Several Complex Variables – Exercise 2

Due by November 25th, 2009. Write your id (or student number) on your solutions, but do NOT write your name. Please let me know immediately when you find a mistake or a misprint.

1. Let $A = \{z \in \mathbb{C}; 1 \leq |z| \leq 2\}$ be an annulus. Prove that it is impossible to approximate the function $1/z$ by polynomials (in z) uniformly on A. Same for the circle $T = \{z \in \mathbb{C}; |z| = 1\}$.

2. Let $\Omega \subset \mathbb{C}$ be an open set. Prove that there is a holomorphic function $f : \Omega \to \mathbb{C}$ that does not admit an analytic continuation to any domain of the form $\Omega \cup B(z, \varepsilon)$ with $z \in \partial\Omega$ and $\varepsilon > 0$.

3. Let $\Omega \subset \mathbb{C}^n$ be an open set, and suppose $u : \Omega \to \mathbb{R}$ is a C^2-function. The *complex hessian* of u at the point $z_0 \in \Omega$ is the matrix

$$Hess_{\mathbb{C}} f(z_0) = \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0) \right)_{j,k=1,\dots,n}.$$

   A function $u : \mathbb{C}^n \to \mathbb{R}$ is *pluriharmonic* if its complex hessian vanishes.

   (a) Prove that the complex hessian is an hermitian matrix, and that the the real and imaginary parts of a holomorphic function are pluriharmonic.

   (b) How would you call a pluriharmonic function on $\mathbb{C}^n$ that depends only on the real part of the variables $z_1, \dots, z_n$?

   (c) Prove that a pluriharmonic function is harmonic. Prove that for any function u pluriharmonic in a polydisc, there is a holomorphic function f whose real part is u.

4. Let $\Omega \subset \mathbb{C}^n$ be a logarithmically-convex, complete Reinhardt domain. Show that there exists a power series $\sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ whose domain of convergence is exactly Ω.

5. Prove that a pluriharmonic function on a polydisc (continuous up to the boundary) attains its maximum and minimum on the skeleton. (hint: Mimic the proof for holomorphic functions)

6. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Its *Shilov boundary* is defined as the minimal closed set $S \subset \partial\Omega$ such that for any holomorphic function $f : \Omega \to \mathbb{C}$ (continuous up to the boundary),

$$\sup_{z \in \Omega} |f(z)| = \sup_{z \in S} |f(z)|$$

(a) ★★ Explain the meaning of the word "minimal" (i.e., the intersection of all sets for which...), and prove that a Shilov boundary always exists.

(b) Compute the Shilov boundary (not only an upper bound for the Shilov boundary) of a polydisc and of a ball.

(c) Consider the following tube domain

$$\Omega = \{z = (z_1, z_2) \in \mathbb{C}^2; Im(z) \in C\}$$

where $C \subset \mathbb{R}^2$ is the cone $C = \{(x, y) \in \mathbb{R}^2; |y| \leq x\}$. Suppose that f is a holomorphic function on a neighborhood of Ω, and that $\sup_{z \in \Omega} |f(z)|$ is attained on $\partial\Omega$ (this holds when f is bounded). Prove that the maximum of $|f(z)|$ is actually attained on $\{z \in \mathbb{C}^2; Im(z) = 0\}$. Is there an even smaller set? (hint: partition the boundary into pieces, each looks like the upper half plane, and use the Phargmén-Lindelöf theorem, which).