Smooth functions – List of Exercises

January 26, 2011

Week 1

1. Suppose p is a polynomial of degree d in n real variables. Assume that $p(x) > 0$ for any $0 \neq x \in \mathbb{R}^n$. Do there exist $c, \varepsilon > 0$ such that

$$p(x) \geq c|x|^d \qquad \text{for all } |x| < \varepsilon$$

   (a) When $d = 2$.

   (b) When d is an arbitrary even number.

2. (a) Find an example for a function that is differentiable of order two at a point $p \in \mathbb{R}^n$, but is discontinuous at all points except p.

   (b) Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear transformation, and $f \in C^m(\mathbb{R}^n)$. Prove that

$$\|f \circ T\|_{C^m} \leq C\|f\|_{C^m}$$

   where C is a constant depending only on m and n.

3. (a) Suppose $f : K \to \mathbb{R}$ is a continuous function, where $K \subset \mathbb{R}^n$ is compact and convex. Prove that f has an optimal (i.e., minimal) modulus of continuity ω, and that the optimal one is equivalent, up to a factor of two, to a regular modulus of continuity (i.e., $1/2 \leq \omega/\tilde{\omega} \leq 2$ for a regular modulus of continuity $\tilde{\omega}$).

   (b) Prove that the optimal is also equivalent, up to factor 10, to a concave modulus of continuity.

(c) What happens in a non-convex domain (but still compact)?

4. (a) Prove Taylor's theorem for $C^{m,\omega}$.

   (b) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an $(m+1)$-times continuously differentiable function. What can you say about the relation between the $C^{m,1}$ norm and the C^{m+1} norm of f? What can you say about the relation between the spaces of functions $C^{m,1}(\mathbb{R}^n)$ and $C^{m+1}(\mathbb{R}^n)$?

5. Suppose x_n^j $(j = 0, \ldots, m, n = 1, 2, \ldots)$ are real numbers, such that for any $j = 0, \ldots, m$,

$$x_n^j \xrightarrow{n\to\infty} 0.$$

Assume furthermore that for any n, the numbers $x_n^0, \ldots, x_n^m$ are all distinct. Prove that there exist coefficients a_n^j $(j = 0, \ldots, m, n = 1, 2, \ldots)$ with the following property: For any C^m function $f : \mathbb{R} \to \mathbb{R}$,

$$f^{(m)}(0) = \lim_{n\to\infty} \sum_{j=0}^{m} a_n^j f(x_n^j).$$

(October 24, 2010: Thanks to Nadav Yesha and Yinon Spinka for correcting errors in the first week's exercises).

Week 2

6. (a) Suppose $f : (0,1) \to \mathbb{R}$ is a C^2 function with $M = \sup |f''| < \infty$. Prove that for any $\varepsilon > 0$, the set of critical values of f may be covered by $\lceil \alpha/\varepsilon \rceil$ intervals of length $\beta\varepsilon^2$, where $\alpha, \beta > 0$ depend only on M.

   (b) Prove that the countable set $\left\{\frac{1}{\log n}; n \geq 2\right\}$ is never contained in the set of critical values of a C^2 function $f : (0,1) \to \mathbb{R}$ with $\|f\|_{C^2} < \infty$.

   (c) For any d, find a bounded, countable set A of real numbers with the following property: For any C^∞ function $f : B^d \to \mathbb{R}$ that admits a C^∞ extension to $2B^d$ (the ball of radius two centered at zero), the set A is not contained in the set of critical values of f.

7. Suppose $E \subset \mathbb{R}^n$ is a closed set, $\delta(x) = d(x, E)$. Recall the a dyadic cube Q is "good" if

$$Diam(Q) \leq \inf_{x\in Q^*} \delta(x)$$

where $Diam(Q) = \sqrt{n}\delta_Q$ is the diameter of Q, and Q^* is the dilation of Q by factor three around its center. A cube Q belongs to the CZ-decomposition if it is "good", and if either its parent Q^+ is bad, or $\delta_Q = 1$. Prove that

(a) For any $Q \in CZ$ with $\delta_Q < 1$, for any $x \in Q*$,

$$Diam(Q) \leq \delta(x) \leq C Diam(Q),$$

where $C > 0$ depends solely on n.

(b) Two cubes $Q, \tilde{Q} \in CZ$ are "neighbors" if $\overline{Q} \cap \overline{\tilde{Q}} \neq \emptyset$. Prove that when Q and $\tilde{Q}$ are neighbors,

$$\frac{1}{2}\delta_Q \leq \delta_{\tilde{Q}} \leq 2\delta_Q.$$

8. What is the C^m-stratification that was constructed in the proof of Sard's lemma of the following sets:

(a) $K \times K \subset \mathbb{R}^2$, where $K \subset [0, 1]$ is the usual Cantor set.

(b) $S = \{0\} \cup \{1/n; n, \geq 1\} \subset \mathbb{R}$. Is there a stratification of this set in which each stratum A is a stratum with respect to itself?

(November 9, 2010: Thanks to Shahar Karmeli and Lev Radziviloski for correcting errors in the second week's exercises).

Week 3 – no class

Week 4

9. Fix $x \in \mathbb{R}^n$. Recall that for $P_1, P_2 \in \mathcal{P}$, we set $P_1 \odot_x P_2 = J_x(P_1 P_2)$.

(a) Prove that $\odot_x$ is a multiplication on $\mathcal{P}$, that makes it a commutative ring.

(b) Find a continuum of ideals in $\mathcal{P}$. For which m, n is it possible?

(c) Describe all ideals generated by (a few) monomials, and prove that when $n > 1$ there are at least, say, $2^{n+m}/(n+m)$ of them.

10. (a) Suppose $r_1 \leq r_2 \leq 1, x \in \mathbb{R}^n$. Prove that

$$B(x, r_1) \odot_x B(x, r_2) \subseteq C r_2^m \omega(r_2) B_{C^{m,\omega}}(x, r_1)$$

where $E \odot_x F = \{p_1 \odot_x p_2; p_1 \in E, p_2 \in F\}$ and $C > 0$ depends solely on m and n.

(b) Suppose $x, y \in \mathbb{R}^n$, $|x - y| \leq r \leq 1$ and $P_1, P_2 \in B(x, r)$. Prove that

$$P_1 \odot_y P_2 - P_1 \odot_x P_2 \in C r^m \omega(r) B(x, y).$$

11. Suppose $\{p_x\}_{x \in \mathbb{R}^n} \subseteq \mathcal{P}$ is a collection of polynomials, and $M > 0$ is such that

$$p_x - p_y \in M B(x, y)$$

for all $x, y \in \mathbb{R}^n$ with $|x - y| \leq 1$. Prove that $F(x) = p_x(x)$ is a $C^{m,\omega}$ function, with $\|F\|_{\dot{C}^{m,\omega}} \leq CM$, such that

$$J_x(F) = p_x$$

for any $x \in \mathbb{R}^n$.

Week 5

12. Write down the proof of Whitney's extension theorem for the homogenous $\dot{C}^{m,1}$ norm (No need to re-prove statements about the Calderón-Zygmund decomposition or the partitions of unity).

13. In the proof of Whitney's extension theorem in class, we considered

$$\tilde{P}_x = J_x \left(\sum_{Q \in CZ} \theta_Q P_Q \right) \in \mathcal{P}.$$

On the other hand, since we need to interpolate polynomials in some way, we could have tried to take $\tilde{P}_x = \sum_{Q \in CZ} \theta_Q(x) P_Q \in \mathcal{P}$ (that is, $\tilde{P}_x(y) = \sum_{Q \in CZ} \theta_Q(x) P_Q(y)$). Do you think it would work? why?

14. Suppose $E \subset \mathbb{R}^n$ is a finite set, $\{P_x\}_{x \in E} \subset \mathcal{P}_{n,m}$. Prove that

$$c \left\| \{P_x\}_{x \in E} \right\|_{C^{m+1}} \leq \left\| \{P_x\}_{x \in E} \right\|_{C^{m,1}} \leq C \left\| \{P_x\}_{x \in E} \right\|_{C^{m+1}}$$

for some constants $c, C > 0$ depending only on m and n.

Week 6

15. Suppose $E \subset \mathbb{R}^n$ is a finite set with $\#(E) = N, \varepsilon > 0$ and $f : E \to \mathbb{R}$. Suppose we are given a Callahan-Kosaruju decomposition of E with parameter $\varepsilon > 0$, whose length is at most CN/ε^n. How would you efficiently compute the Lipschitz constant of f? Recall that

$$Lip(f) = \sup_{\substack{x,y \in E \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

Week 7

16. (a) Suppose (X, ρ) is a metric space, $E \subseteq X$, and $f : E \to \mathbb{R}$ is a λ-Lipschitz function. Show that

$$F(x) = \inf_{y \in E} \{f(y) + \lambda \rho(x, y)\} \tag{1}$$

is a λ-Lipschitz extension of f to the entire space X.

(b) Suppose (X, ρ) is a metric space, $F : X \to \mathcal{I}(\mathbb{R})$ where $\mathcal{I}(\mathbb{R})$ is the collection of all bounded, closed intervals in $\mathbb{R}$. Use formulae such as (1) in order to prove: If we have a 1-Lipschitz selection for all $S \subset X$ with $\#(S) \leq 2$, then we have a 1-Lipschitz selection for the entire X.

17. Use Zorn's lemma in order to deduce Kirszbraun's theorem from the following finitary statement proved in class: For any finite set S in a Hilbert space H and a point $x \notin S$ – any 1-Lipschitz function from S to H may be extended to a 1-Lipschitz function from $S \cup \{x\}$ to H.

18. For a convex set $A \subset \mathbb{R}^2$ we write $R(A)$ for the smallest rectangle, parallel to the axes, that contains A. Prove that

(a) $R\left(\bigcap_{\alpha \in I} K_\alpha\right) = \bigcap_{\alpha, \beta \in I} R\left(K_\alpha \cap K_\beta\right).$

(b) For parallel rectangles $\{A_\alpha\}_{\alpha \in I}, \{B_\beta\}_{\beta \in J}$, we have

$$d\left(\bigcap_\alpha A_\alpha, \bigcap_\beta B_\beta\right) = \sup_{\alpha, \beta} d(A_\alpha, B_\beta)$$

whenever the intersections in the left-hand side are non-empty, where here $d(A, B) = \inf_{x \in A, y \in B} \|x - y\|_\infty$.

Week 8

19. Explain how to adapt the proof of the finiteness principle using Lipschitz selection for $\dot{C}^{1,1}(\mathbb{R}^2)$, to the case of $C^{1,1}(\mathbb{R}^2)$.

    That is, prove the following statement: There exists a universal constant $C > 0$ with the following property: Let $E \subset \mathbb{R}^n$ be a closed set, $f : E \to \mathbb{R}$. Suppose that for any $S \subset E$ with $\#(S) \leq C$, we have

    $$\|f|_S\|_{C^{1,1}(S)} \leq M.$$

    Then $\|f\|_{C^{1,1}(E)} \leq CM$.

20. Denote $A = (-1 - \varepsilon, 0), B = (-1 + \varepsilon, 0), C = (-1, -\varepsilon^2), D = (1 + \varepsilon, 0), E = (1 - \varepsilon, 0), F = (1, \varepsilon^2)$. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ vanishes at five of these six points, and $f(F) = \varepsilon$.

    We explained in class that $\|f\|_{\dot{C}^{1,1}} \geq c/\varepsilon$. Prove that if we remove one of these six points (any point), then there is a $\dot{C}^{1,1}$ extension whose $\dot{C}^{1,1}$-norm is bounded by a universal constant.

Week 9

21. Suppose $\mathcal{A} \subseteq \mathcal{M}$ is a subset of multi-indices. Suppose $\phi : \mathcal{A} \to \mathcal{M}$ satisfies

    - For any $\alpha \in \mathcal{A}$, we have $\phi(\alpha) \leq \alpha$.
    - If $\phi(\alpha) \neq \alpha$, then $\phi(\alpha) \notin \mathcal{A}$.

    Prove that $\phi(\mathcal{A}) \leq \mathcal{A}$, with equality iff ϕ is the identity map.

Week 10

22. Fix $\mathcal{A} \subseteq \mathcal{M}, x \in \mathbb{R}^n, \delta > 0$. Denote

$$\mathcal{R}_{\mathcal{A}}(x,\delta) = \left\{ P \in \mathcal{P}; \forall \beta \geq \alpha_{\mathcal{A},x}(P), |\partial^\beta P(x)| \leq \delta^{m+1-|\beta|} \right\},$$

where $\alpha_{\mathcal{A},x}(P) = \max\{\alpha \in \mathcal{A}; \partial^\alpha P(x) \neq 0\}$. Prove that for any $K \geq 1$ and a centrally-symmetric convex set $\Omega \subseteq \mathcal{P}$,

$$\mathcal{B}_{\mathcal{A}}(\delta) \subseteq K\pi_{\mathcal{A},x} \left\{ \Omega \cap \mathcal{R}_{\mathcal{A}}(x,\delta) \right\}$$

if and only if there exist polynomials $\{P_\alpha\}_{\alpha \in \mathcal{A}}$ with the following properties:

(a) $\partial^\beta P_\alpha(x) = \delta_{\alpha,\beta}$ for any $\alpha, \beta \in \mathcal{A}$.

(b) $|\partial^\beta P_\alpha(x)| \leq K\delta^{|\alpha|-|\beta|}$ for any $\mathcal{M} \ni \beta \geq \alpha \in \mathcal{A}$.

(c) $\delta^{m+1-|\alpha|} P_\alpha \in K\Omega$.

23. Suppose that A is a $n \times n$ matrix, with ones on the main diagonal, such that the sum of the absolute values of the off-diagonal elements in each row does not exceed $1/2$.

Prove that A^{-1} exists, and that all of its elements are at most 2 in absolute value.

Week 11

24. Suppose $Q_0 \in CZ(\mathcal{A}_0) \setminus CZ(\mathcal{A}_0^-)$. Let $x \in E \cap Q_0^{***}$. Prove that

$$\mathcal{B}_{\mathcal{A}_0}\left(A_2\delta_{Q_0}\right) \subseteq CA_1(\mathcal{A}_0)\pi_{\mathcal{A}_0,x} \left\{ \sigma(x, \ell(\mathcal{A}_0) - 1) \cap B(x, A_2\delta_{Q_0}) \right\},$$

where $C > 0$ is a constant depending solely on m and n.