Approximately Gaussian marginals and the hyperplane conjecture

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Lecture based on a joint work with Ronen Eldan.
Many open problems

This talk is concerned with convex bodies in high dimension.

- Despite recent progress, even the simplest questions remain unsolved:

**Question [Bourgain, 1980s]**

Suppose $K \subset \mathbb{R}^n$ is a convex body of volume one. Does there exist an $(n-1)$-dimensional hyperplane $H \subset \mathbb{R}^n$ such that

$$\text{Vol}_{n-1}(K \cap H) > c$$

where $c > 0$ is a universal constant?

- Known: $\text{Vol}_{n-1}(K \cap H) > cn^{-1/4}$ (Bourgain ’91, K. ’06).
- Affirmative answer for: unconditional convex bodies, zonoids, their duals, random convex bodies, outer finite volume ratio, few vertices/facets, subspaces/quotients of $L^p$, Schatten class, ...
As was observed by K. Ball, the hyperplane conjecture is most naturally formulated in the class of log-concave densities.

A probability density on $\mathbb{R}^n$ is log-concave if it takes the form $\exp(-H)$ for a convex function $H : \mathbb{R}^n \to [-\infty, \infty)$.

Examples of log-concave densities:
The Gaussian density, the uniform density on a convex body.

1. Product of log-concave densities is (proportional to) a log-concave density.
2. Prékopa-Leindler: If $X$ is a log-concave random vector, so is the random vector $T(X)$ for any linear map $T$. 
Isotropic Constant

For a log-concave density \( \rho : \mathbb{R}^n \to [0, \infty) \) set

\[
L_\rho = \sup_{x \in \mathbb{R}^n} \rho^{\frac{1}{n}}(x) \det \text{Cov}(\rho)^{\frac{1}{2n}}
\]

the isotropic constant of \( \rho \). The isotropic constant is affinely invariant. What’s its meaning?

- **Normalization:** Suppose \( X \) is a random vector in \( \mathbb{R}^n \) with density \( \rho \). We say that \( X \) (or that \( \rho \)) is **isotropic** if
  \[
  \mathbb{E}X = 0, \quad \text{Cov}(X) = \text{Id}
  \]
  That is, all of the marginals have mean zero and var. one.

- For an isotropic, log-concave density \( \rho \) in \( \mathbb{R}^n \), we have
  \[
  L_\rho \sim \rho(0)^{1/n} \sim \int_{\mathbb{R}^n} \rho^{1 + \frac{1}{n}} \sim \exp \left( \frac{1}{n} \int_{\mathbb{R}^n} \rho \log \rho \right) > c
  \]
  where \( A \sim B \) means \( cA < B < CA \) for universal \( c, C > 0 \).
An equivalent formulation of the slicing problem

The hyperplane conjecture is directly equivalent to the following:

**Slicing problem, again:**

Is it true that for any $n$ and an isotropic, log-concave $\rho : \mathbb{R}^n \rightarrow [0, \infty)$,

$$L_\rho < C$$

where $C > 0$ is a universal constant?

(the equivalence follows from works by Ball, Bourgain, Fradelizi, Hensley, Milman, Pajor and others, uses Brunn-Minkowski).

- For a uniform density on $K \subset \mathbb{R}^n$, $L_K = Vol_n(K)^{-1/n}$. Can we have the same covariance as the Euclidean ball, in a substantially smaller convex set?
It is straightforward to show that $L_\rho > c$, for a universal constant $c > 0$.

To summarize, define

$$L_n = \sup_{\rho : \mathbb{R}^n \to [0, \infty)} L_\rho.$$

It is currently known that

$$L_n \leq C n^{1/4}.$$

It is enough to consider the uniform measure on centrally-symmetric convex bodies (Ball ’88, K. ’05):

$$L_n \leq C \sup_{K \subset \mathbb{R}^n} L_K$$

where $K \subset \mathbb{R}^n$ is convex with $K = -K$. 

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Further open problems

Theorem (“Central Limit Theorem for Convex Bodies”, K. ’07)

Most of the volume of a log-concave density in high dimensions, with the isotropic normalization, is concentrated near a sphere of radius $\sqrt{n}$.

Define

$$\sigma_n^2 = \sup_X \text{Var}(|X|) \sim \sup_X \mathbb{E} \left(|X| - \sqrt{n}\right)^2,$$

where the supremum runs over all log-concave, isotropic random vectors $X$ in $\mathbb{R}^n$.

The theorem states that

$$\sigma_n \ll \sqrt{n}$$
Approximately Gaussian marginals

The importance of $\sigma_n$ stems from:

**Theorem (Sudakov ’78, Diaconis-Freedman ’84,...)**

Suppose $X$ is an isotropic random vector in $\mathbb{R}^n$, $\varepsilon > 0$. If

$$\mathbb{P}\left(\left|\frac{|X|}{\sqrt{n}} - 1\right| \geq \varepsilon\right) \leq \varepsilon.$$

Then for most $\theta \in S^{n-1}$,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(X \cdot \theta \leq t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2} ds \right| \leq C \left( \varepsilon + \frac{1}{n^{1/5}} \right).$$

Many marginals are approx. standard Gaussians.

Therefore, most of the 1D marginals of a high-dimensional normalized convex body are approx. standard Gaussians.
How thin is the shell?

- Current best bound, due to Fleury ’10:
  \[ \sigma_n \leq C n^{3/8} \]
  (improving on a previous bound of \( \sigma_n \leq C n^{0.401} \), K. ’07).
- Typical marginals of an isotropic log-concave random vector in \( \mathbb{R}^n \), are \( C \sigma_n / \sqrt{n} \)-close to Gaussian.

Conjecture [Antilla-Ball-Perissinaki ’03]

Perhaps

\[ \sigma_n \leq C \]

for a universal constant \( C > 0 \)?

1. Corresponds to a philosophy that “Convexity is as good as independent random variables”, in view of Berry-Esseen.
2. True in some cases, including unconditional convex bodies (K. ’09) and random convex bodies (Fleury ’10).
Thin shell vs. Slicing problem

Theorem (Eldan, K. ’10)

There is a universal constant $C$ such that

$$L_n \leq C \sigma_n.$$ 

Remarks:

1. Pushing $\sigma_n$ much below $n^{1/4}$ might be hard (at the moment only $n^{3/8}$ is known).

2. If $L_n$ is not bounded, then CLT for convex bodies is weaker than the classical CLT.

3. Strengthens a result announced by K. Ball ’06 (the larger spectral-gap instead of $\sigma_n$, exponential dependence).
Suppose $K \subset \mathbb{R}^n$ a convex body, barycenter at the origin, $X$ is uniformly distributed in $K$.

The **logarithmic Laplace transform** is the convex function

$$\Lambda(\xi) = \log \mathbb{E} \exp(X \cdot \xi) \quad (\xi \in \mathbb{R}^n).$$

The logarithmic Laplace transform helps relate the covariance matrix and the volume of $K$. 
Differentiating the logarithmic Laplace transform

Recall that \( \Lambda(\xi) = \log \mathbb{E} \exp(X \cdot \xi) \)

- For \( \xi \in \mathbb{R}^n \), denote by \( X_\xi \) the “tilted” log-concave random vector in \( \mathbb{R}^n \) whose density is proportional to

\[
x \mapsto 1_K(x) \exp(\xi \cdot x).
\]

(any idea for a good coupling when \( n \geq 2 \)?)

Then,

1. \( \nabla \Lambda(\xi) = \mathbb{E} X_\xi \in K \).
2. The hessian \( \nabla^2 \Lambda(\xi) = \text{Cov}(X_\xi) \).
3. Third derivatives? A bit complicated. With \( b_\xi = \mathbb{E} X_\xi \),

\[
\partial^i \log \det \nabla^2 \Lambda(\xi) = \text{Tr} \left[ \text{Cov}(X_\xi)^{-1} \mathbb{E}(X_\xi^i - b_\xi^i)(X_\xi - b_\xi) \otimes (X_\xi - b_\xi) \right].
\]
Transportation of measure

The function $\Lambda(\xi)$ is strictly convex, so $\nabla \Lambda$ is one-to-one. Recall that $\nabla \Lambda(\xi) \in K$ for all $\xi$.
From the change of variables formula,

\[
Vol_n(K) \geq Vol_n(\nabla \Lambda(\mathbb{R}^n)) = \int_{\mathbb{R}^n} \det \nabla^2 \Lambda(\xi) \, d\xi \geq \int_{nK^\circ} \det \nabla^2 \Lambda
\]

- In particular, there exists $\xi \in nK^\circ$ with

\[
\det \nabla^2 \Lambda(\xi) = \det \text{Cov}(X_\xi) \leq \frac{Vol_n(K)}{Vol_n(nK^\circ)}.
\]

Since $e^{-n} \leq \exp(\xi \cdot x) \leq e^n$ for $x \in K$, then for such $\xi \in nK^\circ$,

\[
L_{X_\xi} \leq \frac{C}{Vol_n(K)^{1/n}} \left( \frac{Vol_n(K)}{Vol_n(nK^\circ)} \right)^{1/(2n)} \sim \left( \frac{1}{Vol_n(K) Vol_n(nK^\circ)} \right)^{1/(2n)}.
\]

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Theorem (Bourgain-Milman ’87)

\[ Vol_n(K) Vol_n(nK^\circ) \geq c^n \]

where \( c > 0 \) is a universal constant.

Therefore \( L_{\xi} < \text{Const} \) for most \( \xi \in nK^\circ \).

There is a correspondence between centered log-concave densities and convex bodies due to K. Ball:

Suppose \( f : \mathbb{R}^n \to [0, \infty) \) is a log-concave. Denote

\[ K(f) = \left\{ x \in \mathbb{R}^n; (n+1) \int_0^\infty f(rx)r^n dr \geq 1 \right\}, \]

the convex body associated with \( f \).
Isomorphic version of the slicing problem

- When $f$ is log-concave, the body $K(f)$ is convex – closely related to Busemann inequality.
- When $f$ has barycenter at the origin, $K(f)$ and $f$ have roughly the same volume and covariance matrix. So $L_{K(f)} \sim L_f$.
- Suppose $f$ is supported on a convex body $K$. Denote $a = \inf_K f^{1/n}$ and $b = \sup_K f^{1/n}$. Then
  \[ aK \subseteq K(f) \subseteq bK. \]

Applying this construction to $X_\xi$, we deduce:

**Corollary [K. ’06]**

For any convex body $K \subset \mathbb{R}^n$ and $0 < \varepsilon < 1$, there exists another convex body $T \subset \mathbb{R}^n$ with

1. $(1 - \varepsilon)K \subseteq T \subseteq (1 + \varepsilon)K$.
2. $L_T \leq C/\sqrt{\varepsilon}$, where $C > 0$ is a universal constant.
Using Paouris large deviations Theorem

### Theorem (Paouris ’06)

Suppose $X$ is an isotropic, log-concave random vector in $\mathbb{R}^n$. Then for any $t \geq C\sqrt{n}$,

$$\mathbb{P}(|X| \geq t) \leq C \exp(-ct)$$

where $c, C > 0$ are universal constants.

Stability of the isotropic constant: It follows immediately that when $K$ and $T$ are convex bodies of volume one, such that

$$\text{Vol}_n(K \cap T) \geq e^{-\sqrt{n}},$$

then necessarily $L_K \sim L_T$.

- This leads to the bound $L_n \leq Cn^{1/4}$. 

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What's the connection to thin shell?

Paouris theorem is about large deviations. How can we use the thin shell estimates?

Suppose $K \subset \mathbb{R}^n$ convex body, $X$ uniform in $K$, isotropic.

- To prove $L_n \leq C\sigma_n$, we need the lower bound:

$$Vol_n(K) = \int_{\mathbb{R}^n} \det \nabla^2 \Lambda(\xi) d\xi \geq \left( \frac{1}{C\sigma_n} \right)^n$$

Note that $\det \nabla^2 \Lambda(0) = \det \text{Cov}(X) = 1$.

- Third derivatives of $\Lambda$ again, at the origin:

$$\nabla \log \det \nabla^2 \Lambda(\xi) \Big|_{\xi = 0} = \mathbb{E} X|X|^2 = \mathbb{E} X(|X|^2 - n).$$
Relation to thin shell

Recalling that $\sigma_n^2 \sim \mathbb{E} \left( |X|^2 - n \right)^2 / n$, from Cauchy-Schwartz,

$$\left| \nabla \log \det \nabla^2 \Lambda(\xi) \right|_{\xi=0} = \mathbb{E} |X(\xi)|^2 \leq C \sqrt{n} \sigma_n.$$

- In fact, throughout the proof, in place of $\sigma_n$ we work with the smaller

$$\sigma_n = \frac{1}{\sqrt{n}} \sup_X \mathbb{E} |X|^2$$

where the supremum runs over all isotropic, log-concave random vectors $X$ in $\mathbb{R}^n$.

To proceed, we have to work with third derivatives at non-zero $\xi$ (or take higher order derivatives at zero and use Taylor’s theorem – this could be explained in another talk...).
A Riemannian metric

Computing the third derivatives for $\xi \neq 0$ is slightly easier with respect to a suitable Riemannian metric.

**Definition**

For $\xi \in \mathbb{R}^n$, consider the positive-definite quadratic form

$$g_\xi(u, v) = \text{Cov}(X_\xi)u \cdot v \quad (u, v \in \mathbb{R}^n)$$

- This Riemannian metric lets $X_\xi$ “feel isotropic”.
- This metric does not depend on the Euclidean structure:

$$g_\xi(u, v) = \mathbb{E}u(X_\xi - b_\xi) \cdot v(X_\xi - b_\xi) \quad (u, v \in \mathbb{R}^{n*})$$

where $b_\xi = \mathbb{E}X_\xi$ and $u, v$ are viewed as linear functionals.

- The absolute values of the sectional curvatures are bounded by a universal constant. They vanish when $X_1, \ldots, X_n$ are independent r.v.’s.
A Riemannian metric

Our only use of this Riemannian structure is to ease manipulations of third derivatives. Also for a non-zero $\xi \in \mathbb{R}^n$, we have

$$|\nabla_g \log \det \text{Cov}(X_\xi)|_g \leq C \sqrt{n} \sigma_n$$

Consequently, for $\xi \in \mathbb{R}^n$ with $d_g(0, \xi) \leq \sqrt{n}/\sigma_n$,

$$\det \text{Cov}(X_\xi) \geq e^{-n}.$$ 

We need a lower bound for an integral of $\det \text{Cov}(X_\xi)$.

• How big is the Riemannian ball of radius $\sqrt{n}/\sigma_n$ around the origin?
Lemma

\[ d_g(0, \xi) \leq \sqrt{\Lambda(2\xi)} \]

Proved by inspecting the Riemannian length of the (Euclidean) segment \([0, \xi]\): By convexity,

\[ d_g(0, \xi) \leq \int_0^1 \sqrt{\frac{\partial^2}{\partial \xi^2} \Lambda(r\xi)} \, dr \leq \sqrt{\Lambda(2\xi)} \]

Therefore,

\[ \left( \frac{1}{L_K} \right)^n \geq c^n \text{Vol}_n \left( \left[ \Lambda \leq n/\sigma_n^2 \right] \right) \]

Now we forget about the Riemannian metric. We were not really able to deeply exploit the Riemannian geometry.
Recall that $K \subset \mathbb{R}^n$ is a convex body whose barycenter is at the origin, $X$ uniform in $K$.

**Lemma**

There exist universal $c, C > 0$ such that

$$cnK^\circ \subseteq [\Lambda \leq n] \subseteq CnK^\circ$$

Proved by standard log-concave tricks, nothing more than asymptotics of 1D integrals. Bourgain-Milman: When $X$ is isotropic,

$$\text{Vol}_n ([\Lambda < n]) \geq \frac{c^n}{\text{Vol}_n(K)} = (cL_K)^n \geq \tilde{c}^n,$$

Suppose $X$ is isotropic, and take an integer $1 \leq k \leq n$. Then, for any $k$-dimensional subspace $E \subset \mathbb{R}^n$,

$$\text{Vol}_k ([\Lambda < k] \cap E) \geq c^k.$$
Completing the proof

Recall that

\[
\left(\frac{1}{L_K}\right)^n \geq c^n \text{Vol}_n\left([\Lambda \leq n/\sigma_n^2]\right).
\]

Take an integer \( k \sim n/\sigma_n^2 \). Then, for any \( k \)-dimensional subspace \( E \subset \mathbb{R}^n \),

\[
\text{Vol}_k\left([\Lambda < k] \cap E\right) \geq c^k.
\]

Without confusion, we deduce

\[
\text{Vol}_n\left([\Lambda < k]\right) \geq \left(\frac{c\sqrt{k}}{\sqrt{n}}\right)^n \geq \left(\frac{\tilde{c}}{\sigma_n}\right)^n.
\]

This completes the proof of

\[
L_n \leq C\sigma_n.
\]
If $\Lambda^* : K \to \mathbb{R}$ is the Legendre transform of $\Lambda$, then the hessian
\[ \nabla^2 \Lambda^*(x) \quad (x \in K) \]
defined a Riemannian structure on $K$, isometric to the one described above, linearly invariant.

The expression
\[ \text{Vol}_n(K) = \int \det \text{Cov}(X_\xi) d\xi \]
reminds us of the Riemannian volume (a square root is missing!). One may construct a very similar Kähler metric on $\mathbb{C}^n/i\mathbb{Z}^n$, whose volume is exactly $\text{Vol}_n(K)$. Perhaps it allows a more intrinsic analysis? I have no idea.
Thank you!