Approximately gaussian marginals and the hyperplane conjecture

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Abstract

We discuss connections between certain well-known open problems related to the uniform measure on a high-dimensional convex body. In particular, we show that the "thin shell conjecture" implies the "hyperplane conjecture". This extends a result by K. Ball, according to which the stronger "spectral gap conjecture" implies the "hyperplane conjecture".

1 Introduction

Little is currently known about the uniform measure on a general high-dimensional convex body. Many aspects of the Euclidean ball or the unit cube are easy to analyze, yet it is difficult to answer even some of the simplest questions regarding arbitrary convex bodies, lacking symmetries and structure. For example,

Question 1.1 *Is there a universal constant $c > 0$ such that for any dimension n and a convex body $K \subset \mathbb{R}^n$ with $Vol_n(K) = 1$, there exists a hyperplane $H \subset \mathbb{R}^n$ for which $Vol_{n-1}(K \cap H) > c$?*

Here, of course, Vol_k stands for k-dimensional volume. A convex body is a bounded, open convex set. Question 1.1 is referred to as the "slicing problem" or the "hyperplane conjecture", and was raised by Bourgain [5, 6] in relation to the maximal function in high dimensions. It was demonstrated by Ball [2] that Question 1.1 and similar questions are most naturally formulated in the broader class of logarithmically concave densities.

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A probability density $\rho : \mathbb{R}^n \to [0,\infty)$ is called *log-concave* if it takes the form $\rho = \exp(-H)$ for a convex function $H : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. A probability measure is log-concave if it has a log-concave density. The uniform probability measure on a convex body is an example of a log-concave probability measure, as well as the standard gaussian measure on $\mathbb{R}^n$. A log-concave probability density decays exponentially at infinity (e.g., [17, Lemma 2.1]), and thus has moments of all orders. For a probability measure μ on $\mathbb{R}^n$ with finite second moments, we consider its barycenter $b(\mu) \in \mathbb{R}^n$ and covariance matrix $Cov(\mu)$ defined by

$$b(\mu) = \int_{\mathbb{R}^n} x d\mu(x), \qquad Cov(\mu) = \int_{\mathbb{R}^n} (x - b(\mu)) \otimes (x - b(\mu)) d\mu(x)$$

where for $x \in \mathbb{R}^n$ we write $x \otimes x$ for the $n \times n$ matrix $(x_i x_j)_{i,j=1,\ldots,n}$. A log-concave probability measure μ on $\mathbb{R}^n$ is *isotropic* if its barycenter lies at the origin and its covariance matrix is the identity matrix. For an isotropic, log-concave probability measure μ on $\mathbb{R}^n$ we denote

$$L_\mu = L_f = f(0)^{1/n}$$

where f is the log-concave density of μ. It is well-known (see, e.g., [17, Lemma 3.1]) that $L_f > c$, for some universal constant $c > 0$. Define

$$L_n = \sup_\mu L_\mu$$

where the supremum runs over all isotropic, log-concave probability measures μ on $\mathbb{R}^n$. As follows from the works of Ball [2], Bourgain [5], Fradelizi [11], Hensley [12] and Milman and Pajor [20], Question 1.1 is directly equivalent to the following:

Question 1.2 *Is it true that* $\sup_n L_n < \infty$*?*

See also Milman and Pajor [20] and the second author's paper [16] for a survey of results revolving around this question. For a convex body $K \subset \mathbb{R}^n$ we write μ_K for the uniform probability measure on K. A convex body $K \subset \mathbb{R}^n$ is centrally-symmetric if $K = -K$. It is known that

$$L_n \leq C \sup_{K \subset \mathbb{R}^n} L_{\mu_K} \tag{1}$$

where the supremum runs over all centrally-symmetric convex bodies $K \subset \mathbb{R}^n$ for which μ_K is isotropic, and $C > 0$ is a universal constant. Indeed, the reduction from log-concave distributions to convex bodies was proven by Ball [2] (see [16] for the straightforward generalization to the non-symmetric case), and the reduction from general convex bodies to centrally-symmetric ones was outlined, e.g., in the last paragraph of [15]. The best estimate known to date is $L_n < Cn^{1/4}$ for a universal constant $C > 0$ (see [16]), which slightly sharpens an earlier estimate by Bourgain [7, 8, 9].

Our goal in this note is to establish a connection between the slicing problem and another open problem in high-dimensional convex geometry. Write $|\cdot|$ for the standard Euclidean norm in $\mathbb{R}^n$, and denote by $x \cdot y$ the scalar product of $x, y \in \mathbb{R}^n$. We say that a random vector X in $\mathbb{R}^n$ is isotropic and log-concave if it is distributed according to an isotropic, log-concave probability measure. Let $\sigma_n \geq 0$ satisfy

$$\sigma_n^2 = \sup_X \mathbb{E}(|X| - \sqrt{n})^2 \tag{2}$$

where the supremum runs over all isotropic, log-concave random vectors X in $\mathbb{R}^n$. The parameter σ_n measures the width of the "thin spherical shell" of radius $\sqrt{n}$ in which most of the mass of X is located. See (5) below for another definition of σ_n, equivalent up to a universal constant, which is perhaps more common in the literature. It is known that $\sigma_n \leq Cn^{0.41}$ where $C > 0$ is a universal constant (see [19]), and it is suggested in the works of Anttila, Ball and Perissinaki [1] and of Bobkov and Koldobsky [4] that perhaps

$$\sigma_n \leq C \tag{3}$$

for a universal constant $C > 0$. Again, up to a universal constant, one may restrict attention in (2) to random vectors that are distributed uniformly in centrally-symmetric convex bodies. This essentially follows from the same technique as in the case of the parameter L_n mentioned above.

The importance of the parameter σ_n stems from the central limit theorem for convex bodies [18]. This theorem asserts that most of the one-dimensional marginals of an isotropic, log-concave random vector are approximately gaussian. The Kolmogorov distance to the standard gaussian distribution of a typical marginal has roughly the order of magnitude of $\sigma_n/\sqrt{n}$. Therefore, the conjectured bound (3) actually concerns the quality of the gaussian approximation to the marginals of high-dimensional log-concave measures. Our main result reads as follows:

Inequality 1.1 *For any $n \geq 1$,*

$$L_n \leq C\sigma_n \tag{4}$$

where $C > 0$ is a universal constant.

Inequality 1.1 states, in particular, that an affirmative answer to the slicing problem follows from the *thin shell conjecture* (3). This sharpens a result announced by Ball [3], according to which a positive answer to the slicing problem is implied by the stronger conjecture suggested by Kannan, Lovász and Simonovits [13]. The quick argument leading from the latter conjecture to (3) is explained in Bobkov and Koldobsky [4]. Write $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ for the unit sphere, and denote

$$\underline{\sigma}_n = \frac{1}{\sqrt{n}} \sup_X \left| \mathbb{E} X |X|^2 \right| = \frac{1}{\sqrt{n}} \sup_X \sup_{\theta \in S^{n-1}} \mathbb{E}(X \cdot \theta)|X|^2,$$

where the supremum runs over all isotropic, log-concave random vectors X in $\mathbb{R}^n$.

Lemma 1.3 *For any $n \geq 1$,*

$$\sigma_n^2 \leq \frac{1}{n} \sup_X \mathbb{E}(|X|^2 - n)^2 \leq C\sigma_n^2, \tag{5}$$

where the supremum runs over all isotropic, log-concave random vectors X in $\mathbb{R}^n$. Furthermore,

$$1 \leq \underline{\sigma}_n \leq C\sigma_n \leq \tilde{C} n^{0.41}.$$

Here, $C, \tilde{C} > 0$ are universal constants.

Inequality 1.1 may be sharpened, in view of Lemma 1.3, to the bound

$$L_n \leq C \underline{\sigma}_n,$$

for a universal constant $C > 0$. This is explained in the proof of Inequality 1.1 in Section 3. Our argument involves a certain Riemannian structure, which is presented in Section 2.

As the reader has probably already guessed, we use the letters $c, \tilde{c}, c', C, \tilde{C}, C'$ to denote positive universal constants, whose value is not necessarily the same in different appearances. Further notation and facts to be used throughout the text: The support $Supp(\mu)$ of a Borel measure μ on $\mathbb{R}^n$ is the minimal closed set of full measure. When μ is log-concave, its support is a convex set. For a Borel measure μ on $\mathbb{R}^n$ and a Borel map $T : \mathbb{R}^n \to \mathbb{R}^k$ we define the push-forward of μ under T to be the measure $\nu = T_*(\mu)$ on $\mathbb{R}^k$ with

$$\nu(A) = \mu(T^{-1}(A)) \quad \text{for any Borel set } A \subset \mathbb{R}^k.$$

Note that for any log-concave probability measure μ on $\mathbb{R}^n$, there exists an invertible affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T_*(\mu)$ is isotropic. When

T is a linear function and $k < n$, we say that $T_*(\mu)$ is a marginal of μ. The Prékopa-Leindler inequality implies that any marginal of a log-concave probability measure is itself a log-concave probability measure. The Euclidean unit ball is denoted by $B_2^n = \{x \in \mathbb{R}^n; |x| \leq 1\}$, and its volume satisfies

$$\frac{c}{\sqrt{n}} \leq Vol_n(B_2^n)^{1/n} \leq \frac{C}{\sqrt{n}}.$$

We write $\nabla\varphi$ for the gradient of the function φ, and $\nabla^2\varphi$ for the hessian matrix. For $\theta \in S^{n-1}$ we write ∂_θ for differentiation in direction θ, and $\partial_{\theta\theta}(\varphi) = \partial_\theta(\partial_\theta\varphi)$.

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2 A Riemannian metric associated with a convex body

The main mathematical idea presented in this note is a certain Riemannian metric associated with a convex body $K \subset \mathbb{R}^n$. Our construction is affinely invariant: We actually associate a Riemannian metric with any affine equivalence class of convex bodies (two convex bodies in $\mathbb{R}^n$ are affinely equivalent if there exists an invertible affine transformation that maps one to the other. Thus, all ellipsoids are affinely equivalent).

Begin by recalling the technique from [16]. Suppose that μ is a compactly-supported Borel probability measure on $\mathbb{R}^n$ whose support is not contained in a hyperplane. Denote by $K \subset \mathbb{R}^n$ the interior of the convex hull of $Supp(\mu)$, so K is a convex body. The *logarithmic Laplace transform* of μ is

$$\Lambda(\xi) = \Lambda_\mu(\xi) = \log \int_{\mathbb{R}^n} \exp(\xi \cdot x) d\mu(x) \qquad (\xi \in \mathbb{R}^n). \tag{6}$$

The function Λ is strictly convex and C^∞-smooth on $\mathbb{R}^n$. For $\xi \in \mathbb{R}^n$ let μ_ξ be the probability measure on $\mathbb{R}^n$ for which the density $d\mu_\xi/d\mu$ is proportional to $x \mapsto \exp(\xi \cdot x)$. Differentiating under the integral sign, we see that

$$\nabla\Lambda(\xi) = b(\mu_\xi), \qquad \nabla^2\Lambda(\xi) = Cov(\mu_\xi) \qquad (\xi \in \mathbb{R}^n),$$

where $b(\mu_\xi)$ is the barycenter of the probability measure μ_ξ and $Cov(\mu_\xi)$ is the covariance matrix. We learned the following lemma from Gromov's work [10]. A proof is provided for the reader's convenience.

Lemma 2.1 *In the above notation,*

$$\int_{\mathbb{R}^n} \det \nabla^2 \Lambda(\xi) d\xi = Vol_n(K).$$

Proof: It is well-known that the open set $\nabla\Lambda(\mathbb{R}^n) = \{\nabla\Lambda(\xi); \xi \in \mathbb{R}^n\}$ is convex, and that the map $\xi \mapsto \nabla\Lambda(\xi)$ is one-to-one (see, e.g., Rockafellar [22, Theorem 26.5]). Denote by $\overline{K}$ the closure of K. Then,

$$\nabla\Lambda(\mathbb{R}^n) \subseteq \overline{K} \tag{7}$$

since for any $\xi \in \mathbb{R}^n$, the point $\nabla\Lambda(\xi) \in \mathbb{R}^n$ is the barycenter of a certain probability measure supported on the compact, convex set $\overline{K}$. Next we show that $\overline{\nabla\Lambda(\mathbb{R}^n)}$ contains all of the exposed points of $Supp(\mu)$. Let $x_0 \in Supp(\mu)$ be an exposed point, i.e., there exists $\xi \in \mathbb{R}^n$ such that

$$\xi \cdot x_0 > \xi \cdot x \qquad \text{for all } x_0 \neq x \in Supp(\mu). \tag{8}$$

We claim that

$$\lim_{r \to \infty} \nabla\Lambda(r\xi) = x_0. \tag{9}$$

Indeed, (9) follows from (8) and from the fact that x_0 belongs to the support of μ: The measure $\mu_{r\xi}$ converges weakly to the delta measure δ_{x_0} as $r \to \infty$, hence the barycenter of $\mu_{r\xi}$ tends to x_0. Therefore $x_0 \in \overline{\nabla\Lambda(\mathbb{R}^n)}$. Any exposed point of $\overline{K}$ is an exposed point of $Supp(\mu)$, and we conclude that all of the exposed points of $\overline{K}$ are contained in $\overline{\nabla\Lambda(\mathbb{R}^n)}$. From Straszewicz's theorem (see, e.g., Schneider [23, Theorem 1.4.7]) and from (7) we deduce that

$$\overline{K} = \overline{\nabla\Lambda(\mathbb{R}^n)}.$$

The set $\nabla\Lambda(\mathbb{R}^n)$ is open and convex, hence necessarily $\nabla\Lambda(\mathbb{R}^n) = K$. Since Λ is strictly-convex, its hessian is positive-definite everywhere, and according to the change of variables formula,

$$Vol_n(K) = Vol_n\left(\nabla\Lambda(\mathbb{R}^n)\right) = \int_{\mathbb{R}^n} \det \nabla^2\Lambda(\xi) d\xi.$$

$\square$

Recall that μ is any compactly-supported probability measure on $\mathbb{R}^n$ whose support is not contained in a hyperplane. For each $\xi \in \mathbb{R}^n$ the hessian matrix $\nabla^2 \Lambda(\xi) = Cov(\mu_\xi)$ is positive definite. For $\xi \in \mathbb{R}^n$ set

$$g(\xi)(u,v) = g_\mu(\xi)(u,v) = Cov(\mu_\xi) u \cdot v \qquad (u,v \in \mathbb{R}^n). \tag{10}$$

Then $g_\mu(\xi)$ is a positive-definite bilinear form for any $\xi \in \mathbb{R}^n$, and thus g_μ is a Riemannian metric on $\mathbb{R}^n$. We also set

$$\Psi_\mu(\xi) = \log \frac{\det \nabla^2 \Lambda(\xi)}{\det \nabla^2 \Lambda(0)} = \log \frac{\det Cov(\mu_\xi)}{\det Cov(\mu)} \qquad (\xi \in \mathbb{R}^n). \tag{11}$$

We say that $X_\mu = (\mathbb{R}^n, g_\mu, \Psi_\mu, 0)$ is the "Riemannian package associated with the measure μ".

Definition 2.2 *A "Riemannian package of dimension n" is a quadruple $X = (U, g, \Psi, x_0)$ where $U \subset \mathbb{R}^n$ is an open set, g is a Riemannian metric on U, $x_0 \in U$ and $\Psi : U \to \mathbb{R}$ is a function with $\Psi(x_0) = 0$.*

Suppose $X = (U, g, \Psi, x_0)$ and $Y = (V, h, \Phi, y_0)$ are Riemannian packages. A map $\varphi : U \to V$ is an isomorphism of X and Y if the following conditions hold:

1. φ is a Riemannian isometry between the Riemannian manifolds (U, g) and (V, h).
2. $\varphi(x_0) = y_0$.
3. $\Phi(\varphi(x)) = \Psi(x)$ for any $x \in U$.

When such an isomorphism exists we say that X and Y are isomorphic, and we write $X \cong Y$.

Let us describe an additional construction of the same Riemannian package associated with μ, a construction which is dual to the one mentioned above. Consider the Legendre transform

$$\Lambda^*(x) = \sup_{\xi \in \mathbb{R}^n} \left[\xi \cdot x - \Lambda(\xi) \right] \qquad (x \in K).$$

Then $\Lambda^* : K \to \mathbb{R}$ is a strictly-convex C^∞-function, and $\nabla \Lambda^* : K \to \mathbb{R}^n$ is the inverse map of $\nabla \Lambda : \mathbb{R}^n \to K$ (see Rockafellar [22, Chapter V]). Define

$$\Phi_\mu(x) = \log \frac{\det \nabla^2 \Lambda^*(b(\mu))}{\det \nabla^2 \Lambda^*(x)} \qquad (x \in K),$$

and for $x \in K$ set

$$h(x)(u,v) = h_\mu(x)(u,v) = \left[\nabla^2 \Lambda^* \right](x) u \cdot v \qquad (u,v \in \mathbb{R}^n).$$

Then h_μ is a Riemannian metric on K. Note the identity

$$\left[\nabla^2\Lambda(\xi)\right]^{-1} = \left[\nabla^2\Lambda^*\right](\nabla\Lambda(\xi)) \qquad\qquad (\xi \in \mathbb{R}^n).$$

Using this identity, it is a simple exercise to verify that the Riemannian package $\tilde{X}_\mu = (K, h_\mu, \Phi_\mu, b(\mu))$ is isomorphic to the Riemannian package $X_\mu = (\mathbb{R}^n, g_\mu, \Psi_\mu, 0)$ described earlier, with $x = \nabla\Lambda(\xi)$ being the isomorphism.

The constructions X_μ and $\tilde{X}_\mu$ are equivalent, and each has advantages over the other. It seems that X_μ is preferable when carrying out computations, as the notation is usually less heavy in this case. On the other hand, the definition $\tilde{X}_\mu$ is perhaps easier to visualize: Suppose that μ is the uniform probability measure on K. In this case $\tilde{X}_\mu$ equips the convex body K itself with a Riemannian structure. One is thus tempted to imagine, for instance, how geodesics look on K, and what is a Brownian motion in the body K with respect to this metric. The following lemma shows that this Riemannian structure on K is invariant under linear transformations.

Lemma 2.3 *Suppose μ and ν are compactly-supported probability measures on $\mathbb{R}^n$ whose support is not contained in a hyperplane. Assume that there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\nu = T_*(\mu).$$

Then $X_\mu \cong X_\nu$.

Proof: It is straightforward to check that the linear map T^t (the transposed matrix) is the required isometry between the Riemannian manifolds $(\mathbb{R}^n, g_\nu)$ and $(\mathbb{R}^n, g_\mu)$. However, perhaps a better way to understand this isomorphism, is to note that the construction of X_μ may be carried out in a more abstract fashion: Suppose that V is an n-dimensional linear space, denote by V^* the dual space, and let μ be a compactly-supported Borel probability measure on V whose support is not contained in a proper affine subspace of V. The logarithmic Laplace transform $\Lambda : V^* \to \mathbb{R}$ is well-defined, as is the family of probability measures μ_ξ $(\xi \in V^*)$ on the space V. For a point $\xi \in V^*$ and two tangent vectors $\eta, \zeta \in T_\xi V^* \equiv V^*$, set

$$g_\xi(\eta,\zeta) = \int_V \eta(x)\zeta(x)d\mu_\xi(x) - \left(\int_V \eta(x)d\mu_\xi(x)\right)\left(\int_V \zeta(x)d\mu_\xi(x)\right). \tag{12}$$

A moment of reflection reveals that the definition (12) of the positive-definite bilinear form g_ξ is equivalent to the definition (10) given above. Additionally, there exists a linear operator $A_\xi : V^* \to V^*$, which is self-adjoint and positive-definite with respect to the bilinear form g_0, that satisfies

$$g_\xi(\eta, \zeta) = g_0(A_\xi \eta, \zeta) \qquad \text{for all } \eta, \zeta \in V^*.$$

Hence we may define $\Psi(\xi) = \log \det A_\xi$, which coincides with the definition (11) of Ψ_μ above. Therefore, $X_\mu = (V^*, g, \Psi, 0)$ is the Riemannian package associated with μ. Back to the lemma, we see that X_μ is constructed from exactly the same data as X_ν, hence they must be isomorphic. $\square$

Corollary 2.4 *Suppose μ and ν are compactly-supported probability measures on $\mathbb{R}^n$ whose support is not contained in a hyperplane. Assume that there exists an affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\nu = T_*(\mu).$$

Then $X_\mu \cong X_\nu$.

Proof: The only difference from Lemma 2.3 is that the map T is assumed to be affine, and not linear. It is clearly enough to deal with the case where T is a translation, i.e.,

$$T(x) = x + x_0 \qquad\qquad (x \in \mathbb{R}^n)$$

for a certain vector $x_0 \in \mathbb{R}^n$. From the definition (6) we see that

$$\Lambda_\nu(\xi) = \xi \cdot x_0 + \Lambda_\mu(\xi) \qquad (\xi \in \mathbb{R}^n).$$

Adding a linear functional does not influence second derivatives, hence $g_\mu = g_\nu$ and also $\Psi_\mu = \Psi_\nu$. Therefore $X_\mu = (\mathbb{R}^n, g_\mu, \Psi_\mu, 0)$ is trivially isomorphic to $X_\nu = (\mathbb{R}^n, g_\nu, \Psi_\nu, 0)$. $\square$

An n-dimensional Riemannian package is of "log-concave type" if it is isomorphic to the Riemannian package X_μ associated with a compactly-supported, log-concave probability measure μ on $\mathbb{R}^n$. Note that according to our terminology, a log-concave probability measure is absolutely-continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, hence its support is never contained in a hyperplane.

Lemma 2.5 *Suppose $X = (U, g, \Psi, \xi_0)$ is an n-dimensional Riemannian package of log-concave type. Let $\xi_1 \in U$. Denote*

$$\tilde{\Psi}(\xi) = \Psi(\xi) - \Psi(\xi_1) \qquad (\xi \in U). \tag{13}$$

Then also $Y = (U, g, \tilde{\Psi}, \xi_1)$ is an n-dimensional Riemannian package of log-concave type.

Proof: Let μ be a compactly-supported log-concave probability measure on $\mathbb{R}^n$ whose associated Riemannian package $X_\mu = (\mathbb{R}^n, g_\mu, \Psi_\mu, 0)$ is isomorphic to X. Thanks to the isomorphism, we may identify ξ_1 with a certain point in $\mathbb{R}^n$, which will still be denoted by ξ_1 (with a slight abuse of notation). We now interpret the definition (13) as

$$\tilde{\Psi}(\xi) = \Psi(\xi) - \Psi(\xi_1) \qquad (\xi \in \mathbb{R}^n).$$

In order to prove the lemma, we need to demonstrate that

$$Y = (\mathbb{R}^n, g_\mu, \tilde{\Psi}, \xi_1) \tag{14}$$

is of log-concave type. Recall that μ_{ξ_1} is the compactly-supported probability measure on $\mathbb{R}^n$ whose density with respect to μ is proportional to $x \mapsto \exp(\xi_1 \cdot x)$. A crucial observation is that μ_{ξ_1} is log-concave. Set $\nu = \mu_{\xi_1}$, and note the relation

$$\Lambda_\nu(\xi) = \Lambda_\mu(\xi + \xi_1) - \Lambda_\mu(\xi_1) \qquad (\xi \in \mathbb{R}^n). \tag{15}$$

It suffices to show that the Riemannian package Y in (14) is isomorphic to $X_\nu = (\mathbb{R}^n, g_\nu, \Psi_\nu, 0)$. We claim that an isomorphism φ between X_ν and Y is simply the translation

$$\varphi(\xi) = \xi + \xi_1 \qquad (\xi \in \mathbb{R}^n).$$

In order to see that φ is indeed an isomorphism, note that (15) yields

$$\nabla^2 \Lambda_\nu(\xi) = \nabla^2 \Lambda_\mu(\xi + \xi_1) \qquad (\xi \in \mathbb{R}^n), \tag{16}$$

hence φ is a Riemannian isometry between $(\mathbb{R}^n, g_\nu)$ and $(\mathbb{R}^n, g_\mu)$, with $\varphi(0) = \xi_1$. The relation (16) implies that $\tilde{\Psi}(\varphi(\xi)) = \Psi_\nu(\xi)$ for all $\xi \in \mathbb{R}^n$. Hence φ is an isomorphism between Riemannain packages, and the lemma is proven. $\square$

Remark. When μ is a product measure on $\mathbb{R}^n$ (such as the uniform probability measure on the cube, or the gaussian measure), straightforward computations of curvature show that the manifold $(\mathbb{R}^n, g_\mu)$ is flat (i.e., all sectional curvatures vanish). We were not able to extract meaningful information from the local structure of the Riemannian manifold $(\mathbb{R}^n, g_\mu)$ in the general case.

3 Inequalities

Proof of Lemma 1.3: First, note that for any random vector X in $\mathbb{R}^n$ with finite fourth moments,

$$\mathbb{E}(|X| - \sqrt{n})^2 \leq \frac{1}{n}\mathbb{E}(|X| - \sqrt{n})^2(|X| + \sqrt{n})^2 = \frac{1}{n}\mathbb{E}(|X|^2 - n)^2.$$

This proves the inequality on the left in (5). Regarding the inequality on the right, we use the bound

$$\mathbb{E}|X|^4 1_{|X|>C\sqrt{n}} \leq C\exp\left(-\sqrt{n}\right) \tag{17}$$

which follows from Paouris theorem [21]. Here $1_{|X|>C\sqrt{n}}$ is the random variable that equals one when $|X| > C\sqrt{n}$ and vanishes otherwise. Apply again the identity $|X|^2 - n = (|X| - \sqrt{n})(|X| + \sqrt{n})$ to conclude that

$$\begin{aligned}\mathbb{E}(|X|^2-n)^2 &= \mathbb{E}(|X|^2-n)^2 1_{|X|\leq C\sqrt{n}} + \mathbb{E}(|X|^2-n)^2 1_{|X|>C\sqrt{n}} \\ &\leq \ (C+1)^2 n\mathbb{E}(|X|-\sqrt{n})^2 + \mathbb{E}|X|^4 1_{|X|>C\sqrt{n}},\end{aligned} \tag{18}$$

where $C \geq 1$ is the universal constant from (17). A simple computation shows that $\sigma_n \geq \sqrt{2}$, as is witnessed by the standard gaussian random vector in $\mathbb{R}^n$, or by the example in the next paragraph. Thus the inequality on the right in (5) follows from (17) and (18). Our proof of (5) utilized the deep Paouris theorem. Another possibility could be to use [19, Theorem 4.4] or the deviation inequalities for polynomials proved first by Bourgain [7].

In order to prove the second assertion in the lemma, observe that since $\mathbb{E}X = 0$,

$$\mathbb{E}(X\cdot\theta)|X|^2 = \mathbb{E}(X\cdot\theta)(|X|^2-n) \leq \sqrt{\mathbb{E}(X\cdot\theta)^2\mathbb{E}(|X|^2-n)^2} \leq C\sqrt{n}\sigma_n,$$

where we used the Cauchy-Schwartz inequality, the fact that $\mathbb{E}(X\cdot\theta)^2 = 1$ and (5). It remains to prove that $\underline{\sigma}_n \geq 2$. To this end, consider the case where $Y_1, \ldots, Y_n$ are independent random variables, all distributed according to the density $t \mapsto e^{-I(t+1)}$ on the real line, where $I(a) = a$ for $a \geq 0$ and $I(a) = +\infty$ for $a < 0$. Then $Y = (Y_1, \ldots, Y_n)$ is a random vector distributed according to an isotropic, log-concave probability measure on $\mathbb{R}^n$, and

$$\mathbb{E}\frac{\sum_{j=1}^n Y_j}{\sqrt{n}}|Y|^2 = 2\sqrt{n}.$$

This completes the proof. $\square$

When φ is a smooth real-valued function on a Riemannian manifold (M, g), we denote its gradient at the point $x_0 \in M$ by $\nabla_g\varphi(x_0) \in T_{x_0}(M)$. Here $T_{x_0}(M)$ stands for the tangent space to M at the point x_0. The subscript g in $\nabla_g\varphi(x_0)$ means that the gradient is computed with respect to the Riemannian metric g. The usual gradient of a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}^n$ is denoted by $\nabla\varphi(x_0) \in \mathbb{R}^n$, without any subscript. The length of a tangent vector $v \in T_{x_0}(M)$ with respect to the metric g is $|v|_g = \sqrt{g_{x_0}(v, v)}$.

Lemma 3.1 *Suppose $X = (U, g, \Psi, \xi_0)$ is an n-dimensional Riemannian package of log-concave type. Then, for any $\xi \in U$,*

$$|\nabla_g \Psi(\xi)|_g \leq \sqrt{n}\underline{\sigma}_n.$$

Proof: Suppose first that $\xi = \xi_0$. We need to establish the bound

$$|\nabla_g \Psi(\xi_0)|_g \leq \sqrt{n}\underline{\sigma}_n \tag{19}$$

for any log-concave package $X = (U, g, \Psi, \xi_0)$ of dimension n. Any such package X is isomorphic to $X_\mu = (\mathbb{R}^n, g_\mu, \Psi_\mu, 0)$ for a certain compactly-supported log-concave probability measure μ on $\mathbb{R}^n$. Furthermore, according to Corollary 2.4, we may apply an appropriate affine map and assume that μ is isotropic. Thus our goal is to prove that

$$|\nabla_{g_\mu} \Psi_\mu(0)|_{g_\mu} \leq \sqrt{n}\underline{\sigma}_n. \tag{20}$$

Since μ is isotropic, $\nabla^2 \Lambda_\mu(0) = Cov(\mu) = Id$, where Id is the identity matrix. Consequently, the desired bound (20) is equivalent to

$$|\nabla \Psi_\mu(0)| \leq \sqrt{n}\underline{\sigma}_n.$$

Equivalently, we need to show that

$$\partial_\theta \log \frac{\det \nabla^2 \Lambda_\mu(\xi)}{\det \nabla^2 \Lambda_\mu(0)}\bigg|_{\xi=0} \leq \sqrt{n}\underline{\sigma}_n \qquad \text{for all } \theta \in S^{n-1}.$$

A straightforward computation shows that $\partial_\theta \log \det \nabla^2 \Lambda_\mu(\xi)$ equals the trace of the matrix $\left(\nabla^2 \Lambda_\mu(\xi)\right)^{-1} \nabla^2 \partial_\theta \Lambda_\mu(\xi)$. Since μ is isotropic,

$$\partial_\theta \log \frac{\det \nabla^2 \Lambda_\mu(\xi)}{\det \nabla^2 \Lambda_\mu(0)}\bigg|_{\xi=0} = \triangle \partial_\theta \Lambda_\mu(0) = \int_{\mathbb{R}^n} (x \cdot \theta)|x|^2 d\mu(x) \leq \sqrt{n}\underline{\sigma}_n,$$

according to the definition of $\underline{\sigma}_n$, where $\triangle$ stands for the usual Laplacian in $\mathbb{R}^n$. This completes the proof of (19). The lemma in thus proven in the special case where $\xi = \xi_0$.

The general case follows from Lemma 2.5: When $\xi \neq \xi_0$, we may consider the log-concave Riemannian package $Y = (U, g, \tilde{\Psi}, \xi)$, where $\tilde{\Psi}$ differs from Ψ by an additive constant. Applying (19) with the log-concave package Y, we see that

$$|\nabla_g \Psi(\xi)|_g = |\nabla_g \tilde{\Psi}(\xi)|_g \leq \sqrt{n}\underline{\sigma}_n.$$

□

The next lemma is a crude upper bound for the Riemannian distance, valid for any Hessian metric (that is, a Riemannian metric on $U \subset \mathbb{R}^n$ induced by the hessian of a convex function).

Lemma 3.2 *Let μ be a compactly-supported probability measure on $\mathbb{R}^n$ whose support is not contained in a hyperplane. Denote by Λ its logarithmic Laplace transform, and let $X_\mu = (\mathbb{R}^n, g_\mu, \Psi_\mu, 0)$ be the associated Riemannian package. Then for any $\xi, \eta \in \mathbb{R}^n$,*

$$d(\xi, \eta) \leq \sqrt{\Lambda(2\xi - \eta) - \Lambda(\eta) - 2\nabla\Lambda(\eta) \cdot (\xi - \eta)}, \tag{21}$$

where $d(\xi, \eta)$ is the Riemannian distance between ξ and η, with respect to the Riemannian metric g_μ. In particular, when the barycenter of μ lies at the origin,

$$d(\xi, 0) \leq \sqrt{\Lambda(2\xi)}. \tag{22}$$

Proof: The bound (21) is obvious when $\xi = \eta$. When $\xi \neq \eta$, we need to exhibit a path from η to ξ whose Riemannian length is at most the expression on the right in (21). Set $\theta = (\xi - \eta)/|\xi - \eta|$ and $R = |\xi - \eta|$. Consider the interval

$$\gamma(t) = \eta + t\theta \qquad (0 \leq t \leq R).$$

This path connects η and ξ, and its Riemannian length is

$$\begin{aligned} \int_0^R \sqrt{g_\mu(\gamma(t))\,(\theta,\theta)}dt &= \int_0^R \sqrt{[\partial_{\theta\theta}\Lambda](\eta + t\theta)}dt \\ = \quad \int_0^R \sqrt{\frac{d^2\Lambda(\eta + t\theta)}{dt^2}}dt &\leq \sqrt{\int_0^{2R} (2R - t)\frac{d^2\Lambda(\eta + t\theta)}{dt^2}dt \int_0^R \frac{dt}{2R - t}}, \end{aligned}$$

according to the Cauchy-Schwartz inequality. Clearly, $\int_0^R dt/(2R - t) = \log 2 \leq 1$. Regarding the other integral, recall Taylor's formula with integral remainder:

$$\int_0^{2R} (2R - t)\frac{d^2\Lambda(\eta + t\theta)}{dt^2}dt = \Lambda(\eta + 2R\theta) - [\Lambda(\eta) + 2R\theta \cdot \nabla\Lambda(\eta)]\,.$$

The inequality (21) is thus proven. Furthermore, $\Lambda(0) = 0$, and when the barycenter of μ lies at the origin, also $\nabla\Lambda(0) = 0$. Thus (22) follows from (21). □

The volume-radius of a convex body $K \subset \mathbb{R}^n$ is

$$v.rad.(K) = (Vol_n(K)/Vol_n(B_2^n))^{1/n}.$$

This is the radius of the Euclidean ball that has exactly the same volume as K. When $E \subseteq \mathbb{R}^n$ is an affine subspace of dimension ℓ and $K \subset E$ is a convex body, we interpret $v.rad.(K)$ as $(Vol(K)/Vol(B_2^\ell))^{1/\ell}$. For a subspace $E \subset \mathbb{R}^n$, denote by $Proj_E : \mathbb{R}^n \to E$ the orthogonal projection operator onto E in $\mathbb{R}^n$. A Borel measure μ on $\mathbb{R}^n$ is *even* or *centrally-symmetric* if $\mu(A) = \mu(-A)$ for any measurable $A \subset \mathbb{R}^n$.

Lemma 3.3 *Let μ be an even, isotropic, log-concave probability measure on $\mathbb{R}^n$. Let $1 \leq t \leq \sqrt{n}$ and denote by $B_t \subset \mathbb{R}^n$ the collection of all $\xi \in \mathbb{R}^n$ with $d(0,\xi) \leq t$, where $d(0,\xi)$ is as in Lemma 3.2. Then,*

$$Vol_n(B_t)^{1/n} \geq c\frac{t}{\sqrt{n}}, \tag{23}$$

where $c > 0$ is a universal constant. Here, as elsewhere, Vol_n stands for the Lebesgue measure on $\mathbb{R}^n$ (and not the Riemannian volume).

Proof: It suffices to prove the lemma under the additional assumption that t is an integer. According to Lemma 3.2,

$$K_t := \{\xi \in \mathbb{R}^n; \Lambda(2\xi) \leq t^2\} \subseteq B_t.$$

Let $E \subset \mathbb{R}^n$ be any t^2-dimensional subspace, and denote by $f_E : \mathbb{R}^n \to [0,\infty)$ the density of the isotropic probability measure $(Proj_E)_*\mu$. Then f_E is a log-concave function, according to the Prékopa-Leindler inequality, and f_E is also an even function. According to the definition above,

$$f_E(0)^{1/t^2} = L_{f_E} \geq c.$$

Note that the restriction of Λ to the subspace E is the logarithmic Laplace transform of $(Proj_E)_*\mu$. It is proven in [17, Lemma 2.8] that

$$v.rad.(K_t \cap E) \geq ct f_E(0)^{1/t^2} \geq c't. \tag{24}$$

The bound (24) holds for any subspace $E \subset \mathbb{R}^n$ of dimension t^2. From [14, Corollary 3.1] we deduce that

$$v.rad.(K_t) \geq \tilde{c}t.$$

Since $K_t \subseteq B_t$, the bound (23) follows. $\square$

Lemma 3.4 *Let μ be a compactly-supported, even, isotropic, log-concave probability measure on $\mathbb{R}^n$. Denote by K the interior of the support of μ, a convex body in $\mathbb{R}^n$. Then,*

$$Vol_n(K)^{1/n} \geq c/\underline{\sigma}_n,$$

where $c > 0$ is a universal constant.

Proof: Set $t = \max\{\sqrt{n}/\underline{\sigma}_n, 1\}$. Then $1 \leq t \leq \sqrt{n}$ and $\underline{\sigma}_n \leq C\sqrt{n}$, according to Lemma 1.3. Recall the definition of the set $B_t \subset \mathbb{R}^n$ from Lemma 3.3. Consider the Riemannian package $X_\mu = (\mathbb{R}^n, g_\mu, \Psi_\mu, 0)$ that is associated with the measure μ. According to Lemma 3.1, for any $\xi \in B_t$,

$$\Psi_\mu(0) - \Psi_\mu(\xi) \leq \sqrt{n}\underline{\sigma}_n d(0, \xi) \leq t\sqrt{n}\underline{\sigma}_n \leq Cn.$$

Since $\Psi_\mu(\xi) = \log\det \nabla^2 \Lambda_\mu(\xi)$ and $\Psi_\mu(0) = 0$, then

$$\det \nabla^2 \Lambda_\mu(\xi) \geq e^{-Cn} \qquad \text{for any } \ \xi \in B_t.$$

From Lemma 2.1,

$$Vol_n(K) = \int_{\mathbb{R}^n} \det \nabla^2 \Lambda_\mu(\xi) d\xi \geq \int_{B_t} \det \nabla^2 \Lambda_\mu(\xi) d\xi \geq e^{-Cn} Vol_n(B_t)$$

as Λ_μ is convex and hence $\det \nabla^2 \Lambda_\mu(\xi) \geq 0$ for all ξ. Lemma 3.3 yields that

$$Vol_n(K)^{1/n} \geq e^{-C}\left(c\frac{t}{\sqrt{n}}\right) \geq \frac{c'}{\underline{\sigma}_n}.$$

The lemma is proven. □

Proof of Inequality 1.1: Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body such that the uniform probability measure μ_K is isotropic. Then,

$$L_{\mu_K} = \frac{1}{Vol_n(K)^{1/n}} \leq C\underline{\sigma}_n$$

thanks to Lemma 3.4. In view of (1), the bound $L_n \leq C\underline{\sigma}_n$ is proven. The desired inequality (4) now follows from Lemma 1.3. □

The following proposition is not applied in this article. It is nevertheless included as it may help understand the nature of the elusive quantity $\left|\mathbb{E}X|X|^2\right|$ for an isotropic, log-concave random vector X in $\mathbb{R}^n$.

Proposition 3.5 *Suppose X is an isotropic random vector in $\mathbb{R}^n$ with finite third moments. Then,*

$$\left|\mathbb{E}X|X|^2\right|^2 \leq Cn^3 \int_{S^{n-1}} \left(\mathbb{E}(X\cdot\theta)^3\right)^2 d\sigma_{n-1}(\theta)$$

where σ_{n-1} is the uniform Lebesgue probability measure on the sphere S^{n-1}, and $C > 0$ is a universal constant.

Proof: Denote $F(\theta) = \mathbb{E}(X \cdot \theta)^3$ for $\theta \in \mathbb{R}^n$. Then $F(\theta)$ is a homogenous polynomial of degree three, and its Laplacian is

$$\triangle F(\theta) = 6\mathbb{E}(X \cdot \theta)|X|^2.$$

Denote $v = \mathbb{E}X|X|^2 \in \mathbb{R}^n$. The function

$$\theta \mapsto F(\theta) - \frac{6}{2n+4}|\theta|^2(\theta \cdot v) \qquad (\theta \in \mathbb{R}^n)$$

is a homogenous, harmonic polynomial of degree three. In other words, the restriction $F|_{S^{n-1}}$ decomposes into spherical harmonics as

$$F(\theta) = \frac{6}{2n+4}(\theta \cdot v) + \left(F(\theta) - \frac{6}{2n+1}(\theta \cdot v) \right) \qquad (\theta \in S^{n-1}).$$

Since spherical harmonics of different degrees are orthogonal to each other,

$$\int_{S^{n-1}} F^2(\theta) d\sigma_{n-1}(\theta) \geq \frac{36}{(2n+4)^2} \int_{S^{n-1}} (\theta \cdot v)^2 d\sigma_{n-1}(\theta) = \frac{36}{n(2n+4)^2}|v|^2.$$

□

Remark. According to Proposition 3.5, if we could show that $\left|\mathbb{E}(X \cdot \theta)^3\right| \leq C/n$ for a typical unit vector $\theta \in S^{n-1}$, we would obtain a positive answer to Question 1.1. It is interesting to note that the function

$$F(\theta) = \mathbb{E}|X \cdot \theta| \qquad (\theta \in S^{n-1})$$

admits tight concentration bounds. For instance,

$$\int_{S^{n-1}} (F(\theta)/E - 1)^2 d\sigma_{n-1}(\theta) \leq C/n^2$$

where $E = \int_{S^{n-1}} F(\theta) d\sigma_{n-1}(\theta)$, whenever X is distributed according to a suitably normalized log-concave probability measure on $\mathbb{R}^n$. The normalization we currently prefer here is slightly different from the isotropic normalization. The details will be explained elsewhere, as well as some relations to the problem of stability in the Brunn-Minkowski inequality.

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