Several Complex Variables – Exercise 4

Due by February 14th, 2010 – there will be no extensions this time. Please let me know immediately when you find a mistake or a misprint. Question 4 is the hardest, requires integration by parts, a long computation, and the identities used in the proof of the Bochner-Lichnerowicz-Hörmander formula in class.

1. Suppose a holomorphic disc $D(z)$ ($z \in U \subset \mathbb{C}$) is tangent at $z = 0$ to the boundary of a smooth domain $\Omega \subset \mathbb{C}^n$. Prove that $D'(0) \in T^{\mathbb{C}}_{D(0)}(\partial\Omega)$.

2. Smooth Urysohn lemma: Suppose $\Omega \subset \mathbb{R}^n$ is an open set, and $A, B \subset \Omega$ are relatively closed sets in Ω with $A \cap B = \emptyset$. Prove that there exists a C^∞ function $\Psi : \Omega \to [0,1]$ that equals one on a relative neighborhood of A and zero on a relative neighborhood of B. [Think, for example, on the case $\Omega = \mathbb{R}^2 \setminus \{0\}$ and A and B being the intersection of Ω with lines of different slopes through the origin]

3. (a) Prove that Dolbeat closed forms are a closed subspace of $L^2_{(0,1)}(e^{-\phi})$.
   (b) If $u \in L^2$ and $\bar\partial u$ exists in $L^2_{(0,1)}(e^{-\phi})$, then $\bar\partial u$ is Dolbeat closed.
   (c) If a form f is not Dolbeat closed, then there is α such that $\bar\partial^*_\phi \alpha \equiv 0$, yet $f \cdot \bar\alpha$ is not identically zero.

4. A variation on the Bochner-Lichnerowicz-Hörmander identity: Suppose $\Omega \subset \mathbb{C}^n, 0 \in U \subset \mathbb{C}$ are open bounded sets with smooth boundary, $f(z, w_1, \ldots, w_n)$ a real valued, smooth, bounded function on $U \times \Omega$. Suppose $\Omega = [\rho < 0]$ for a defining function ρ with $|\nabla\rho| = 1$ on $\partial\Omega$. Abbreviate $\varphi(w) = f(0, w)$. We decompose in $L^2(e^\varphi)$,

$$\frac{\partial f}{\partial z}(0, w) = c + H(w) - \mathcal{L}g$$

for some function smooth g. Here $\mathcal{L} = \bar\partial^\varphi_* \circ \bar\partial$, the function H is holomorphic with $\int H e^{-\varphi} = 0$ and c is a constant. Denote

$$e^{-\Phi(z)} = \int_\Omega e^{-f(z,w)} d\lambda(w) \qquad (z \in U).$$

Prove that $e^{-\Phi(0)} \frac{\partial^2 \Phi}{\partial z \partial \bar z}(0)$ is equal to

$$\int_\Omega \left[-|H|^2 + \sum_{j,k} \left| g_{\bar j \bar k} \right|^2 + \mathcal{H}_{(0,w)} f \left(1, \bar\partial g\right) \right] e^{-\varphi(w)} d\lambda(w) + \int_{\partial\Omega} \mathcal{H}_w \rho(\bar\partial g) e^\varphi.$$

Here, $\mathcal{H}_a(\varphi)(z) = \sum_{j,k} \varphi_{j\bar k}(a) z_j \overline{z_k}$ is the Levi form.