Several Complex Variables – Exercise 3

Due by December 16th, 2009. Please let me know immediately when you find a mistake or a misprint. Write your id (or student number) on your solutions, but do NOT write your name.

1. Suppose we would define, for a domain $\Omega \subset \mathbb{C}^n$,

$$\widehat{\Omega} = \{z \in \mathbb{C}^n; |\ell(z)| \le \|\|\ell\|_{L^\infty(\Omega)} \text{ for all complex-linear } \ell : \mathbb{C}^n \to \mathbb{C}\}.$$

   How would you characterize $\widehat{\Omega} \subset \mathbb{C}^n$ geometrically? Same if we replace complex-linear by complex-affine.

2. Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. The Jacobian of f at the point $a \in \Omega$ is the matrix

$$J_a f = \left(\frac{\partial f_j}{\partial z_k} \right)_{j,k=1\ldots,n}.$$

   (The differential of f at a is the map $h \mapsto (J_a f)h$, for $h \in \mathbb{C}^n$).

   (a) Suppose that $\det J_a(f) \neq 0$ at $a \in \Omega$. Prove that locally, there is an holomorphic inverse. (You may use any real-variable theorem, like the implicit function theorem).

   (b) Suppose f is one-to-one, and $n = 1$. Recall that $\det J_a f \neq 0$ for $a \in \Omega$.

   (c) Suppose f is one-to-one, and $n = 2$. Prove that $\det J_a f \neq 0$ for $a \in \Omega$. Explain why your proof generalizes to higher dimensions.

3. (a) Schwarz lemma: Suppose f is holomorphic from the unit ball $B(0, 1)$ to the polydisc $P(0, 1)$ with $f(0) = 0$. Prove that $\|f(z)\|_\infty \le \|z\|_2$ for any $z \in B(0, 1)$.

   (b) Prove that there is no biholomorphic map between the unit ball and the polydisc (hint: suppose first that $f(0) = 0$).

4. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain of holomorphy.

   (a) Prove that $\mathbb{C}^n \setminus \Omega$ is connected.

   (b) Let $K \subset\subset \Omega$. Prove that there exists an analytic polyhedron P in Ω (i.e., an analytic polytope with respect to finitely many holomorphic functions in Ω) such that

$$K \subset\subset P \subset\subset \Omega.$$

   Is the assumption that Ω is a domain of holomorphy necessary?

(c) ★ Same, but now take an analytic polyhedron P in $\mathbb{C}^n$ (with respect to entire functions).

5. Let $\Omega \subset \mathbb{C}^n$ be a domain with a smooth boundary, $a \in \partial\Omega$. In which dimensions the following statement is true: Ω is pseudoconvex at a if and onlf if $\Omega \cap H$ is pseudoconvex at a for any complex hyperplane $H \subset \mathbb{C}^n$ containing a.

6. At which boundary points is the domain

$$\Omega = \{(z, w) \in \mathbb{C}^2; |z|^2 + \sqrt{|w|} < 1\}$$

pseudo-convex? strictly pseudo-convex? convex?