Real analysis - Exam 2016b - Solution

1. Let $f_1, f_2, \ldots$ be a sequence of measurable functions on a finite measure space $(\Omega, \mathcal{F}, \mu)$ and assume that f_n converges in measure to f. Set $N_0 := 0$ and then, inductively, for each $k \in \mathbb{N}$, choose $N_k > N_{k-1}$ such that
$$\mu(\{x \in \Omega \ : \ |f_n(x) - f(x)| \geq 1/k\}) \leq 2^{-k} \qquad \text{for all } n \geq N_k.$$
Denote $A := \limsup_{k\to\infty} A_k$, where
$$A_k := \{x \in \Omega \ : \ |f_{N_k}(x) - f(x)| \geq 1/k\}.$$
Since $\sum_k \mu(A_k) \leq \sum_k 2^{-k} < \infty$, the Borel-Cantelli lemma implies that $\mu(A) = 0$. Note that for any $x \in \Omega \setminus A$, we have $\lim_{k\to\infty} f_{N_k}(x) = f(x)$. Therefore, $f_{N_k} \to f$ almost everywhere.

2. Let $A \subset \mathbb{R}$ be a set of Lebesgue measure zero. For each $n \in \mathbb{N}$, choose an open set $A_n \subset \mathbb{R}$ such that $A \subset A_n$ and $m(A_n) \leq 1/n^2$. Define $f_n(x) := m((-\infty, x] \cap A_n)$ and $f := \sum_n f_n$. Since $f \geq 0$ and
$$f(x) = \sum_{n=1}^{\infty} m((-\infty, x] \cap A_n) \leq \sum_{n=1}^{\infty} m(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$
we see that f is well-defined. Moreover, since f_n is non-decreasing, f is clearly also non-decreasing. It remains to show that $f'(x) = \infty$ for all $x \in A$. Let $x \in A$ and let $N \in \mathbb{N}$. Since f_n is non-decreasing, we have
$$\begin{aligned} \liminf_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \liminf_{h\to 0} \sum_{n=1}^{\infty} \frac{f_n(x+h) - f_n(x)}{h} \\ &\geq \liminf_{h\to 0} \sum_{n=1}^{N} \frac{f_n(x+h) - f_n(x)}{h} \\ &= \sum_{n=1}^{N} \liminf_{h\to 0} \frac{f_n(x+h) - f_n(x)}{h} = \sum_{n=1}^{N} f_n'(x). \end{aligned}$$
Since A_n is open and $x \in A \subset A_n$, we clearly have $f_n'(x) = 1$ for all n. In particular, $\sum_{n=1}^{N} f_n'(x) = N$. Since N was arbitrary, we obtain
$$\liminf_{h\to 0} \frac{f(x+h) - f(x)}{h} \geq \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} f_n'(x) = \sup_{N\in\mathbb{N}} N = \infty.$$
We have therefore shown that $f'(x) = \infty$ and the proof is complete.

3. First note that since C is a compact set and g is continuous, $g(C)$ is also a compact set and so it is measurable. Note also that since g is strictly increasing and continuous, it is a bijection with its image $[0, 2]$. Let (C_n) be the sequence of sets defined in the construction of the Cantor set C. In particular, C_n is a union of 2^n disjoint closed intervals each of length 3^{-n}, $C = \cap_n C_n$ and f is constant on any interval contained in $[0, 1] \setminus C$. Thus, for any interval $I = (a, b) \subset [0, 1] \setminus C$, we have $g(I) = (g(a), g(b))$ so that
$$m(g(I)) = g(b) - g(a) = f(b) - f(a) + b - a = b - a = m(I).$$

Therefore, since $[0,1]\backslash C$ is a disjoint union of open intervals and g is a bijection, g preserves its measure so that

$$m(g([0,1] \setminus C)) = m([0,1] \setminus C) = m([0,1]) - m(C) = 1.$$

Using again that g is a bijection, we obtain

$$m(g(C)) = m(g([0,1])) - m(g([0,1] \setminus C)) = m([0,2]) - 1 = 1.$$

4. **(a).** Since $\frac{\partial f}{\partial y}$ is a Lipschitz function (in two variables), it follows that $\frac{\partial f}{\partial y}(\cdot, y)$ is a Lipschitz function (in the first variable) and so it also has bounded variation. Therefore, it is differentiable almost everywhere so that h is well-defined almost everywhere.

   Since $\frac{\partial f}{\partial y}(\cdot, y)$ is Lipschitz, it is also absolutely continuous. Thus, for any $x_0 < x_1$ and y,

$$\frac{\partial f}{\partial y}(x_1, y) - \frac{\partial f}{\partial y}(x_0, y) = \int_{[x_0,x_1]} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(s, y)ds = \int_{[x_0,x_1]} h(s, y)ds.$$

   In particular, for any $x > 0$ and y, since $\frac{\partial f}{\partial y}$ is zero on ∂U,

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, y) = \int_{[0,x]} h(s, y)ds.$$

   Similarly, $f(x, \cdot)$ is absolutely continuous, so that for any x and $y > 0$,

$$f(x, y) - f(x, 0) = \int_{[0,y]} \frac{\partial f}{\partial y}(x, t)dt = \int_{[0,y]} \left(\int_{[0,x]} h(s, t)ds\right) dt.$$

   Therefore, since h is bounded (and hence integrable), we may apply Fubini to obtain

$$f(x, y) - f(x, 0) = \int_{[0,x]\times[0,y]} h.$$

   (b). Using (a), $\frac{\partial f}{\partial x}|_{\partial U} = 0$ and Fubini, for any $(x, y) \in U$, we have

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x}\bigg|_{(x,y)} \int_{[0,x]} \left(\int_{[0,y]} h(s, t)dt\right) ds.$$

   By Lebesgue's differentiation theorem, for any $0 < y < 1$, for almost all $0 < x < 1$,

$$\frac{\partial f}{\partial x}(x, y) = \int_{[0,y]} h(x, t)dt.$$

   Finally, the fact that this holds for almost all $(x, y) \in U$ follows from Fubini, since the subset $U' \subset U$ on which this equation holds is measurable.

   (c). By (b), by Lebesgue's differentiation theorem and by Fubini, for almost all $(x, y) \in U$,

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x, y) = h(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x, y).$$