Poincaré Inequalities and Moment Maps

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Poincaré-type inequalities

- Poincaré-type inequalities (in this lecture): Bounds for the **variance** of a function in terms of the **gradient**.

Wirtinger inequality (1904)

$$\int_0^\pi f = 0 \qquad \Rightarrow \qquad \int_0^\pi f^2 \leq \int_0^\pi (f')^2.$$

Here it is easy to describe the eigenfunctions. More generally,

Poincaré inequality for Convex Domains (Payne & Weinberger, 1960)

For any convex body $K \subset \mathbb{R}^n$ and $f : K \to \mathbb{R}$,

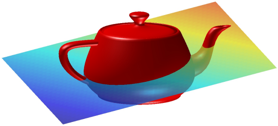
$$\int_K f = 0 \qquad \Rightarrow \qquad \int_K f^2 \leq \frac{D^2}{\pi^2} \int_K |\nabla f|^2,$$

where $D = diameter(K) = \sup_{x,y \in K} |x - y|$.

Spectral gap

- Consider the **Laplacian** (with Neumann boundary conditions) on a convex body $K \subset \mathbb{R}^n$. Then $\lambda_0 = 0$, and according to Payne-Weinberger,
$$\lambda_1 = \lambda_1(K) \geq \pi^2/D^2$$
for $D = diameter(K)$.
- The Payne-Weinberger bound is optimal for a thin cylinder, but not for a Euclidean ball or a cube (off by a factor of n).

For an arbitrary convex body, it is difficult to describe the eigenfunction(s). Original proof uses **bisections** of convex bodies. The idea was further developed by Gromov and Milman, and by Kannan, Lovász and Simonovits.



Isoperimetry and spectral gap

- Intimately related to geometry, and to the **isoperimetric coefficient**: For an open set $K \subset \mathbb{R}^n$ define

$$h(K) = \inf \left\{ Vol_{n-1}(\partial A \cap K) \, ; \, A \subseteq K, \; Vol_n(A) = Vol_n(K)/2 \right\}.$$

Theorem (Maz'ya/Cheeger '70, Buser/Ledoux '82/'04)

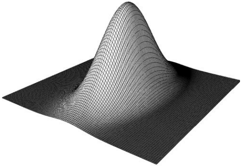
For any open, convex body $K \subset \mathbb{R}^n$,

$$\frac{h^2(K)}{4} \leq \lambda_1(K) \leq 10 h^2(K).$$

- Lower bound is from Maz'ya-Cheeger inequality (no convexity needed), upper bound is due to Buser and to Ledoux.
- May be used to estimate the **rate of mixing** of Brownian motion and various Markov chains on a convex domain.

Our motivation

- A probability measure μ on $\mathbb{R}^n$ is **log-concave** if its density is $\exp(-\Psi)$ for a convex function $\Psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.
- The uniform measure on a convex body is log-concave, as is the Gaussian density.
- We say that μ is **isotropic** when
$$\int_{\mathbb{R}^n} x d\mu(x) = 0 \quad \text{and} \quad Cov(\mu) = Id.$$



Thin shell problem (Antilla, Ball & Perissinaki '03, Bobkov & Koldobsky '03)

Suppose μ is log-concave and isotropic. Is it true that

$$\int_{\mathbb{R}^n} \left(|x| - \sqrt{n} \right)^2 d\mu(x) \leq C$$

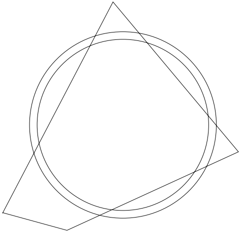
for a universal constant $C > 0$?

Why care about Thin Shell?

A positive answer would imply...

1. An optimal rate of convergence in the convex CLT.
2. K.-Eldan '11: The hyperplane conjecture.
3. Eldan '12: The Kannan-Lovasz-Simonovits conjecture, up to logarithmic factors! (work in progress)

What is the width of the **thin spherical shell** that contains most of the volume of an isotropic convex body?



- The best available thin shell bound is

$$\mathbb{E}\left(\frac{|X|}{\sqrt{n}} - 1\right)^2 \leq \frac{C}{n^\alpha}$$

with $\alpha = 1/3$ (Guédon-Milman '11), $\alpha = 1/4$ (Fleury '10) and $\alpha = 1/6$ (K., '07). Ideas of Paouris are relevant.

How would you prove thin shell bound?

Suppose μ is a log-concave measure on $\mathbb{R}^n$, isotropic. Need:

$$\int_{\mathbb{R}^n} \left(\frac{|x|^2}{n} - 1 \right)^2 d\mu(x) \ll 1.$$

All current proofs rely on the following idea: Projecting the measure to a random subspace, it becomes approximately the uniform measure on a sphere (à la Dvoretzky's theorem).

- Another possible approach:
  Try to prove a Poincaré-type inequality

$$\int_K \varphi^2 \leq \int_K Q(\nabla \varphi) d\mu$$

  for **all** functions φ with $\int \varphi d\mu = 0$, for some quadratic expression Q. We only need $\varphi(x) = |x|^2/n - 1$.
- The Poincaré inequalities mentioned above are not sufficient for obtaining non-trivial bounds.

Unconditional convex bodies

A convex body $K \subset \mathbb{R}^n$ is called **unconditional** when

$$(x_1, \ldots, x_n) \in K \qquad \Longleftrightarrow \qquad (|x_1|, \ldots, |x_n|) \in K.$$

(i.e., invariant under coordinate reflections)

Theorem (Poincaré Inequality for unconditional convex bodies, K. '09+'11)

Suppose $K \subset \mathbb{R}^n$ is convex and unconditional. Then for any f,

$$\int_K f = 0 \qquad \Longrightarrow \qquad \int_K f^2 \leq \int_K \sum_{i=1}^n (4x_i^2 + V_i)|\partial^i f|^2 dx$$

where $V_i = \int_K x_i^2 / Vol_n(K)$.

- Generalizes directly to log-concave measures that are **unconditional** (i.e., invariant under coordinate reflections).
- Convexity assumptions are essential (coordinate-wise monotonicity does not suffice).

Unconditional convex bodies

Suppose K is either the **Euclidean ball** of volume one (and radius $c\sqrt{n}$), or else **the cube** $[-1/2, 1/2]^n$. The bound

$$\int_K f^2 \leq \int_K \sum_{i=1}^n (4x_i^2 + 1)|\partial^i f|^2 dx, \tag{1}$$

for f with $\int_K f = 0$ is typically better than Payne-Weinberger's bound

$$\int_K f^2 \leq \frac{D^2(K)}{\pi^2} \int_K \sum_{i=1}^n |\partial^i f|^2 dx, \tag{2}$$

because $D(K) = diameter(K) \sim \sqrt{n}$, while typically $x_i^2 \approx 1$.

- Substituting $f(x) = |x|^2/n - 1$ into (1), we completely solve the "thin shell" problem for unconditional convex bodies.

Convexity and symmetry

- Other types of symmetries work as well, including those of the simplex (Barthe and Cordero-Erausquin '11).

Theorem (unconditional log-concave measures)

Suppose μ is a log-concave, unconditional probability measure on $\mathbb{R}^n$. Then for any f, denoting $Var_\mu(f) = \int f^2 d\mu - \left(\int f d\mu\right)^2$,

$$Var_\mu(f) \leq \int_{\mathbb{R}^n} \sum_{i=1}^{n} (4x_i^2 + V_i) |\partial^i f(x)|^2 d\mu(x)$$

where $V_i = \int x_i^2 d\mu$.

Let $\exp(-\Psi)$ be the density of μ. Then $\triangle_\mu u := \triangle u - \nabla u \cdot \nabla \Psi$ is the "μ-Laplacian", as it satisfies

$$\int_{\mathbb{R}^n} (u \triangle_\mu v) d\mu = -\int_{\mathbb{R}^n} (\nabla u \cdot \nabla v) d\mu.$$

Dualizing Bochner's formula

- The proof uses the **Bochner-Lichnerowicz-Weitzenböck identity:** For any φ,
$$\int_{\mathbb{R}^n} (\triangle_\mu \varphi)^2 d\mu = \int_{\mathbb{R}^n} |\nabla^2 \varphi|^2 d\mu + \int_{\mathbb{R}^n} (\nabla^2 \Psi)(\nabla \varphi) \cdot \nabla \varphi d\mu$$
where $|\nabla^2 \varphi|$ is the Hilbert-Schmidt norm of the Hessian.
- Our only use of convexity: The second term is non-negative as Ψ is **convex.** Hence,
$$\int_{\mathbb{R}^n} (\triangle_\mu \varphi)^2 d\mu \geq \int_{\mathbb{R}^n} |\nabla^2 \varphi|^2 d\mu.$$

Proposition ("dual Bochner")

For any log-concave probability measure μ on $\mathbb{R}^n$ and a function f,
$$Var_\mu(f) \leq \sum_{i=1}^n \|\partial^i f\|^2_{H^{-1}(\mu)}.$$

What is the $H^{-1}(\mu)$ norm?

- We define, for a function g,

$$\|g\|_{H^{-1}(\mu)} = \sup\left\{\int_{\mathbb{R}^n} g\varphi d\mu\,;\ \int_{\mathbb{R}^n} |\nabla\varphi|^2 d\mu \leq 1\right\}.$$

- This is the dual of the $H^1(\mu)$-norm.

Observe that $\|g\|_{H^{-1}(\mu)} = +\infty$ when $\displaystyle\int g d\mu \neq 0$.

- Hence, the bound

$$Var_\mu(f) \leq \int_{\mathbb{R}^n} \sum_{i=1}^n \|\partial^i f\|^2_{H^{-1}(\mu)}$$

is useless when $\int \nabla f d\mu \neq 0$.

Transportation of measure

- The $H^{-1}(\mu)$-norm has a **geometric interpretation.** Suppose μ, ν are prob. measures on $\mathbb{R}^n$. Define

$$W_2^2(\mu,\nu) = \inf_T \int_{\mathbb{R}^n} |x - T(x)|^2 d\mu(x)$$

where the infimum runs over T that pushes forward μ to ν. This is the L^2-Monge-Kantorovich-Wasserstein distance.

From Brenier, Otto and Villani we learn:

Proposition ("H^{-1}-norm is the infinitesimal transportation cost")

If $\int g d\mu = 0$, then (under mild regularity assumptions)

$$\|g\|_{H^{-1}(\mu)} = \lim_{\varepsilon \to 0} \frac{W_2(\mu_\varepsilon, \mu)}{\varepsilon}$$

where $d\mu_\varepsilon / d\mu = 1 + \varepsilon g$.

Use of symmetries

Decompositions: If $\mu = \int \mu_\alpha d\lambda(\alpha)$, then,

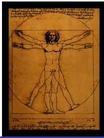
$$\|g\|^2_{H^{-1}(\mu)} \leq \int \|g\|^2_{H^{-1}(\mu_\alpha)} d\lambda(\alpha).$$

- Recall "dual Bochner": When μ is log-concave, for any f,

$$Var_\mu(f) \leq \int_{\mathbb{R}^n} \sum_{i=1}^n \|\partial^i f\|^2_{H^{-1}(\mu)} = \lim_{\varepsilon \to 0} \sum_{i=1}^n \frac{W_2^2((1+\varepsilon \partial^i f)\mu, \mu)}{\varepsilon^2}.$$

Suppose μ is uniform on K. Then we need to efficiently transport the uniform measure on K to the measure, supported on K, whose density on K is $1 + \varepsilon \partial^i f$.

- Assume that both f and K are **unconditional**.
  Then $\partial^i f$ is **odd** with respect to the i^{th} coordinate.
  A naïve transportation that comes to one's mind is...



Transportation along parallel segments

Note that for any line ℓ parallel to the i^{th} unit vector,

$$\int_{K \cap \ell} \partial^i f = 0.$$

We may thus apply the (non-optimal) transportation of $1_K dx$ to $(1 + \varepsilon \partial^i f) 1_K dx$, separately on each line segment!

- Simple computation: For $d\mu_R = 1_{[-R,R]} dx$ on the line,

$$\|g\|^2_{H^{-1}(\mu_R)} \leq 4 \int_{-R}^{R} x^2 g^2(x) dx$$

  for any odd function g.
- Hence, when K and f are unconditional (e.g., $f(x) = |x|^2$),

$$\|\partial^i f\|^2_{H^{-1}(K)} \leq 4 \int_K x_i^2 |\partial^i f(x)|^2 dx.$$

Symmetry seems crucial

- To conclude, we obtained the following Poincaré inequality for unconditional K and f:

$$Var_K(f) \leq \sum_{i=1}^n \|\partial^i f\|^2_{H^{-1}(K)} \leq 4 \int_K \sum_{i=1}^n x_i^2 |\partial^i f(x)|^2 dx.$$

- Standard tricks reduce the case of any function f to an unconditional function f.
- The most important case is $f(x) = |x|^2$. Even here, we must utilize the symmetries of K. We obtain

$$\int_K \left(\frac{|x|^2}{n} - 1 \right)^2 \frac{dx}{Vol_n(K)} \leq \frac{C}{n^2} \sum_{i=1}^n \int_K x_i^4 \frac{dx}{Vol_n(K)}$$

when K is convex, unconditional and isotropic (e.g., the cube $[-\sqrt{3}, \sqrt{3}]^n$).

A digression: multiplicity of the Neumann eigenvalue

Proposition (K. '09 – was it known before?)

Let $K \subset \mathbb{R}^n$ be a convex body. Let $\varphi \not\equiv 0$ be an eigenfunction of the Neumann Laplacian, corresponding to λ_1. Then,

$$\int_K \nabla\varphi \neq 0.$$

Consequently, the multiplicity of λ_1 is at most n.

Proof: If $\int_K \nabla\varphi = 0$, then by Bochner,

$$\lambda_1^2 \int_K \varphi^2 = \int_K |\triangle\varphi|^2 > \int_K \sum_{i=1}^n |\nabla\partial^i\varphi|^2 \geq \lambda_1 \int_K \sum_{i=1}^n |\partial^i\varphi|^2.$$

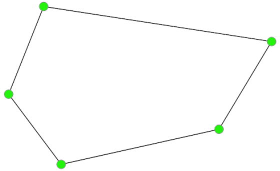
- A deeper fact (Nadirashvili '88): For any simply-connected planar domain, the multiplicity of λ_1 is at most 2.

Convexity without symmetry

- In the central case where $f(x) = |x|^2$, our method boils down to transporting

$$(1 + \varepsilon x_i) 1_K(x) dx \qquad \text{to} \qquad 1_K(x) dx.$$

How can we bound the transportation cost without symmetries?



While trying to find a solution, I came across unfamiliar territory: Moment maps and Kähler geometry.

- The basic idea: Introduce additional symmetries by considering a certain transportation of measure from a space of twice or thrice the dimension.

The simplest example

- Suppose $K = [0,1] \subset \mathbb{R}$. Denote $D = \{z \in \mathbb{R}^2; |z|^2 \leq 1\}$. The map

$$\pi(z) = |z|^2$$

pushes forward the uniform probability measure on D to the uniform probability measure on K.

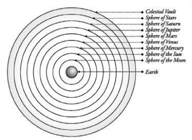
Now, for $f : K \to \mathbb{R}$ with $\int_K f = 0$, set

$$g(z) = f(\pi(z)) \qquad (z \in \mathbb{R}^2).$$

- Since D is convex, we can use "dual Bochner":

$$Var_K(f) = Var_D(g) \leq \left\| \frac{\partial g}{\partial x} \right\|_{H^{-1}(D)}^2 + \left\| \frac{\partial g}{\partial y} \right\|_{H^{-1}(D)}^2$$

The example of the interval



- The integral of ∇g is zero on each **circle** of the form $\pi^{-1}(x)$ (because g is even).
- Transport
$$\left(1+\varepsilon\frac{\partial g}{\partial x}\right)1_D(x)dx \qquad \text{to} \qquad 1_D(x)dx$$
separately along each circle. Observe that
$$\nabla g\left(re^{i\theta}\right) = 2f'(r^2)re^{i\theta},$$
which is (up to a constant) one specific function on the circle.

This leads to the inequality: For any $f:[0,1]\to\mathbb{R}$,
$$\int_0^1 f = 0 \quad \Longrightarrow \quad \int_0^1 f^2 \le 4\int_0^1 x^2(f'(x))^2dx$$

A straightforward generalization

Fix integers $\ell, n \geq 1$. For $z = (z_1, \ldots, z_n) \in (\mathbb{R}^\ell)^n$ set

$$\pi(z) = \left(|z_1|^\ell, \ldots, |z_n|^\ell \right).$$

Then $\pi : \mathbb{R}^{\ell n} \to \mathbb{R}^n$ is a proper map, that pushes forward the Lebesgue measure on $\mathbb{R}^{\ell n}$ to a (multiple of) the Lebesgue measure on $\mathbb{R}^n_+$.

- The case where $\ell = 2$ is the **moment map** from $\mathbb{C}^n$ to $\mathbb{R}^n_+$.

An example: Suppose $\ell = 2$ and $\Omega \subset \mathbb{C}^n$ is the unit ball

Then π pushes forward the uniform measure on Ω to the uniform measure on the simplex, with $\pi^{-1}(x)$ being a torus.

- The actually argument works for prob. measures in $\mathbb{R}^n_+$ with certain **convexity properties**. No symmetries required!

The convexity assumptions

Therefore, if we have enough **convexity** for the Bochner argument, then we obtain an interesting Poincaré inequality.

Theorem (K. '11)

Let $\ell, n > 1$ be integers and let ν be a probability measure on $\mathbb{R}_+^n$ whose density is **$(1/\ell)$-log-concave.** Then, for any f,

$$Var_\nu(f) \leq \frac{\ell^2}{\ell - 1} \sum_{i=1}^n \int_{\mathbb{R}_+^n} x_i^2 |\partial^i f|^2 d\nu(x).$$

- The probability density $\rho : \mathbb{R}_+^n \to \mathbb{R}$ is q-log-concave, for $0 < q \leq 1$, if the function
$$(x_1, \ldots, x_n) \mapsto \rho\left(x_1^q, \ldots, x_n^q\right)$$
is log-concave on $\mathbb{R}_+^n$.
- $(1/\ell)$-log-concavity seems **weaker** than log-concavity (it is certainly so in the unconditional case).

CLT for non-convex bodies

- The theorem holds also for non-integer $\ell > 1$.
- The uniform measure on

$$B_p^n = \left\{ x \in \mathbb{R}_+^n ; \sum_{i=1}^n |x_i|^p \leq 1 \right\}$$

for $0 < p \leq 1$ is p-log-concave.

Corollary

Let $n \geq 1$ be an integer, let $\ell > 1$ and let $K \subset \mathbb{R}_+^n$ be such that

$$\left\{ (x_1^{1/\ell}, \ldots, x_n^{1/\ell}) \; ; \; (x_1, \ldots, x_n) \in K \right\}$$

is a convex set. Then, for any function f

$$\int_K f = 0 \quad \Rightarrow \quad \int_K f^2 \leq \frac{\ell^2}{\ell - 1} \sum_{i=1}^n \int_K x_i^2 |\partial^i f(x)|^2 dx.$$

Remarks about our "weak convexity"

- We obtain a good "thin shell estimate" for convex bodies such as B_p^n for $p = 1/2$, or $p > 0$ in general.
- The uniform measure on a convex body in $\mathbb{R}_+^n$ is p-log-concave for $p = 1$ (the useless case in our corollary).
- If $K \subset \mathbb{R}^n$ is **unconditional** and convex, then the uniform measure on $K \cap \mathbb{R}_+^n$ is $1/2$-log-concave (and also p-log-concave for any $0 < p < 1$).
- We thus recover the result discussed earlier for **unconditional, convex sets**.

Unconditionality is used only to deduce p-log-concavity from q-log-concavity when $p < q$.

The symmetry assumptions return through the back door.

Toric Kähler manifolds

- The most important case where $\ell = 2$ corresponds to the moment map

$$\mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (|z_1|^2, \dots, |z_n|^2) \in \mathbb{R}^n_+.$$

- Be wise, generalise?!

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, convex function. Then,

$$K = \nabla\psi(\mathbb{R}^n)$$

is a convex set. Consider the complex torus $\mathbb{T}^n_{\mathbb{C}} = \mathbb{C}^n / (\sqrt{-1}\mathbb{Z}^n)$, and the Kähler form on $\mathbb{T}^n_{\mathbb{C}}$

$$\omega_\psi = 2\sqrt{-1}\partial\bar{\partial}\psi = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} \psi_{ij} dz_i \wedge d\bar{z}_j.$$

Kähler manifolds for analphabets

- Hmmm... a Kähler form? This means that on $\mathbb{T}^n_{\mathbb{C}} = \mathbb{C}^n / (\sqrt{-1}\mathbb{Z}^n)$ we have a Riemannian metric g_ψ with

$$g_\psi\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g_\psi\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \psi_{ij} \qquad (i, j = 1, \ldots, n)$$

  while $g_\psi\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = 0$ for any i, j.
- The associated moment map

$$\mathbb{T}^n_{\mathbb{C}} \ni (x + \sqrt{-1}y) \mapsto \nabla\psi(x) \in \mathbb{R}^n$$

  pushes forward the Riemannian volume to the uniform measure on the convex body $K \subset \mathbb{R}^n$.
- For $x \in \mathbb{R}^n$, the real torus

$$\left\{x + \sqrt{-1}y;\, y \in \mathbb{R}^n/\mathbb{Z}^n\right\}$$

  is the pre-image of a point in K.

Simple formula for Ricci curvature

We would like to apply the technique demonstrated earlier. Two ingredients:

1. We need "enough convexity/curvature" to apply Bochner.
2. We need to estimate the H^{-1}-norm on each torus.

Regarding convexity: Bochner's identity has a **Ricci** term. Our Riemannian manifold has a non-negative Ricci form when

$$x \mapsto \det \nabla^2 \psi(x)$$

is log-concave on $\mathbb{R}^n$. This is the condition we will assume.

- In other words, we need to transport an arbitrary log-concave measure on $\mathbb{R}^n$, to the uniform measure on K.

  For each such μ, we get a Poincaré-type inequality on K.

Messy expressions

The infinitesimal **transportation cost** along each torus is easily computable, yet the formula is messy.

1. For a smooth, convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ set

$$Q^*_{\psi,x}(V) = \sum_{i,j,k,\ell,m,p=1}^{n} V^i V^j \psi^{\ell m} \psi_{jkm} \psi^{kp} \psi_{i\ell p} \qquad (V \in \mathbb{R}^n).$$

2. The dual quadratic form is

$$Q_{\psi,x}(U) = \sup \left\{ 4 \left(\sum_{i,j=1}^{n} \psi_{ij} U^i V^j \right)^2 \ ; \ V \in \mathbb{R}^n, \ Q^*_{\psi,x}(V) \leq 1 \right\},$$

for $x \in K$ and $U \in \mathbb{R}^n$, where ψ_{ij} is evaluated at $\nabla \psi^*(x)$.

An almost-inapplicable, too general theorem

Theorem

Let $K \subset \mathbb{R}^n$ be a convex set. Suppose $\psi : \mathbb{R}^n \to \mathbb{R}$ is a smooth, convex function, with

1. $\nabla\psi(\mathbb{R}^n) = K$.
2. $x \mapsto \det \nabla^2\psi(x)$ is log-concave on $\mathbb{R}^n$.
3. The function ψ should be "regular at infinity", a mild technical assumption.

Then, for any function $f : K \to \mathbb{R}$,

$$\int_K f = 0 \quad \Rightarrow \quad \int_K f^2 \leq \int_K Q_{\psi,x}(\nabla f)dx.$$

- We may thus take the infimum over all such functions ψ.
- We may replace the convex body $K \subset \mathbb{R}^n$ by a log-concave probability measure μ. Condition 2 is replaced by "$\nabla\psi$ transports a log-concave measure to μ." (Kolesnikov, '11)

Well, it's not entirely inapplicable...

Say, for the case where K is a Euclidean ball, we may construct reasonable functions ψ with $\det \nabla^2 \psi$ log-concave...

- We may test our general theorem in the case where $K \subset \mathbb{R}^n$ is a simplex, and

$$\psi(x) = \log\left(1 + e^{x_1} + \ldots + e^{x_n}\right).$$

- The Kähler manifold obtained is $\mathbb{CP}^n$ (more precisely, it is the open, dense subset of a full measure in $\mathbb{CP}^n$, where the toric action is free).
- In this case, we get wonderful Poincaré type inequalities.

We were not able to write down reasonable, explicit formulae in any case, other than $\mathbb{C}^n$ and $\mathbb{CP}^n$.

Another approach, without complexification

Let us inspect again the "dual Bochner" inequality, in the context of a Riemmanian manifold with a measure (M, g, μ).

- The $L^2(\mu)$-norm is defined also for vector fields:

$$\langle V, W \rangle_{L^2(\mu)} = \int_M \langle V, W \rangle_g d\mu.$$

- The operator $\triangle_\mu$ satisfies $\langle f, \triangle_\mu g \rangle_{L^2(\mu)} = -\langle \nabla f, \nabla g \rangle_{L^2(\mu)}$.

For any Banach (semi-)norm $\| \cdot \|$ on vector fields:

$$\forall u, \ \int (\triangle_\mu u)^2 d\mu \geq \|\nabla u\|^2 \quad \Longleftrightarrow \quad \forall f, \ Var_\mu(f) \leq \|\nabla f\|_*^2.$$

Recall that Bochner's formula tells us:

$$\int_M (\triangle_\mu u)^2 d\mu = \int_M \|\nabla_M^2 u\|_{L^2(\mu)}^2 d\mu + \int_M Ric_{M,\mu}(\nabla u, \nabla u) d\mu$$

Anti-climax?

- The toric Kähler manifold $(\mathbb{T}^n_{\mathbb{C}}, \omega_\psi)$ is rather similar to the measure-metric space
$$X = (\mathbb{R}^n, \nabla^2 \psi, \mu)$$
where $d\mu = \det \nabla^2 \psi(x) dx$.
(the Riemannian volume is $d\mu = \sqrt{\det \nabla^2 \psi(x)} dx$).

Simply apply the Bochner formula for X, play with the various terms, and deduce the dual version of our “too general theorem”.

Our results are thus trivially “realized” (i.e., the inverse operation to “complexify”).

- In particular, the amusing theorem for measures on $\mathbb{R}^n_+$ follows directly by Brascamp-Lieb and change of variables (this proof works for non-integer ℓ as well...)

Many questions remain

- Is this point of view of "symmetries" and "transportation cost" helpful? Does it reduce to trivial manipulations of Bochner's formula?

Another idea whose origin lies in complex geometry: **Kähler-Einstein manifolds.**

Theorem (Wang-Zhu '04, Donaldson '08, E. Legendre '11, Berman-Berndtsson '12)

Suppose $K \subset \mathbb{R}^n$ is a convex body, barycenter at the origin. Then there exists convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that

(a) $\det \nabla^2 \psi(x) = C_K e^{-\psi(x)}$.

(b) $\nabla \psi(\mathbb{R}^n) = K$.

- The Kähler manifold whose potential is ψ, has Ricci tensor which equals half of the metric. (Hence we have log Sobolev inequality, Gaussian concentration etc.)
- Related to the Mongé-Ampere eigenfunction (Lions '86).

A bit more on Kähler-Einstein transportation

In the language of measure-probability spaces: Suppose $K \subset \mathbb{R}^n$ is convex, barycenter at the origin.

- Then there is a Riemmanian metric-measure space

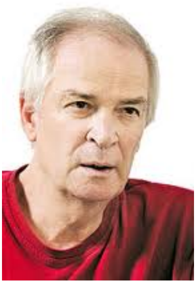
$$X_K = (K, \nabla^2 \varphi, \mu_K)$$

  where μ_K is the uniform measure on K, and the Riemannian metric is induced by φ (the Legendre transform of ψ from the previous theorem).
- The Laplacian's spectral gap is one, with multiplicity n. The eigenspace consists of all linear functionals!
- The Ricci tensor equals half the metric tensor. This space "resembles the Gaussian" in so many respects.

This space X_K must be good for something, isn't it?

Thank you!

Kähler



Einstein

Thank you!

Kähler

Einstein