$Im(\nabla F_K) = int(K),$$

the interior of K. Furthermore, for any $x \in \mathbb{R}^n$ denote by $\mu_{K,x}$ the probability measure on $\mathbb{R}^n$ whose density at $y \in \mathbb{R}^n$ equals

$$\frac{e^{\langle x,y \rangle} 1_K(y)}{\int_K e^{\langle x,z \rangle} dz}.$$

Then, for any $x \in \mathbb{R}^n$,

$$\nabla F_K(x) = bar(\mu_{K,x}) = \int_{\mathbb{R}^n} y \ d\mu_{K,x}(y),$$

the barycenter of $\mu_{K,x}$. Additionally,

$$Hess(F_K)(x) = Cov(\mu_{K,x}) = \int_{\mathbb{R}^n} y \otimes y \ d\mu_{K,x}(y) - \left[\int_{\mathbb{R}^n} y \ d\mu_{K,x}(y) \right] \otimes \left[\int_{\mathbb{R}^n} y \ d\mu_{K,x}(y) \right],$$

the covariance matrix of $\mu_{K,x}$. Here $Hess$ stands for Hessian, and $x \otimes x$ stands for the matrix whose entries are $(x_i x_j)_{i,j=1,\dots,n}$.

Proof: The smoothness of F_K is clear, as we are integrating a smooth function on a compact set. The strict convexity of F_K follows from the Cauchy-Schwartz inequality, since for any $x_1 \neq x_2 \in \mathbb{R}^n$,

$$\int_K e^{\left\langle \frac{x_1+x_2}{2}, y \right\rangle} \frac{dy}{Vol(K)} < \sqrt{\int_K e^{\langle x_1,y \rangle} \frac{dy}{Vol(K)}} \sqrt{\int_K e^{\langle x_2,y \rangle} \frac{dy}{Vol(K)}}. \quad (23)$$

Taking the logarithm of both sides in (23), we obtain that $F_K\left(\frac{x_1+x_2}{2}\right) < \frac{F(x_1)+F(x_2)}{2}$. Next, we differentiate under the integral sign to get that for any $x \in \mathbb{R}^n$,

$$\nabla F_K(x) = \frac{\int_K y e^{\langle x,y \rangle} dy}{\int_K e^{\langle x,y \rangle} dy} = \int_{\mathbb{R}^n} y \ d\mu_{K,x}(y). \quad (24)$$

Thus $\nabla F_K(x)$ is the barycenter of the measure $\mu_{K,x}$. Since $\mu_{K,x}$ is supported on the compact, convex set K, its barycenter $bar(\mu_{K,x}) \in K$. Therefore

$$\nabla F_K(x) \in K \quad \text{for any } x \in \mathbb{R}^n. \tag{25}$$

Next, let $y \in \partial K$ be an extremal point of K (i.e. there is no interval centered at y that is contained in K). There exists a supporting hyperplane for K, such that y is its only contact point with K. Thus, there exist $x \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\langle x, y \rangle = b, \quad \forall z \in K,\ z \neq y \Rightarrow \langle x, z \rangle < b.$$

Consider the measure $\mu_{K,rx}$ for large $r > 0$. Its density is proportional to $z \mapsto e^{r\langle x,z \rangle} 1_K(z)$, and it attains its unique maximum at y. Furthermore, it is straightforward to verify that as $r \to \infty$,

$$\mu_{K,rx} \stackrel{w^*}{\longrightarrow} \delta_y$$

where δ_y is the delta measure supported on y. Therefore, by (24),

$$\nabla F_K(rx) \stackrel{r \to \infty}{\longrightarrow} y$$

and $y \in \overline{Im(\nabla F_K)}$. Since y was an arbitrary extremal point of K, we conclude that $\overline{Im(\nabla F_K)}$ contains all extremal points of K. Recall that $Im(\nabla F_K)$ is convex [14, Lemma 2.3], and that K is the convex hull of its extremal points. Therefore,

$$K \subset \overline{Im(\nabla F_K)}. \tag{26}$$

Since $Im(\nabla F_K)$ is open (e.g., Lemma 2.2 in [14]), by combining (25) and (26) we conclude that $Im(\nabla F_K)$ is the interior of K. This proves the first part of the lemma. It remains to compute the Hessian matrix of F_K. Fix $1 \leq i, j \leq n$. Differentiation of (24) yields,

$$\frac{\partial^2 F_K(x)}{\partial x_i \partial x_j} = \frac{\int_K y_i y_j e^{\langle x,y \rangle} dy \int_K e^{\langle x,y \rangle} dy - \int_K y_i e^{\langle x,y \rangle} dy \int_K y_j e^{\langle x,y \rangle} dy}{\left(\int_K e^{\langle x,y \rangle} dy \right)^2}$$

and the lemma is proven. □

Suppose μ_1, μ_2 are two Borel measures on $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable map. We say that T transports μ_1 to μ_2 if for any Borel set $A \subset \mathbb{R}^n$,

$$\mu_2(A) = \mu_1(T^{-1}(A)).$$

Equivalently, for any continuous, non-negative function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \varphi(x) d\mu_2(x) = \int_{\mathbb{R}^n} \varphi(Tx) d\mu_1(x).$$

Lemma 3.2 *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a strictly-convex, C^2-smooth function. Denote $K = Im(\nabla F)$, let λ_K be the restriction of the Lebesgue measure to K, and define μ to be the measure whose density at $x \in \mathbb{R}^n$ equals $\frac{d\mu}{dx} = \det HessF(x)$.*
Then $\nabla F : \mathbb{R}^n \to \mathbb{R}^n$ transports μ to λ_K.

Proof: Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuous, non-negative function. Since F is strictly convex, then $\nabla F : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one. Changing variables $x = \nabla F(y)$, we obtain

$$\int_{Im(\nabla F)} \varphi(x)dx = \int_{\mathbb{R}^n} \varphi(\nabla F(y)) \det(Hess(F(y)))dy = \int \varphi(\nabla F(y))d\mu(y).$$

This completes the proof. □

Denote by μ_K the measure on $\mathbb{R}^n$ whose density at x is $\det Cov(\mu_{K,x})$. Lemma 3.1 and Lemma 3.2 tell us that ∇F_K transports the measure μ_K to the uniform measure on K. In particular, $\mu_K(\mathbb{R}^n) = Vol(K)$. Thus, we may transfer volumetric computations on K to corresponding questions on the measure μ_K.

4 Proof of the main results

Proof of Theorem 1.1: By translating and rescaling K, we may assume that $Vol(K) = 1$ and that the barycenter of K lies at the origin. In particular,

$$conv(K, -K) \subset K - K \tag{27}$$

where $conv(A, B)$ denotes the convex hull of A and B. By the Rogers-Shephard theorem [28],

$$Vol(K - K) \leq \binom{2n}{n} Vol(K) < 4^n. \tag{28}$$

Let $K' = [conv(K, -K)]^\circ$, the polar body of $conv(K, -K)$. Then

$$K' = \{x \in \mathbb{R}^n; \forall y \in K, |\langle x, y\rangle| \leq 1\}. \tag{29}$$

According to the Bourgain-Milman theorem [9], followed by (27) and (28),

$$Vol(K')^{\frac{1}{n}} > \frac{c}{nVol(conv(K, -K))^{\frac{1}{n}}} > \frac{c}{nVol(K - K)^{\frac{1}{n}}} > \frac{4c}{n}. \tag{30}$$

Next, Recall the definition of the measure μ_K from Section 3. That is, for any $x \in \mathbb{R}^n$, we define a probability measure $\mu_{K,x}$ whose density at $y \in \mathbb{R}^n$ equals

$$\frac{e^{\langle x,y\rangle} 1_K(y)}{\int_K e^{\langle x,z\rangle}dz}.$$

Then, we define μ_K to be the measure whose density at x equals $\det Cov(\mu_{K,x}) = \det Hess(F_K)(x)$. By Lemma 3.1, $Im(\nabla F_K)$ is the interior of K. According to Lemma 3.2, there exists a map that transports the measure μ_K to the uniform measure on K. In particular,

$$\mu_K(\varepsilon nK') < \mu_K(\mathbb{R}^n) = Vol(K) = 1.$$

Thus,

$$Vol(\varepsilon nK') \min_{x \in \varepsilon nK'} \det Cov(\mu_{K,x}) \leq \int_{\varepsilon nK'} \det Cov(\mu_{K,x})dx = \mu_K(\varepsilon nK') < 1. \tag{31}$$

According to (30) and (31),

$$\min_{x \in \varepsilon n K'} \det Cov(\mu_{K,x}) < \left(\frac{C}{\varepsilon} \right)^n .$$

Let $x \in \varepsilon n K'$ be such that

$$\det Cov(\mu_{K,x}) < \left(\frac{C}{\varepsilon} \right)^n . \tag{32}$$

The measure $\mu_{K,x}$ is log-concave; Indeed, its density is proportional to $f(y) := e^{\langle x,y \rangle} 1_K(y)$, which is the product of $e^{\langle x,y \rangle}$ and $1_K(y)$, both log-concave. Also, by the definition of the isotropic constant (3),

$$\det Cov(\mu_{K,x}) = \left(\frac{\int_{\mathbb{R}^n} f(y)dy}{\sup_{y \in \mathbb{R}^n} f(y)} \right)^2 L_f^{2n}. \tag{33}$$

Since $x \in \varepsilon n K'$ and $f(y) = e^{\langle x,y \rangle} 1_K(y)$, then by (29),

$$\sup_{y \in \mathbb{R}^n} f(y) = \sup_{y \in K} e^{\langle x,y \rangle} \leq e^{\varepsilon n}. \tag{34}$$

Also, by Jensen's inequality,

$$\int_{\mathbb{R}^n} f(y)dy = \int_K e^{\langle x,y \rangle} dy \geq \exp \left(\int_K \langle x,y \rangle dy \right) = 1. \tag{35}$$

Now (32), (33), (34) and (35) imply that

$$L_f^{2n} < e^{2\varepsilon n} \left(\frac{C}{\varepsilon} \right)^n \quad \text{and hence} \quad L_f < \frac{c'}{\sqrt{\varepsilon}}. \tag{36}$$

The function $f : K \to [0, \infty)$ is log-concave, and

$$e^{-\varepsilon n} \leq \inf_{y \in K} f(y) \leq \sup_{y \in K} f(y) \leq e^{\varepsilon n}. \tag{37}$$

We may invoke Lemma 2.8, based on the estimate (37). By the conclusion of that lemma there exists a convex set $T \subset \mathbb{R}^n$, with $L_T \asymp L_f$ such that

$$d(K,T) < e^{\varepsilon} \leq 1 + e\varepsilon, \qquad (0 < \varepsilon < 1).$$

However, by (36) we know that $L_T < cL_f < \frac{C}{\sqrt{\varepsilon}}$. This completes the proof. □

Next we prove Corollary 1.2. We begin by quoting Paouris theorem [25, 26].

Theorem 4.1 (Paouris) *Let $K \subset \mathbb{R}^n$ be an isotropic convex body. Then for any $t > 1$,*

$$Vol(K \setminus ct\sqrt{n} L_K D) < e^{-t\sqrt{n}},$$

where $D = \{x \in \mathbb{R}^n ; |x| \leq 1\}$ is the unit Euclidean ball, and $c > 0$ is a universal constant.

Our next lemma is a consequence of Theorem 4.1.

Lemma 4.2 *Let $K, T \subset \mathbb{R}^n$ be convex bodies, and $t \geq 1$. Suppose that*

$$d(K,T) < 1 + \frac{t}{\sqrt{n}}. \tag{38}$$

Then,

$$L_T < ctL_K,$$

where $c > 0$ is a universal constant.

Proof: We may assume that $t < \sqrt{n}$, as otherwise the conclusion of the lemma is trivial, since it is easy to prove that $L_T < c\sqrt{n}$. (For example, if $Vol(T) = 1$, then there exists a direction in which the width of T is smaller than $c\sqrt{n}$, and thus there exists a hyperplane section whose volume is larger than $\frac{1}{c\sqrt{n}}$). According to (38) there exist $x_0, y_0 \in \mathbb{R}^n$ with

$$\frac{1}{1+\frac{t}{\sqrt{n}}}(K + x_0) \subset (T + y_0) \subset \left(1 + \frac{t}{\sqrt{n}}\right)(K + x_0). \tag{39}$$

Applying an affine transformation to both K and T, we may suppose that $Vol(K) = 1$, that the barycenter of K is at the origin, and that K is isotropic. Let us set $\tilde{T} = \frac{1}{1+\frac{t}{\sqrt{n}}}(T + y_0) - x_0$. By (39), $\tilde{T} \subset K$. Additionally, again from (39),

$$Vol(\tilde{T}) = \frac{1}{\left(1 + \frac{t}{\sqrt{n}}\right)^n} Vol(T) \geq \frac{1}{\left(1 + \frac{t}{\sqrt{n}}\right)^{2n}} Vol(K) > e^{-2t\sqrt{n}}. \tag{40}$$

According to Theorem 4.1, we know that

$$Vol(K \setminus ct\sqrt{n}L_K D) < e^{-4t\sqrt{n}} \tag{41}$$

for some universal constant $c > 0$. Since $\tilde{T} \subset K$, then (40) and (41) imply that

$$Vol(\tilde{T} \cap ct\sqrt{n}L_K D) \geq \frac{1}{2} Vol(\tilde{T}).$$

Therefore, the median of the function $x \mapsto |x|$ on $\tilde{T}$, with respect to the uniform measure on $\tilde{T}$, is not larger than $ct\sqrt{n}L_K$. Since $\tilde{T}$ is convex, by (6),

$$\sqrt{\frac{\int_{\tilde{T}} |x|^2 dx}{Vol(\tilde{T})}} < Ct\sqrt{n}L_K \tag{42}$$

for some universal constant $C > 0$. According to (4) and (40),

$$L_T = L_{\tilde{T}} = L_{1_{\tilde{T}}} \leq C \frac{tL_K}{Vol(\tilde{T})^{\frac{1}{n}}} < c' tL_K.$$

The lemma is proven. □

Proof of Corollary 1.2: Let $K \subset \mathbb{R}^n$ be a convex body, and let us set $\varepsilon = \frac{1}{\sqrt{n}}$. According to Theorem 1.1, there exists a convex body $T \subset \mathbb{R}^n$ with

$$d(K,T) < 1 + \varepsilon = 1 + \frac{1}{\sqrt{n}} \tag{43}$$

and

$$L_T < \frac{c}{\sqrt{\varepsilon}} = cn^{1/4}. \tag{44}$$

We may apply Lemma 4.2 based on (43) and (44). By the conclusion of that lemma, $L_K < c'n^{1/4}$. □

Corollary 4.3 *Let* $f : \mathbb{R}^n \to [0,\infty)$ *be a log-concave function with* $0 < \int f < \infty$. *Then,*

$$L_f < cn^{1/4},$$

where $c > 0$ *is a universal constant.*

Proof: Translating f if necessary, we may assume that the barycenter of f lies at the origin. Let $T = K_{n+1}(f)$. The set T is convex, by Theorem 2.2, and hence $L_T < cn^{1/4}$, by Corollary 1.2. We also know that $L_T \asymp L_f$, according to Lemma 2.7. Thus $L_f < cn^{1/4}$. □

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