Real analysis - Exam 2016a - Solution

1. Denote $A := \{x \in [0,1) : f'(x) = 0\}$. By the assumption, $m(A) = 1$. Let $\epsilon > 0$. For every $x \in A$ there exists $0 < \delta_x < 1 - x$ such that for every $0 < h < \delta_x$, we have

$$\Delta_{[x,x+h]}(f) = |f(x+h) - f(x)| \leq \epsilon h.$$

Since f is absolutely continuous, there exists $\delta > 0$ such that for any pairwise disjoint intervals $J_1, \dots, J_n \subset [0,1]$, we have

$$\sum_{k=1}^{n} m(J_k) \leq \delta \qquad \Longrightarrow \qquad \sum_{k=1}^{n} \Delta_{J_k}(f) \leq \epsilon.$$

Define

$$\mathcal{F} := \{[x, x+h] : x \in A,\ 0 < h < \delta_x\}$$

and note that F is a Vitali cover of A. Hence, by the Vitali covering lemma, there exist pairwise disjoint intervals $I_1, \dots, I_m \in \mathcal{F}$ such that $m(I) \geq 1 - \delta$, where $I := I_1 \cup \dots \cup I_m$. Write $[0,1] \setminus I = J_1 \cup \dots \cup J_n$, where $J_1, \dots, J_n$ are pairwise disjoint intervals. Then, since $m(J_1) + \dots + m(J_n) = 1 - m(I) \leq \delta$, we have

$$|f(1) - f(0)| \leq \sum_{i=1}^{m} \Delta_{I_i}(f) + \sum_{k=1}^{n} \Delta_{J_k}(f) \leq \epsilon \sum_{i=1}^{m} m(I_k) + \epsilon \leq 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $f(1) = f(0)$. By the same reasoning, it follows that $f(x) = f(0)$ for all $x \in [0,1]$.

2. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap (0,1)$ and let $\epsilon_n := 2^{-n-2}$. Denote

$$A := \bigcup_{n=1}^{\infty} (r_n - \epsilon_n, r_n + \epsilon_n) \cap (0,1).$$

Note that A is a dense open subset of $(0,1)$ and that

$$m(A) \leq \sum_{n=1}^{\infty} 2\epsilon_n = 1/2.$$

Define $f \colon [0,1] \to \mathbb{R}$ by

$$f(x) := m(A \cap [0,x]) = \int_{[0,x]} \mathbb{1}_A.$$

Then f is strictly increasing since

$$0 \leq x < y \leq 1 \qquad \Longrightarrow \qquad f(y) - f(x) = m(A \cap (x,y]) > 0,$$

where we used the fact that A is open and dense in $(0,1)$. Clearly f is absolutely continuous, since $G(x) = \int_{[0,x]} g$ is absolutely continuous for any integrable function $g \colon [0,1] \to \mathbb{R}$. It remains to check that f' is zero on a set of positive measure. We first show that $f'(x) = 1$ for all $x \in A$. By Lebesgue's differentiation theorem, $f'(x) = \mathbb{1}_A(x)$ for almost all $x \in [0,1]$. In particular, $f'(x) = 0$ for almost all $x \in [0,1] \setminus A$. Thus,

$$m(\{x \in [0,1] : f'(x) = 0\}) \geq m([0,1] \setminus A) = 1 - m(A) \geq 1/2.$$

3. Let $s > 0$. Since $t \mapsto e^{st}$ is strictly increasing, we have

$$f(x) > t \qquad \Longleftrightarrow \qquad e^{sf(x)} > e^{st}.$$

Since $e^{sf(x)}$ is positive, Markov's inequality implies that

$$m(\{x \in [0,1] : f(x) > t\}) = m(\{x \in [0,1] : e^{sf(x)} > e^{st}\}) \leq \frac{\int_{[0,1]} e^{sf(x)} dx}{e^{st}} \leq e^{s^2 - st}.$$

Substituting $s = t/2$, we obtain

$$m(\{x \in [0,1] : f(x) > t\}) \leq e^{s^2 - st} = e^{-t^2/4}.$$

4. Denote $f_s := \mathbb{1}_{\{g<s\}}$ for $s \in \mathbb{R}$. Note that

$$f_s \in \mathcal{F} \qquad \Longleftrightarrow \qquad \int f_s = \int \mathbb{1}_{\{g<s\}} = m(\{g < s\}) = 1.$$

Let us check that such an s exists. Denote $\phi(s) := \int f_s$. Clearly, $\phi(-1) = 0$ and $\phi(s) \to \infty$ as $s \to \infty$. Thus, it suffices to show that ϕ is continuous. This will follow if we show that $m(\{g = s\}) = 0$ for all $s \in \mathbb{R}$. Indeed, $\{g = s\}$ is countable for all $s \in \mathbb{R}$, as one may easily check (for instance by using that $\{g'' = 0\} = a + b\mathbb{Z}$ for some $a, b > 0$). We have thus shown that $f_{s_0} \in \mathcal{F}$ for some $s_0 \in \mathbb{R}$ (in fact, one may check that ϕ is strictly increasing so that s_0 is uniquely defined). It remains to show that for any $f \in \mathcal{F}$, we have

$$\int fg \geq \int f_{s_0} g.$$

Let $f \in \mathcal{F}$. Then, since $0 \leq f \leq 1$, we have

$$\begin{aligned}
\int fg - \int f_{s_0} g &= \int_{\{g \geq s_0\}} fg + \int_{\{g<s_0\}} fg - \int_{\{g<s_0\}} g \\
&= \int_{\{g \geq s_0\}} fg - \int_{\{g<s_0\}} (1-f)g \\
&\geq \int_{\{g \geq s_0\}} fs_0 - \int_{\{g<s_0\}} (1-f)s_0 \\
&= s_0 \left(\int_{\{g \geq s_0\}} f + \int_{\{g<s_0\}} f - \int_{\{g<s_0\}} 1 \right) \\
&= s_0 \left(\int f - \int f_{s_0} \right) \\
&= 0.
\end{aligned}$$