Lipschitz functions on the infinite-dimensional torus

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Abstract

We discuss the spectrum phenomenon for Lipschitz functions on the infinite-dimensional torus. Suppose that f is a measurable, real-valued, Lipschitz function on the torus $\mathbb{T}^\infty$. We prove that there exists a number $a \in \mathbb{R}$ with the following property: For any $\varepsilon > 0$ there exists a parallel, infinite-dimensional subtorus $M \subseteq \mathbb{T}^\infty$ such that the restriction of the function $f - a$ to the subtorus M has an $L^\infty(M)$-norm of at most ε.

1   Background and Results

One of the most remarkable phenomena in high dimensions is the emergence of a *spectrum* for uniformly continuous functions. It was shown by Milman in his proof of Dvoretzky's theorem [6] that given any 1-Lipschitz function f on the high-dimensional sphere S^n, one may find a section of S^n by a linear subspace of large dimension, on which f is nearly a constant function. The value of this constant may be thought of, approximately, as an element in a spectrum associated with f. An analogous effect in discrete mathematics is Ramsey's theorem [1], according to which any coloring of a large complete graph by a fixed number of colors contains a large induced subgraph which is monochromatic.

There have been several attempts to formulate infinite-dimensional analogs of the Ramsey-Dvoretzky-Milman phenomenon. Let X be an infinite-dimensional Banach space whose unit sphere is denoted by $S(X)$. For a function $f : S(X) \to \mathbb{R}$ one defines its infinite-dimensional spectrum $\sigma(f)$ as the collection of all values $a \in \mathbb{R}$ with the following property: For any $\varepsilon > 0$, there exists an infinite-dimensional subspace $Y \subseteq X$ such that

$$|f(v) - a| < \varepsilon \qquad \text{for all } v \in S(Y),$$

where $S(Y)$ is the unit sphere in the subspace Y. A question that was open for many years was whether the infinite-dimensional spectrum of any Lipschitz function is non-empty. Unfortunately, even when X is a Hilbert space, the answer is decisively negative as was proven

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by Odell and Schlumprecht [7, 8]. On the positive side, Gowers [3] proved that any Lipschitz function on the Banach space c_0 admits a non-empty infinite-dimensional spectrum. The space c_0 is essentially the only separable Banach space for which the answer is positive, as was proven in [7, 8].

Here we investigate the question of existence of an infinite-dimensional spectrum in a different situation, that of the infinite-dimensional torus, or of infinite-dimensional product spaces in general. Lipschitz functions on a finite-dimensional torus were analyzed using probabilistic tools by Faifman, Klartag and Milman [2]. In this paper we will exploit the fact that the infinite-dimensional torus admits a product probability measure, which allows one to use probabilistic arguments akin to the finite-dimensional case.

Let us introduce some terminology and notation and recall a few basic facts that are used throughout the paper. The infinite-dimensional torus is typically denoted by $\mathbb{T}^{\mathbb{N}}$ or by $\mathbb{T}^{\infty}$. An element $x \in \mathbb{T}^{\infty}$ is a sequence $x = (x_i)_{i=1,2,\dots}$ with $x_i \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ for all i. Write σ for the uniform probability measure on $\mathbb{T}^{\infty}$, which is a complete product measure, invariant under translations. When we say that a random point X is distributed uniformly on $\mathbb{T}^{\infty}$ or when we say that a function f on $\mathbb{T}^{\infty}$ is measurable, we always refer to the probability measure σ. For $x, y \in \mathbb{T}^{\infty}$ consider the Euclidean metric

$$\mathrm{dist}(x,y) = \sqrt{\sum_{i=1}^{\infty} \mathrm{dist}^2(x_i, y_i)}$$

where $x = (x_i)_{i \geq 1}, y = (y_i)_{i \geq 1}$ and where $\mathrm{dist}(x_i, y_i)$ is the distance between x_i and y_i in the circle $\mathbb{R}/\mathbb{Z}$. It may happen that $d(x,y) = +\infty$ for some $x, y \in \mathbb{T}^{\infty}$. In fact, the torus $\mathbb{T}^{\infty}$ is split into infinitely-many connected components with respect to the metric dist, all of measure zero. It is explained in Gromov [4, Section 3.1] that for any measurable subsets $A, B \subseteq \mathbb{T}^{\infty}$ with $\sigma(A) \cdot \sigma(B) > 0$,

$$\inf_{x \in a, y \in B} \mathrm{dist}(x,y) \leq C(\sigma(A), \sigma(B)) < \infty$$

for a certain explicit function $C : (0,1] \times (0,1] \to [0,\infty)$. A subset $M \subseteq \mathbb{T}^{\infty}$ is a *parallel infinite-dimensional subtorus* if there exist an infinite subset $A \subseteq \mathbb{N}$ and values $b : \mathbb{N} \backslash A \to \mathbb{T}$ such that

$$M = \left\{ (x_i)_{i \geq 1} \in \mathbb{T}^{\infty} \, ; \, x_i = b_i \text{ for all } i \in \mathbb{N} \setminus A \right\}.$$

Note that the uniform probability measure on the infinite-dimensional subtorus M is well-defined, thus one may speak of the space $L^{\infty}(M)$. Our main result is the following:

Theorem 1. *For any measurable function $f : \mathbb{T}^{\infty} \to \mathbb{R}$ that is Lipschitz with respect to the Euclidean metric* dist*, there exists $a \in \mathbb{R}$ with the following property: For any $\varepsilon > 0$ there exists a parallel infinite-dimensional subtorus $M \subseteq \mathbb{T}^{\infty}$ such that $\|f - a\|_{L^{\infty}(M)} < \varepsilon$.*

Theorem 1 thus implies that any measurable, Lipschitz function on $\mathbb{T}^{\infty}$ has a non-empty spectrum in an appropriate sense. In order to have in mind some concrete examples of measurable functions on $\mathbb{T}^{\infty}$, we mention the function

$$f(x) = \sum_{i=1}^{\infty} a_i \cos(2\pi x_i) \qquad\qquad (x \in \mathbb{T}^{\infty}) \qquad\qquad (1)$$

where $\cos(2\pi x_i)$ is clearly well-defined for $x_i \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. By Kolmogorov's three-series theorem (see, e.g., Kahane [5, Section 3]), the series in (1) converges almost everywhere if and only if $\sum a_i^2 < \infty$. Assuming that indeed $\sum a_i^2 < \infty$, the function f is a well-defined[1], measurable function on $\mathbb{T}^\infty$ which is in fact Lipschitz with respect to the Euclidean metric dist.

We proceed to discuss the necessity of the assumptions of Theorem 1. The condition of measurability is essential: Indeed, fixing a representative x^C for each dist-connected component C of $\mathbb{T}^\infty$, and letting $f(x) = \inf_C \text{dist}(x, x^C)$, we get a Lipschitz function, the restriction of which to every parallel, infinite-dimensional subtorus has arbitrarily large values. An example of a measurable function which is non-Lipschitz and has an empty spectrum may be constructed as follows: It is well-known that there exists a Borel subset $B \subset \mathbb{R}$ such that both B and $\mathbb{R} \setminus B$ intersect any non-empty interval in a set of a positive Lebesgue measure. Consider the set

$$A = \left\{ (x_1, x_2, \ldots) \in \mathbb{T}^\infty \,;\, \sum_{i=1}^\infty \frac{\cos(2\pi x_i)}{i^2} \in B \right\}.$$

Then the indicator function $f = 1_A$ is a measurable function which has no spectrum.

In general, a measurable, dist-Lipschitz function need not be continuous with respect to the usual product topology on $\mathbb{T}^\infty$. The function in (1) is continuous with respect to the product topology only under the stronger requirement that $\sum |a_i| < \infty$. For a function $f : \mathbb{T}^\infty \to \mathbb{R}$ that is continuous in the product topology, its image coincides with its spectrum. This is because every element of the basis of the topology contains a parallel infinite-dimensional subtorus of the form $M = \{(x_i)_{i \geq 1} \in \mathbb{T}^\infty \,;\, x_i = b_i \,\forall i < N\}$.

In addition to the Euclidean metric dist, one defines for $1 \leq p \leq \infty$ and $x, y \in \mathbb{T}^\infty$ the distance dist_p by

$$\text{dist}_p(x, y) = \left(\sum_{i=1}^\infty \text{dist}^p(x_i, y_i) \right)^{1/p}, \tag{2}$$

where the case $p = \infty$ is defined by $\text{dist}_\infty(x, y) = \sup_{i \geq 1} \text{dist}(x_i, y_i)$. Theorem 1 is the case $p = 2$ of the following:

Theorem 2. *For any $1 < p \leq \infty$ and a measurable function $f : \mathbb{T}^\infty \to \mathbb{R}$ which is Lipschitz with respect to the metric dist_p, there exists $a \in \mathbb{R}$ with the following property: For any $\varepsilon > 0$ there exists a parallel infinite-dimensional subtorus $M \subseteq \mathbb{T}^\infty$ such that $\|f - a\|_{L^\infty(M)} < \varepsilon$.*

It is only the product structure of $\mathbb{T}^\infty$ that plays a fundamental role in the proof of Theorem 2 given below. For instance, one may replace the infinite product of circles $\mathbb{T}^\infty$ by the infinite-dimensional cube $[0, 1]^\infty$, or more generally, by an infinite product of the form

$$X = X_1 \times X_2 \times \ldots$$

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[1]Formally, f is well-defined only almost everywhere with respect to σ. For completeness, let us agree that f attains the value zero at the few points x for which the series in (1) diverges.

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where $X_1, X_2, \ldots$ are connected Riemannian manifolds with boundary, all of volume one, that have a "uniformly bounded geometry". By the last phrase we mean that the dimensions, diameters and sectional curvatures of the X_i's should all be uniformly bounded. The distance function dist_p on X is still given by (2). For concreteness, we provide the statement and proof only for the toric case. We believe that the adaptation of our proof of Theorem 2 to the cube $[0, 1]^\infty$ or to the case of a more general product space is rather straightforward.

We are not sure whether the conclusion of Theorem 2 holds true also for $p = 1$. It could be interesting to investigate whether for $p = \infty$, the essential supremum in the conclusion of Theorem 2 may be replaced by a supremum. Let us also comment that the full axiom of choice is not used in the proof of Theorem 2, and that the axiom of dependent choice suffices for our argument.

2 Proofs

Consider the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. The coordinate vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are well-defined on the torus $\mathbb{T}^n$. The metric dist_p on the finite-dimensional torus $\mathbb{T}^n$ is defined via a formula analogous to (2) in which the sum runs only up to n. For a function $f : \mathbb{T}^n \to \mathbb{R}$ we define its oscillation via

$$\text{Osc}(f; \mathbb{T}^n) = \sup_{\mathbb{T}^n} f - \inf_{\mathbb{T}^n} f = \sup_{x,y \in \mathbb{T}^n} |f(x) - f(y)|.$$

According to the Rademacher theorem from real analysis, any function on $\mathbb{T}^n$ which is Lipschitz with respect to dist_p, for some $1 \leq p \leq \infty$, is differentiable almost-everywhere. Let $\omega_{n,p}$ denote the n-dimensional volume of the ℓ_p-ball $B_p^n = \{x \in \mathbb{R}^n ; \sum_{i=1}^n |x_i|^p \leq 1\}$. In this note, all integrals on tori and subtori are carried out with respect to the the uniform probability measure on the torus. We will need the following variant of Morrey's inequality:

Lemma 3. *Let $n \geq 1, p \in (1, \infty], 0 < \varepsilon < 1/2$ and let $f : \mathbb{T}^n \to \mathbb{R}$ be 1-Lipschitz with respect to the metric dist_p. Denote $q = p/(p-1)$, with $q = 1$ in case $p = \infty$. Assume that*

$$\int_{\mathbb{T}^n} \sum_{i=0}^{n-1} \frac{2^{i^2+qi}}{\omega_{i,p} \varepsilon^{i+q}} \cdot \left| \frac{\partial f}{\partial x_{i+1}} \right|^q \leq 1. \tag{3}$$

Then $\text{Osc}(f; \mathbb{T}^n) < 8\varepsilon$.

Proof. Let $\pi : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ be the quotient map. For a point $x \in \mathbb{T}^n$ we denote

$$\ell_i = \pi\left(\mathbb{R}e_i\right), \qquad E_i = \pi\left(Sp(e_1, ..., e_i)\right),$$

where $e_1, \ldots, e_n$ are the standard unit vectors in $\mathbb{R}^n$ and where $Sp(e_1, ..., e_i)$ is the subspace spanned by $e_1, \ldots, e_i$. We also denote $x + A = \{x + y ; y \in A\}$ for a subset $A \subseteq \mathbb{T}^n$ and a point $x \in \mathbb{T}^n$. Thus, $x + \ell_i \subseteq \mathbb{T}^n$ is a one-dimensional torus in $\mathbb{T}^n$ passing through x in

the direction of $\partial/\partial x_i$. The subtorus $x + E_i \subseteq \mathbb{T}^n$ is i-dimensional, and the vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_i$ are tangent to $x + E_i$ at the point x.

Fix a point $P \in \mathbb{T}^n$. For a decreasing index $i = n, \ldots, 0$, we recursively define the random points $P_i, P_i' \in \mathbb{T}^n$ via the following rules:

(i) $P_n = P$.

(ii) The point P_i' is distributed uniformly in the i-dimensional ball $B_p^i(P_{i+1}, \frac{\varepsilon}{2^i})$, where $B_p^i(P_{i+1}, \frac{\varepsilon}{2^i})$ is the dist_p-ball in the subtorus $P_{i+1} + E_i$ centered at P_{i+1} of radius $\frac{\varepsilon}{2^i}$.

(iii) The point P_i is distributed uniformly in the 1-dimensional subtorus $P_i' + \ell_{i+1}$.

Note that our recursive definition has a decreasing index, thus we first define P_n, then P_{n-1}', then P_{n-1}, etc. Since f is Lipschitz, for $i = 0, \ldots, n-1$,

$$\mathbb{E}|f(P_i') - f(P_i)| \leq \mathbb{E} \int_{P_i' + \ell_{i+1}} \left| \frac{\partial f}{\partial x_{i+1}} \right| = \mathbb{E} \left| \frac{\partial f}{\partial x_{i+1}}(P_i) \right|. \tag{4}$$

By an inductive argument, we see that the last $n - i$ coordinates of the random point P_i are independent random variables that are distributed uniformly over the circle $\mathbb{T}$. Let $A_{i+1} \in \mathbb{T}^i$ be the vector which consists of the first i coordinates of P_{i+1}. We also write $B_p^i(A_{i+1}, r)$ for the dist_p-ball of radius r centered at A_{i+1} in the torus $\mathbb{T}^i$. Since $\varepsilon < 1/2$,

$$\mathbb{E} \left| \frac{\partial f}{\partial x_{i+1}}(P_i) \right| = \mathbb{E} \frac{\int_{B_p^i(A_{i+1}, \frac{\varepsilon}{2^i}) \times \mathbb{T}^{n-i}} \left| \frac{\partial f}{\partial x_{i+1}} \right|}{\mathrm{Vol}_i \left(B_p^i \left(A_{i+1}, \frac{\varepsilon}{2^i} \right) \right)} = \frac{\mathbb{E} \int_{B_p^i(A_{i+1}, \frac{\varepsilon}{2^i}) \times \mathbb{T}^{n-i}} \left| \frac{\partial f}{\partial x_{i+1}} \right|}{\omega_{i,p} \left(\frac{\varepsilon}{2^i} \right)^i}. \tag{5}$$

From (4), (5) and the Hölder inequality, for $i = 0, \ldots, n-1$,

$$\mathbb{E}|f(P_i') - f(P_i)| \leq \left(\int_{\mathbb{T}^n} \left| \frac{\partial f}{\partial x_{i+1}} \right|^q \right)^{\frac{1}{q}} \left(\omega_{i,p} \left(\frac{\varepsilon}{2^i} \right)^i \right)^{-\frac{1}{q}}$$

$$\leq \left(\frac{2^{i^2+qi}}{\omega_{i,p}\varepsilon^{i+q}} \right)^{-\frac{1}{q}} \left(\frac{\omega_{i,p}\varepsilon^i}{2^{i^2}} \right)^{-\frac{1}{q}} = \frac{\varepsilon}{2^i},$$

where we used our assumption (3) in the last passage. The function f is 1-Lipschitz with respect to dist_p, and hence $|f(P_i') - f(P_{i+1})| \leq \varepsilon/2^i$ with probability one. Consequently,

$$\mathbb{E}|f(P) - f(P_0)| \leq \mathbb{E} \sum_{i=0}^{n-1} |f(P_i) - f(P_i')| + |f(P_i') - f(P_{i+1})| \leq \sum_{i=0}^{n-1} \frac{2\varepsilon}{2^i} < 4\varepsilon.$$

However, P_0 is distributed uniformly on the torus $\mathbb{T}^n$. Denote $M = \mathbb{E}f(P_0) = \int_{\mathbb{T}^n} f$. We have shown that $|f(P) - M| < 4\varepsilon$. Since $P \in \mathbb{T}^n$ was an arbitrary fixed point, the lemma follows. $\qquad\square$

For $x \in \mathbb{T}^\infty$, denote by $F(x)$ the set of points in $\mathbb{T}^\infty$ that coincide with x in all but finitely many coordinates.

Lemma 4. *Let $A \subseteq \mathbb{T}^\infty$ satisfy $\sigma(A) > 0$. Then $\sigma(\{x \in \mathbb{T}^\infty \, ; \, F(x) \cap A \neq \emptyset\}) = 1$.*

Proof. Denote $B = \{x \in \mathbb{T}^\infty \,;\, F(x) \cap A \neq \emptyset\}$. Then B is a measurable set which is in fact a tail event. Since $A \subseteq B$, Kolmogorov's zero-one law implies that $\sigma(B) = 1$. $\qquad\square$

For a measure space X and a measurable function $f : X \to \mathbb{R}$, we define the essential supremum of f, denoted by $\operatorname{ess\,sup} f$, as the supremum over all $a \in \mathbb{R}$ for which the set $\{x \in X \,;\, f(x) > a\}$ has a non-zero measure. The definition of essential infimum is analogous. Define the *essential oscillation* of f on X by

$$\operatorname{essOsc}(f; X) = \operatorname{ess\,sup} f - \operatorname{ess\,inf} f.$$

Equivalently, $\operatorname{essOsc}(f; X) = \|f(x) - f(y)\|_{L^\infty(X \times X)}$.

Lemma 5. *Let $p \in (1, \infty]$, $0 < \varepsilon < 1/2$ and let $f : \mathbb{T}^\infty \to \mathbb{R}$ be 1-Lipschitz with respect to the metric dist_p. Denote $q = p/(p-1)$, with $q = 1$ in case $p = \infty$. Assume that*

$$\int_{\mathbb{T}^\infty} \sum_{i=1}^\infty c_{\varepsilon,p,i} \cdot \left|\frac{\partial f}{\partial x_i}\right|^q \leq \frac{1}{2},$$

where $c_{\varepsilon,p,i} = 2^{(i-1)^2 + q(i-1)} / \left(\omega_{i-1,p} \cdot \varepsilon^{i-1+q}\right)$. Then

$$\operatorname{essOsc}(f, \mathbb{T}^\infty) < 8\varepsilon. \tag{6}$$

Proof. Let a be a random point, distributed uniformly in $\mathbb{T}^\mathbb{N}$. For a subset $S \subseteq \mathbb{N}$, denote by a_S the restriction of the random point a to the torus $\mathbb{T}^S$. Define $I_n = \{1, \ldots, n\}$. For $b \in \mathbb{T}^{\mathbb{N} \setminus I_n}$ denote

$$\mathbb{T}^n \times \{b\} = \left\{x \in \mathbb{T}^\infty \,;\, x_i = b_i \;\forall i > n\right\}.$$

For $n \geq 1$ we have

$$\mathbb{P}\left(\int_{\mathbb{T}^n \times \{a_{\mathbb{N} \setminus I_n}\}} \sum_{i=1}^n c_{\varepsilon,p,i} \left|\frac{\partial f}{\partial x_i}\right|^q \geq 1\right) \leq \mathbb{E}_a \int_{\mathbb{T}^n \times \{a_{\mathbb{N} \setminus I_n}\}} \sum_{i=1}^\infty c_{\varepsilon,p,i} \left|\frac{\partial f}{\partial x_i}\right|^q$$

$$= \int_{\mathbb{T}^\mathbb{N}} \sum_{i=1}^\infty c_{\varepsilon,p,i} \left|\frac{\partial f}{\partial x_i}\right|^q \leq \frac{1}{2}.$$

Lemma 3 now implies that for any $n \geq 1$,

$$\mathbb{P}\left(\operatorname{Osc}\left(f, \mathbb{T}^n \times \{a_{\mathbb{N} \setminus I_n}\}\right) < 8\varepsilon\right) \geq \frac{1}{2}.$$

Write B_n for the collection of all $b \in \mathbb{T}^\infty$ for which $\operatorname{Osc}(f, \mathbb{T}^n \times \{b_{\mathbb{N} \setminus I_n}\}) < 8\varepsilon$. Obviously $B_{n+1} \subseteq B_n$, and by the above $\sigma(B_n) \geq \frac{1}{2}$ for all $n \geq 1$. Denoting $B = \cap_{n=1}^\infty B_n$, we have $\sigma(B) \geq \frac{1}{2}$. Note that

$$B = \{b \in \mathbb{T}^\mathbb{N} \,;\, |f(x) - f(y)| < 8\varepsilon \;\forall x, y \in F(b)\}. \tag{7}$$

If (6) does not hold, then there exist sets $C, D \subseteq \mathbb{T}^{\mathbb{N}}$ of positive measure such that for all pairs of points $c \in C$ and $d \in D$ one has $|f(c) - f(d)| \geq 8\varepsilon$. Denote

$$\tilde{C} = \{x \in \mathbb{T}^{\mathbb{N}} ; F(x) \cap C \neq \emptyset\} \qquad \text{and} \qquad \tilde{D} = \{x \in \mathbb{T}^{\mathbb{N}} ; F(x) \cap D \neq \emptyset\}.$$

By Lemma 4, we have $\sigma(\tilde{C}) = \sigma(\tilde{D}) = 1$. Thus $\sigma(B \cap \tilde{C} \cap \tilde{D}) \geq \frac{1}{2}$, and there exist a point $b \in B$ and two elements $c \in C \cap F(b)$, $d \in D \cap F(b)$. According to the definition (7) of the set B,

$$|f(c) - f(d)| < 8\varepsilon,$$

in contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 6. *Let $p \in (1, \infty]$, $0 < \varepsilon < 1/2$ and let $f : \mathbb{T}^{\infty} \to \mathbb{R}$ be a measurable function that is 1-Lipschitz with respect to dist_p. Then there exists a parallel infinite-dimensional subtorus $M \subseteq \mathbb{T}^{\mathbb{N}}$ such that the restriction $f|_M$ is measurable and $\mathrm{essOsc}(f; M) \leq 8\varepsilon$.*

Proof. Fix a partition of $\mathbb{N}$ into blocks $B_1, B_2, \ldots \subseteq \mathbb{N}$ of size

$$\#(B_n) = \left\lceil \frac{2^{(n-1)^2 + q(n-1) + (n+1)}}{\omega_{n-1} \cdot \varepsilon^{n-1+q}} \right\rceil \qquad\qquad (n = 1, 2, \ldots). \qquad (8)$$

In each block, choose a random element $i_n \in B_n$, independently and uniformly. Denote $I = \{i_1, i_2, \ldots\} \subseteq \mathbb{N}$. Additionally, let a be a random point, distributed uniformly in $\mathbb{T}^{\mathbb{N}}$, independent of I. As before, write $q = p/(p-1)$ with $q = 1$ in case $p = \infty$. For every fixed n and for every $b \in \mathbb{T}^{\mathbb{N} \backslash B_n}$, the function f restricted to $\mathbb{T}^{B_n} \times \{b\}$ is 1-Lipschitz with respect to dist_p. By Rademacher's theorem, for almost any $x \in \mathbb{T}^{B_n} \times \{b\}$ one has

$$\sum_{i \in B_n} \left| \frac{\partial f}{\partial x_i}(x) \right|^q \leq 1,$$

implying that

$$\mathbb{E}_{i_n} \int_{\mathbb{T}^{B_n} \times \{b\}} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \frac{1}{\#(B_n)}. \qquad (9)$$

Denote by $a_{\mathbb{N} \backslash B_n}$ the restriction of the random point a to the torus $\mathbb{T}^{\mathbb{N} \backslash B_n}$. From (8) and (9),

$$\mathbb{E}_{i_n} \int_{\mathbb{T}^{\mathbb{N}}} \frac{2^{(n-1)^2 + q(n-1)}}{\omega_{n-1,p} \cdot \varepsilon^{n-1+q}} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q = \mathbb{E}_{i_n, a} \int_{\mathbb{T}^{B_n} \times \{a|_{\mathbb{N} \backslash B_n}\}} \frac{2^{(n-1)^2 + q(n-1)}}{\omega_{n-1,p} \cdot \varepsilon^{n-1+q}} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \frac{1}{2^{n+1}}.$$

Denote $c_{\varepsilon,p,n} = 2^{(n-1)^2 + q(n-1)} \big/ \left(\omega_{n-1,p} \cdot \varepsilon^{n-1+q} \right)$. Then,

$$\mathbb{E}_I \int_{\mathbb{T}^{\mathbb{N}}} \sum_{n=1}^{\infty} c_{\varepsilon,p,n} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

That is,

$$\mathbb{E}_{I, a_{\mathbb{N} \backslash I}} \int_{\mathbb{T}^I \times \{a_{\mathbb{N} \backslash I}\}} \sum_{n=1}^{\infty} c_{\varepsilon,p,n} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \frac{1}{2}. \qquad (10)$$

In particular, there exists a subset $I = \{i_1, i_2, \ldots\} \subseteq \mathbb{N}$ and $b \in \mathbb{T}^{\mathbb{N}}$ such that

$$\int_{\mathbb{T}^I \times \{b_{\mathbb{N} \setminus I}\}} \sum_{n=1}^{\infty} c_{\varepsilon,p,n} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \frac{1}{2}, \tag{11}$$

and such that the restriction of f to the subtorus $M := \mathbb{T}^I \times \{b_{\mathbb{N} \setminus I}\}$ is measurable. We may apply Lemma 5 thanks to (11), and conclude that $\mathrm{essOsc}(f; M) < 8\varepsilon$. $\qquad \square$

Proof of Theorem 2. Normalizing, we may assume that f is 1-Lipschitz. Fix a sequence $\varepsilon_n \to 0$ and apply Proposition 6 in order to construct a decreasing sequence of infinite-dimensional parallel tori T_n such that $\mathrm{essOsc}(f; T_n) < \varepsilon_n$. Denote $a_n = \int_{T_n} f$. Then for $m > n$ one has $|a_m - a_n| < \varepsilon_n$, implying that a_n has a limit, denoted by a. It then follows that $a \in \mathbb{R}$ satisfies the conclusion of the theorem. $\qquad \square$

References

[1] Alon, N., Spencer, J. H., *The probabilistic method.* Third Edition. Wiley-Interscience Series in Discrete Math. and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, 2008.

[2] Faifman, D., Klartag, B., Milman V., *On the oscillation rigidity of a Lipschitz function on a high-dimensional flat torus*. Geometric aspects of Funct. Anal. – Israel Seminar. Lecture Notes in Math., Vol. 2116, Springer, (2014), 123–131.

[3] Gowers, W. T., *Lipschitz functions on classical spaces.* European J. Combin., Vol. 13, No. 3, (1992), 141-151.

[4] Gromov, M., *Spaces and questions.* GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part I, 118-161.

[5] Kahane, J.-P., *Some random series of functions.* Second edition. Cambridge Studies in Advanced Mathematics, Vol. 5, Cambridge University Press, Cambridge, 1985.

[6] Milman, V., *Dvoretzky's theorem – thirty years later.* Geom. Funct. Anal. (GAFA), Vol. 2, No. 4, (1992), 455-479.

[7] Odell, E., Schlumprecht, Th., *The distortion of Hilbert space.* Geom. Funct. Anal. (GAFA), Vol. 3, No. 2, (1993), 201-207.

[8] Odell, E., Schlumprecht, Th., *The distortion problem.* Acta Math., Vol. 173, No. 2, (1994), 259-281.

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