Approximate set covering in uniform hypergraphs

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Abstract

The weighted set covering problem, restricted to the class of r-uniform hypergraphs, is considered. We propose a new approach, based on a recent result of Aharoni, Holzman and Krivelevich about the ratio of integer and fractional covering numbers in k-colorable r-uniform hypergraphs. This approach, applied to hypergraphs of maximal degree bounded by Δ, yields an algorithm with approximation ratio $r(1-c/\Delta^{\frac{1}{r-1}})$. Next, we combine this approach with an adaptation of the local ratio theorem of Bar-Yehuda and Even for hypergraphs and present a general framework of approximation algorithms, based on subhypergraph exclusion. An application of this scheme is described, providing an algorithm with approximation ratio $r(1-c/n^{\frac{r-1}{r}})$ for hypergraphs on n vertices. We discuss also the limitations of this approach.

1 Introduction

The *minimum set cover* problem is certainly one of the most central problems of combinatorial optimization. In an instance of this problem, a collection C of subsets of a finite set S and a weight function $w : C \to R^+$ are given, the task is to find a subset $C' \subseteq C$ of minimum weight $w(C') = \sum_{c\in C'} w(c)$ such that every element of S belongs to at least one member of C'. When all weights equal identically 1, we obtain the *unweighted* version of the problem.

For our purposes it is more convenient to reformulate this problem in terms of hypergraphs. A *hypergraph* H is an ordered pair $H = (V, E)$, where V is a finite non-empty set (the set of *vertices*), and E is a collection of distinct non-empty subsets of V (the set of *edges*). H is called *r-uniform* if $|e| = r$ for all edges $e \in E(H)$. The set cover problem can be

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represented as follows: for given hypergraph $H = (V, E)$ and weight function $w : V \to R^+$, solve the optimization problem:

$$\begin{array}{lll} \min & \sum_{v\in V} w(v)g(v) & \\ \text{s.t.} & \sum_{v\in e} g(v) \geq 1 & \text{for every } e \in E(H) , \\ & g(v) \in \{0,1\} & \text{for every } v \in V . \end{array} \tag{1}$$

This formulation leads naturally to the following fractional relaxation of the integer problem (1).

$$\begin{array}{lll} \min & \sum_{v\in V} w(v)g(v) & \\ \text{s.t.} & \sum_{v\in e} g(v) \geq 1 & \text{for every } e \in E(H) , \\ & g(v) \geq 0 & \text{for every } v \in V . \end{array} \tag{2}$$

For a hypergraph $H = (V, E)$, any feasible solution g of (1) (or, alternatively, the set of vertices of weight 1 in a feasible solution g) is called a *cover* of H with *value* $|g| = \sum_{v\in V} w(v)g(v)$. An optimal solution of (1) is an *optimal cover* of H, having the *covering number* of H (which we denote by $\tau(H)$) as its value. Quite similarly, any feasible solution g of (2) is a *fractional cover* of H, an optimal solution of (2) is an *optimal fractional cover* of H with value $\tau^*(H)$, called the *fractional covering number* of H. It is important to note, that the problem (2) is a Linear Programming problem that can be solved in time polynomial in $|V|$ and $|E|$ ([17]). Clearly, the solution $\tau^*(H)$ of the fractional problem (2) may serve as a lower bound on the solution $\tau(H)$ of the more difficult integer problem (1).

In this paper we consider the case of r-uniform hypergraphs, where r is thought to be a *fixed* number.

The above defined problem (1) turns out to be NP-complete even for the unweighted case and $r = 2$ (that is, for the case of graphs, where this problem is usually called the *vertex cover* problem), as shown by Karp [18]. This result motivates studying *approximate* algorithms for this problem.

An *approximation algorithm* A for the problem (1) is a polynomial time algorithm which, for a given input instance (H, w) of (1), produces a feasible solution. The value of the solution obtained by A for an instance H, is denoted by $A(H)$. The *approximation ratio* R_A of an approximation algorithm A on a family of instances $\mathcal{H}$ is $R_A = \sup\left\{\frac{A(H)}{\tau(H)} \ : \ H \in \mathcal{H}\right\}$. The approximation ratio R_A can be viewed as a quantitative measure of the quality of an algorithm A. Here we aim to develop efficient approximation algorithms for the class $\mathcal{H}_r$ of r-uniform hypergraphs, for a given and fixed value of r.

The set cover problem remains hard even as an approximation problem. Speaking explicitly, it is MAX–SNP complete as proven in [22]. Moreover, a commonly believed conjecture (see, e.g., [15], [23]) states that unless P=NP, there does not exist a polynomial time approximation algorithm A with approximation ratio $R_A \leq r - \epsilon$ for any positive constant ϵ.

Let us describe briefly known results in the positive direction. We start with the first non-trivial case $r = 2$. The following very simple algorithm, attributed in [10] to Gavril, can be easily seen to have approximation ratio 2 in the unweighted case. The algorithm proceeds as follows. Given a graph G, find a maximal (under inclusion) matching M in G (for example, by picking edges greedily) and take all vertices in the edges of M as a cover. This idea can be easily generalized to the case of general r. Several algorithms having approximation ratio 2 for the general (weighted) case are known, the algorithm of Bar-Yehuda and Even [3] being the simplest of them. Most (if not all) of the more sophisticated approximation algorithms rely heavily on the theorem of Nemhauser and Trotter [21], enabling to reduce a given problem to the problem restricted to a class of graphs G satisfying $|V(G)|/\tau(G) \leq 2$. This reduction is based on the fact that for any graph $G = (V, E)$ and any weight function $w : V \to R^+$ there always exists an optimal solution g of the corresponding fractional problem (2) with $g(v) \in \{0, 0.5, 1\}$ for every $v \in V(G)$. Unfortunately, this phenomenon does not hold for larger values of r, as it has been shown by Chung, Füredi, Garey and Graham in [6] that for every rational number $0 \leq \alpha < 1$ there exists a 3-uniform hypergraph H such that the fractional part of $\tau^*(H)$ equals α.

An important partial case of the vertex cover problem is when the maximal degree of a graph is bounded from above by a parameter Δ. For this case Hochbaum [15] described an algorithm with approximation ratio $2 - 2/\Delta$, based on a coloring argument. This result has been improved by Halldórsson and Radhakrishnan [13] to the ratio $2 - (\log \Delta + O(1))/\Delta$ by invoking an algorithm of Shearer [25] for finding effectively an independent set of size $\Omega(n \log \Delta/\Delta)$ in a triangle-free graph on n vertices with maximal degree Δ.

For the general case of graphs on n vertices, the best known approximation algorithm is due to Bar-Yehuda and Even [4] (Monien and Speckenmeyer [20] described actually the same algorithm for the unweighted case), it has approximation ratio $2 - \log\log n/2\log n$. Along with the theorem of Nemhauser and Trotter, this algorithm uses the so called Local Ratio Theorem and proceeds by excluding subgraphs $G_0 \subseteq G$, having large covering number relatively to their order.

Returning to the set covering problem for r-uniform hypergraphs for general r, we mention the algorithm of Hall and Hochbaum [11] for a general multicovering problem (where in (1) the first group of restrictions is $\sum_{v \in e} g(v) \geq b(e)$ for each edge $e \in E$). This algorithm has approximation ratio r and performs $O(\max\{|V|, |E|\}\,|V|)$ iterations. Peleg, Schechtman and Wool [23] present several algorithms, having approximation ratio r for the unweighted case, they also describe a randomized approximation algorithm, having approximation ratio $r\left(1-(\frac{c}{m})^{\frac{1}{r}}\right)$, where m is the number of edges in the hypergraph H and c is some absolute constant. This algorithm first solves the fractional problem (2) and then uses the technique of randomized rounding with scaling, developed by Raghavan and Thompson [24].

Now we describe new results presented in this paper and its structure. In Section 2 we gather some results about fractional covers and matchings, used in the subsequent sections. A recent theorem of Aharoni, Holzman and Krivelevich [1], bounding the ratio between integer and fractional covering numbers in k-colorable r-uniform hypergraphs, is presented together with two different proofs in Section 3. Both proofs are algorithmic, the corresponding algorithms are described in the same section. In Section 4 we apply one of these algorithms to the problem of approximating the set covering number for r-uniform hypergraphs of maximal vertex degree bounded by Δ, obtaining an algorithm with approximation ratio $r(1-c/\Delta^{\frac{1}{r-1}})$. In Section 5 we combine the second algorithm of Section 4 with the local ratio approach of Bar-Yehuda and Even, giving a general scheme of an approximation algorithm, parameterized by a family $\mathcal{H}_0$ of excluded subhypergraphs. A particular realization for a concrete choice of $\mathcal{H}_0$ is presented in Section 6, yielding an algorithm with approximation ratio $r(1-c/n^{\frac{r-1}{r}})$ for hypergraphs on n vertices. We discuss also the limitations of the local ratio approach in Section 7.

We close this section with some notation. Given a hypergraph $H=(V,E)$ and a weight function $w: V \to R^+$, the *degree* $d(v)$ of a vertex $v \in V$ is the number of edges of H containing v. For a subset $V_0 \subseteq V$, the notation $H[V_0]$ stands for the subhypergraph of H, induced by V_0. This subhypergraph has set of edges $E(V_0)$, we denote its cardinality by $e(V_0)=|E(V_0)|$. A subset $V_0 \subseteq V$ is called *independent* in H, if $e(V_0)=0$. The *weight* $w(V_0)$ of V_0 is $w(V_0)=\sum_{v\in V_0} w(v)$. For a given hypergraph H_0, a hypergraph H is called H_0*-free*, if H does not contain a (not necessarily induced) copy of H_0. Given a family of hypergraphs $\mathcal{H}_0=\{H_1,\ldots,H_t\}$, H is called $\mathcal{H}_0$*-free*, if it is H_i-free for each member H_i of $\mathcal{H}_0$.

2 Fractional covers and matchings in hypergraphs

In this short section we accumulate some basic facts about fractional covers and matchings, to be used in the sequel.

Let us formulate the LP problem dual to the previously defined problem (2) of determining the fractional covering number of a hypergraph $H=(V,E)$.

$$\max \qquad \sum_{e\in E} f(e)$$

$$\text{s.t.} \quad \textstyle\sum_{e \ni v} f(e) \leq w(v) \qquad \text{for every } v \in V(H)\ , \tag{3}$$
$$f(e) \geq 0 \qquad \text{for every } e \in E\ .$$

A feasible solution f of this problem is a *fractional matching* in H with *value* $|f| = \sum_{e \in E} f(e)$. A fractional matching having the largest possible value, is called *optimal*, and its value is the *fractional matching number* $\nu^*(H)$.

The Duality Theorem of Linear Programming applied to the pair of problems (2), (3) implies

Proposition 1 *For every hypergraph $H = (V, E)$ and every weight function $w : V \to R^+$ the following holds true:*

1. *$\tau^*(H) = \nu^*(H)$;*

2. *If $g : V \to R^+$ is an optimal fractional cover and $f : E \to R^+$ is an optimal fractional matching, then*

$$\begin{aligned} f(e) > 0 \quad &\textit{implies} \quad \sum_{v \in e} g(v) = 1; \\ g(v) > 0 \quad &\textit{implies} \quad \sum_{v \in e} f(e) = w(v)\ . \end{aligned} \tag{4}$$

(These are the so called *complementary slackness conditions*).

We will make a direct use of the following

Proposition 2 *For every r-uniform hypergraph $H = (V, E)$ and every weight function $w : V \to R^+$ the following holds true:*

1. *If g is a fractional cover of H, then for every subset $V_0 \subseteq V$ the function $g' : V_0 \to R^+$, defined by $g'(v) = g(v)$ for every $v \in V_0$ (that is, g' is the restriction of g to V_0), is a fractional cover of the hypergraph $H[V_0]$;*

2. *If g is an optimal fractional cover of H, satisfying $g(v) > 0$ for every $v \in V$, then $\tau^*(H) = w(V)/r$.*

Proof. 1) Follows immediately from the definition of a fractional cover;

2) Let $f : E \to R^+$ be an optimal fractional matching of H with respect to w. Then, by the complementary slackness conditions (4)

$$\begin{aligned} w(V) &= \sum_{v \in V} w(v) = \sum_{v \in V} \sum_{e \ni v} f(e) = \sum_{e \in E} f(e)|e \cap V| \\ &= \sum_{e \in E} f(e) r = r \sum_{e \in E} f(e) = r\nu^*(H) = r\tau^*(H)\ . \qquad \Box \end{aligned}$$

The reader is referred to the survey paper of Füredi [9] for additional results about integer and fractional covers and matchings in hypergraphs.

3 Covers in k-colored hypergraphs

Given a hypergraph $H = (V, E)$, a partition $V = V_1 \cup \ldots \cup V_k$ is a *proper k-coloring* of H with colors $V_1, \ldots, V_k$ if $|e \cap V_i| < |e|$ for each $e \in E(H)$ with $|e| \geq 2$ and $1 \leq i \leq k$, that is, no edge, besides of course singletons, is monochromatic. A hypergraph H is *k-colorable* if it has a proper k-coloring.

Hochbaum [15] suggested the following approach to vertex cover approximation. Given an instance (G, w), first reduce the problem to an instance (G_0, w), for which $\tau(G_0) \geq w(V(G_0))/2$ by applying the Nemhauser – Trotter algorithm, then color G_0 by k colors $V_1, \ldots, V_k$ and take the complement of a color class V_i, having the maximal weight $w(V_i)$, to be an approximate solution, thus obtaining the approximation ratio $2 - 2/k$.

Trying to generalize this idea for the case of r-uniform hypergraphs for general r, we need to bypass the lack of the Nemhauser–Trotter type result for $r \geq 3$. This can be achieved by using the following recent result of Aharoni, Holzman and Krivelevich [1], whose proof can be converted into a polynomial time approximation algorithm.

Theorem 1 *Let $H = (V, E)$ be an r-uniform k-colorable hypergraph and let $w : V \to R^+$ be a weight function on the vertices of H. Denote by $\tau(H)$ and by $\tau^*(H)$ the covering and the fractional covering numbers of H, respectively, with respect to w. Then*

$$\frac{\tau(H)}{\tau^*(H)} \leq \max\left\{r - 1, \frac{k-1}{k} r\right\} .$$

(We remark that in paper [1] bounds on the ratio $\tau(H)/\tau^*(H)$ are proven for various types of r-uniform hypergraphs.)

For the sake of completeness we provide here a proof of the above theorem. Actually, we give two different proofs, to be converted later to two approximation algorithms.

Proof 1. Suppose that the theorem fails, and let $H = (V, E)$ with $w : V \to R^+$ be a counterexample with the smallest number of vertices. Then we must have $\bigcup_{e \in E} e = V$. Fix some proper k-coloring $V_1, \ldots, V_k$ of H. Let $g : V \to R^+$ be an optimal fractional cover of H and let $f : E \to R^+$ be an optimal fractional matching in H, with respect to w. We distinguish between two cases.

Case 1: $g(v) > 0$ for all $v \in V$.

Then, by Proposition 2, part 2, one has

$$\tau^*(H) = \frac{w(V)}{r} . \tag{5}$$

On the other hand, the complement of a heaviest color class is clearly a cover of H with weight not exceeding $\frac{k-1}{k} w(V)$, therefore

$$\tau(H) \leq \frac{k-1}{k} w(V) . \tag{6}$$

Comparing (5) and (6), we derive $\tau(H)/\tau^*(H) \le \frac{k-1}{k}r$ – a contradiction with the choice of H.

Case 2: There exists a vertex $v_0 \in V$ with $g(v_0) = 0$. By our assumption about H, this vertex belongs to some edge $e_0 \in E$. Since $|e_0| = r$ and $\sum_{v \in e_0} g(v) \ge 1$, there exists a vertex $v_1 \in e_0$ with $g(v_1) \ge 1/(r-1)$. If $\{v_1\}$ is a cover of H, then clearly $\tau(H) \le w(v_1)$. On the other hand, by the complementary slackness conditions (4), we have $w(v_1) = \sum_{e \ni v_1} f(e)$, therefore $\tau^*(H) = \nu^*(H) \ge \sum_{e \ni v_1} f(e) = w(v_1)$, so in fact $\tau(H) = \tau^*(H) = w(v_1)$, contradicting the choice of H. So we may consider the hypergraph $H' = H[V - v_1]$. Define a new weight function $w' : V(H') \to R^+$ by $w'(v) = w(v)$ for all $v \in V(H')$. Since H' is also r-uniform and k-colorable, it follows from the minimality of H that the theorem statement is valid for H'. Obviously,

$$\tau(H) \le \tau(H') + w(v_1) \tag{7}$$

(a cover of H' can be extended to a cover of H by adding v_1). On the other hand, by Proposition 2, part 1, we conclude that

$$\tau^*(H') \le \tau^*(H) - g(v_1)w(v_1) \le \tau^*(H) - w(v_1)/(r-1)\ . \tag{8}$$

It follows from (7) and (8) that

$$\begin{aligned}
\tau(H) &\le \tau(H') + w(v_1) \le \max\left\{r-1, \frac{k-1}{k}r\right\}\tau^*(H') + w(v_1)\\
&\le \max\left\{r-1, \frac{k-1}{k}r\right\}\left(\tau^*(H) - \frac{w(v_1)}{r-1}\right) + w(v_1)\\
&\le \max\left\{r-1, \frac{k-1}{k}r\right\}\tau^*(H)\ ,
\end{aligned}$$

again obtaining a contradiction to the choice of H. $\square$

Proof 2. We present this proof in the following probabilistic setting.

Lemma 1 *Let $H = (V, E)$ be an r-uniform hypergraph and let $w : V \to R^+$ be a weight function on the vertices of H. Let $g : V \to R^+$ be an optimal fractional cover of H with respect to w. Suppose $V = V_1 \cup \ldots \cup V_k$ is a partition of V. Suppose further that for some $\delta > 0$ there exists a set $B \subseteq [0, \delta]^k$ such that for every $\bar{x} = (x_1, \ldots, x_k) \in B$ the set*

$$T(\bar{x}) = \bigcup_{i=1}^{k} \{v \in V_i : g(v) \ge x_i\}$$

is a cover of H. If there exists a probability measure μ defined on B ($\mu(B) = 1$) such that all marginal distributions μ_i, $1 \le i \le k$, are uniform on the interval $[0, \delta]$ (that is, if $\bar{x} \in B$

is randomly chosen from B according to the measure μ, then $P[a \le x_i \le b] = (b-a)/\delta$ for every $0 \le a \le b \le \delta$), then

$$\frac{\tau(H)}{\tau^*(H)} \le \frac{1}{\delta} \ .$$

Proof of Lemma 1. Let $\bar{x} \in B$ be randomly chosen from B according to the measure μ. Define a random variable $Y = w(T(\bar{x}))$, where $T(\bar{x})$ is as defined above. Let us estimate the expectation of Y. By linearity of expectation, $E[Y] = \sum_{v \in V} w(v) P[Y_v = 1]$, where Y_v is the indicator random variable for $v \in V$ being selected to T. Since μ has marginal distributions uniform on the interval $[0, \delta]$, for every $1 \le i \le k$ and for every $v \in V_i$ we have

$$P[Y_v = 1] = P[v \in T] = P[g(v) \ge x_i] = \min\{1, g(v)/\delta\} \le g(v)/\delta \ ,$$

hence

$$E[Y] = \sum_{v \in V} w(v) P[Y_v = 1] \le \sum_{v \in V} w(v) \frac{g(v)}{\delta} = \frac{\tau^*(H)}{\delta} \ .$$

We conclude that there exists at least one point $\bar{x} \in B$, for which the corresponding cover $T(\bar{x})$ has weight $w(T)$, satisfying $w(T) \le \tau^*(H)/\delta$. $\square$

Now our strategy is to find, for a k-colored hypergraph H with a given proper k-coloring $V_1, \ldots, V_k$, an appropriate set B and a measure μ and then to apply Lemma 1.

Consider first the case $k < r$. In this case we need to prove $\tau(H)/\tau^*(H) \le r - 1$. Since every k-colorable hypergraph for $k \le r$ is also r-colorable, we may assume without loss of generality that H is an r-colorable hypergraph, thus reducing to the second case $k \ge r$.

Assume now that $k \ge r$. Define first $k+1$ points $Q_0, Q_1, \ldots, Q_k$ in $[0, \frac{k}{(k-1)r}]^k$ as follows.

$$\begin{aligned} Q_0 &= \left(\frac{1}{r}, \ldots, \frac{1}{r}\right) , \\ Q_j &= (0, \ldots, \underbrace{\frac{k}{(k-1)r}}_{j\text{-th coord.}}, \ldots, 0), \quad j = 1, \ldots, k \ . \end{aligned}$$

Now let B_j be the interval in R^k joining Q_0 and Q_j, and let

$$B = \bigcup_{j=1}^{k} B_j \ .$$

Clearly, $B \subseteq [0, \frac{k}{(k-1)r}]^k$. In order to check that $T(\bar{x})$ is a cover for every $\bar{x} \in B$, consider, for example, the interval $B_1 = [Q_0 Q_1]$. It can be easily seen that for each point $\bar{x} = (x_1, \ldots, x_k) \in B_1$ we have $x_1 \ge 1/r$ and $x_1 \ge x_i$ for each $2 \le i \le k$, and the last $k-1$ coordinates have the same value which we denote by y. The equation $(k-1)x_1 + y = k/r$ is satisfied by both endpoints of B_1 and hence by every point of B_1. Consider an edge $e \in E$

and denote $|e \cap V_1| = s_1$, then $s_1 \leq r-1$. If $T(\bar{x}) \cap e = \emptyset$, then $g(v) < x_i$ for each $v \in e \cap V_i$, $1 \leq i \leq k$, therefore

$$\begin{aligned}\sum_{v \in e} g(v) &< s_1 x_1 + (r-s_1)y \leq (r-1)x_1 + y \\ &= (k-1)x_1 + y - (k-r)x_1 \leq \frac{k}{r} - \frac{k-r}{r} = 1 \ ,\end{aligned}$$

obtaining a contradiction to the definition of g.

Now we need do define a probability measure μ on B. To this end, let μ^j be the uniform measure on B_j with $\mu^j(B_j) = 1/k$ and let $\mu = \sum_{j=1}^k \mu^j$. Then μ is a probability measure on B. For a given coordinate $1 \leq i \leq k$, there are $k-1$ intervals B_j, for which the marginal distribution μ_i^j is uniform on $[0, \frac{1}{r}]$ and one interval B_i for which μ_i^i is uniform on $[\frac{1}{r}, \frac{k}{(k-1)r}]$. Hence μ_i is uniform on the interval $[0, \frac{1}{r}]$ with $\mu_i([0, \frac{1}{r}]) = \frac{k-1}{k}$ and also is uniform on the interval $[\frac{1}{r}, \frac{k}{(k-1)r}]$ with $\mu_i([\frac{1}{r}, \frac{k}{(k-1)r}]) = \frac{1}{k}$. Since $\frac{1}{r} \Big/ \left(\frac{k}{(k-1)r} - \frac{1}{r}\right) = k-1$, we derive that μ_i is uniform on the whole interval $[0, \frac{k}{(k-1)r}]$. Therefore the set B and the measure μ, required in Lemma 1, have been found, and we apply the lemma to get the desired result. $\Box$

Now we turn the above proofs into polynomial time approximation algorithms. Taking a closer look at the first proof, we note that it may proceed under a weaker assumption on a hypergraph H. Namely, it suffices to assume that for every subset $V_0 \subseteq V$ the induced subhypergraph $H[V_0]$ contains an independent set I of weight at least $w(V_0)/k$ (this assumption clearly holds true for k-colored hypergraphs). This observation is utilized by the following recursive algorithm.

Algorithm A1

Input: An r-uniform hypergraph $H = (V, E)$, a weight function $w : V \to R^+$, and an algorithm $IND(H', w)$, returning in independent set $I \subseteq V'$ of weight at least $w(V')/k$ in any given induced subhypergraph $H' = (V', E')$ of H.

Output: A cover C of H.

1. $C = \emptyset$;

2. Delete all isolated vertices of H;

3. Find on optimal fractional cover $g : V \to R^+$;

4. if $g(v) > 0$ for every $v \in V$ **begin**

$I = IND(H, w)$;

$C = V \setminus I$;

end;

5. else begin

Find a vertex $v_1 \in V$ with $g(v_1) \geq \frac{1}{r-1}$;

if $\{v_1\}$ is a cover of H, set $C = \{v_1\}$;

else begin

Define $H' = H[V - v_1]$, $w'(v) = w(v)$ for every $v \in V - v_1$;

$C = \{v_1\} \cup \mathbf{A1}(H', w')$;

end;

end;

6. return (C).

Proof 1 of Theorem 1 implies immediately the following

Corollary 1 $R_{A1} \leq \max\left\{r - 1, \frac{k-1}{k} r\right\}$.

The obvious drawback of the above algorithm lies in its recursivity, resulting in a multiple solution of the LP problem (2). If a proper k-coloring $(V_1, \ldots, V_k)$ of H is given, we can use another algorithm based on Proof 2 of Theorem 1. A careful examination of Proof 2 reveals the following facts:

1. The set B is a union of k intervals $B_1, \ldots, B_k$, none of them being parallel to any coordinate axis;

2. While moving along each interval B_j and building the corresponding cover $T(\bar{x})$, one can see that $T(\bar{x})$ may change only at the point $\bar{x} = (x_1, \ldots, x_k)$ for which there exists an index i, $1 \leq i \leq k$, and a vertex $v \in V_i$ for which $g(v) = x_i$. Moreover, given such a pair $(\bar{x}, v)$, we check if B_j contains a point $\bar{y} = (y_1, \ldots, y_k)$ with $y_i > x_i$. If such a point indeed exists, we denote $I_1 = \{1 \leq i \leq k : \; y_i > x_i\}$, $I_2 = [1, k] \setminus I_1$. Making an infinitesimally small step along B_j from $\bar{x}$ towards $\bar{y}$ and obtaining a new point $\overline{x + dx} \in B_j$, we notice that

$$T(\overline{x + dx}) = \bigcup_{i \in I_1} \{v \in V_i : \; g(v) > x_i\} \cup \bigcup_{i \in I_2} \{v \in V_i : \; g(v) \geq x_i\} \; .$$

If x_i is the maximal value of the i-th coordinate in B_j, we consider the set

$$T(\bar{x}) = \bigcup_{i=1}^{k} \{v \in V_i : \ g(v) \geq x_i\} \ .$$

The above two facts show that it suffices to check only a finite number of points from B.

Algorithm A2

Input: An r-uniform hypergraph $H = (V, E)$, a weight function $w : V \to R^+$ and a proper k-coloring $(V_1, \ldots, V_k)$ of H.

Output: A cover C of H.

1. Find an optimal fractional cover $g : V \to R^+$;

2. Define a set $B \subseteq R^k$ as in Proof 2 of Theorem 1;

3. **for** each color V_i **do**

  for each vertex $v \in V_i$ **do**

  for each interval B_j of B **do**

  if there exists a point $\bar{x} = (x_1, \ldots, x_k) \in B_j$ with $g(v) = x_i$ **begin**

   if there exists a point $\bar{y} = (y_1, \ldots, y_k) \in B_j$ with $y_i > x_i$ **begin**

    $I_1 = \{1 \leq i \leq k : \ y_i > x_i\}$;

    $I_2 = \{1, \ldots, k\} \setminus I_1$;

    end;

   else begin

    $I_1 = \emptyset$;

    $I_2 = \{1, \ldots, k\}$;

    end;

   $T = \bigcup_{i \in I_1} \{v \in V_i : \ g(v) > x_i\} \cup \bigcup_{i \in I_2} \{v \in V_i : \ g(v) \geq x_i\}$;

  end;

4. $C =$ a subset $T \subseteq V$ having the smallest weight among the subsets T found in step **3**;

5. **return** (C).

Proof 2 yields the following

Corollary 2 $R_{A2} \le \max\left\{r-1, \frac{k-1}{k}r\right\}$.

We will apply algorithm $A1$ to the set covering problem restricted to r-uniform hypergraphs of bounded maximal degree Δ (Section 4). Algorithm $A2$ will be applied in Sections 5,6 to a general set covering problem for r-uniform hypergraphs.

4 Approximate covers in hypergraphs of bounded degree

Suppose that the family of instances of the set cover problem is restricted to r-uniform hypergraphs of maximal vertex degree at most Δ. We would like to get an approximation algorithm with approximation ratio better than the trivial ratio r.

In order to apply algorithm $A1$ of the preceding section, we need an algorithm IND for finding an independent set of relatively large weight in r-uniform hypergraphs of maximal degree Δ. This is provided by the following

Theorem 2 *Let $H=(V,E)$ be an r-uniform hypergraph of maximal degree at most Δ and let $w: V \to R^+$ be a weight function on the vertices of H. Then H contains an independent set I of weight $w(I) \ge \frac{r-1}{r}w(V)\Big/\Delta^{\frac{1}{r-1}}$. Moreover, such an independent set can be found in time polynomial in $|V|$ and $|E|$.*

Proof. Define first the weight function $w_e: E \to R^+$ on the edges of H by setting $w_e(e)=\sum_{v\in e} w(v)$ for every $e \in E$. Let $V_0 \in V$ be a random subset of V defined by $P[v\in V_0]=p$, the exact value of p will be chosen later. Denote by X the random variable

$$X = w(V_0)\ ,$$

denote also by Y the random variable

$$Y = \frac{1}{r}\sum_{e\in E(V_0)} w_e(e)\ .$$

By linearity of expectation,

$$EX = \sum_{v\in V} w(v)P[v\in V_0] = pw(V)$$

and also

$$\begin{aligned} EY &= \frac{1}{r}\sum_{e\in E} w_e(e)P[e\in E(V_0)] = \frac{1}{r}\sum_{e\in E} w_e(e)p^r = \frac{p^r}{r}\sum_{e\in E} w_e(e) \\ &= \frac{p^r}{r}\sum_{e\in E}\sum_{v\in e} w(v) = \frac{p^r}{r}\sum_{v\in V} w(v)d(v) \le \frac{p^r\Delta w(V)}{r}\ . \end{aligned}$$

Therefore

$$E[X - Y] \geq pw(V) - \frac{p^r \Delta w(V)}{r} \ .$$

Now we choose $p = \frac{1}{\Delta^{\frac{1}{r-1}}}$ to maximize the above expression. Then

$$E[X - Y] \geq \frac{r-1}{r} \frac{w(V)}{\Delta^{\frac{1}{r-1}}} \ .$$

Thus there exists a specific set V_0 for which the difference $X - Y$ is at least $\frac{r-1}{r}\frac{w(V)}{\Delta^{\frac{1}{r-1}}}$. Fix such a set V_0 and for every edge $e \in E(V_0)$ delete from V_0 a vertex $v \in e$, having the smallest weight $w(v)$. Clearly, the total weight of the deleted vertices does not exceed $\frac{1}{r}\sum_{e \in E(V_0)} w(e) = Y$, so the remaining subset $I \subseteq V_0$ is independent and has weight at least $X - Y \geq \frac{r-1}{r}\frac{w(V)}{\Delta^{\frac{1}{r-1}}}$.

The above described randomized algorithm can be easily derandomized using standard techniques of the conditional expectations method (see, e.g., [2], Ch. 15). $\square$

Denote by IND the algorithm described in the above theorem. Incorporating IND into algorithm $A1$ of Section 3, we obtain the algorithm B for which the following result holds.

Corollary 3 $R_B \leq \max\left\{r - 1, r\left(1 - \frac{r-1}{r\Delta^{\frac{1}{r-1}}}\right)\right\}$.

5 Local ratio approach - a general scheme

In this section we develop a general scheme of approximation algorithms, combining the local ratio approach of Bar-Yehuda and Even [4], the idea of Hochbaum [15] based on a coloring argument, and our algorithm $A2$. Our presentation follows closely that of [4].

Lemma 2 *Let $H = (V, E)$ be a hypergraph and let w, w_0 and w_1 be weight functions on the vertices of H such that $w(v) \geq w_0(v) + w_1(v)$ for every $v \in V$. If C^*, C_0^* and C_1^* are optimal covers for the instances (H, w), (H, w_0) and (H, w_1), respectively, then $w(C^*) \geq w_0(C_0^*) + w_1(C_1^*)$.*

Proof.

$$\begin{aligned} w(C^*) &= \sum_{v \in C^*} w(v) \\ &\geq \sum_{v \in C^*} w_0(v) + w_1(v) \\ &= w_0(C^*) + w_1(C^*) \\ &\geq w_0(C_0^*) + w_1(C_1^*) \ . \end{aligned}$$

The last inequality follows from the optimality of the covers C_0^* and C_1^*. $\square$

For a hypergraph H_0 define the *local ratio* $lr(H_0)$ of H_0 as $lr(H_0) = |V(H_0)|/c^*(H_0)$, where $c^*(H_0)$ is the size of a minimal unweighted cover of H_0. Let A be any approximation algorithm for the set covering problem.

Algorithm LOCAL(H_0)

Input: A hypergraph $H = (V, E)$ with a weight function $w : V \to R^+$. It is assumed that H_0 and A have been fixed in advance.

Output: A cover C of H.

1. Find a copy of the hypergraph H_0 in H. Let V_0 be the vertex set of this copy;

2. Set $\delta = \min_{v \in V_0} w(v)$;

3. Define the weight function $w_0 : V \to R^+$ by

$$w_0(v) = \begin{cases} w(v) - \delta, & v \in V_0 \\ w(v), & \text{otherwise} \end{cases} ;$$

4. Run the algorithm A on the instance (H, w_0) to get a cover C for H;

5. **return**(C).

Theorem 3 *(The Local Ratio Theorem)*

$$R_{LOCAL(H_0)}(H, w) \leq \max\{lr(H_0), R_A(H, w_0)\} \ .$$

Proof. Denote $R = \max\{lr(H_0), R_A(H, w_0)\}$. Also, let c^* and c_0^* be the values of optimal solutions for the instances (H, w) and (H, w_0), respectively. Since $|C \cap V_0| \leq |V(H_0)|$, we have

$$\begin{aligned} w(C) &= w_0(C) + \delta|C \cap V_0| \leq w_0(C) + \delta|V(H_0)| \\ &\leq R_A(H, w_0)c_0^* + \delta\, lr(H_0)c^*(H_0) \\ &\leq R(c_0^* + \delta c^*(H_0)) \\ &\leq Rc^* \ . \end{aligned}$$

The last inequality is obtained by defining $w_1(v) = w(v) - w_0(v)$ and then by applying Lemma 2. $\square$

The algorithm $LOCAL(H_0)$ can be easily generalized to a family of fixed hypergraphs $\mathcal{H}_0 = \{H_1, \ldots, H_t\}$, as follows.

Algorithm LOCAL($\mathcal{H}_0$)

Input: A hypergraph $H = (V, E)$ with a weight function $w : V \to R^+$. It is assumed that $\mathcal{H}_0$ and A have been fixed in advance.

Output: A cover C of H.

1. while for some i there exists a copy of H_i in H with vertex set V_0 so that $w(v) > 0$ for every $v \in V_0$ **do begin**

Set $\delta = \min_{v \in V_0} w(v)$;

for all $v \in V_0$ **do** $w(v) = w(v) - \delta$;

end;

end;

2. $C_1 = \{v \in V : w = 0\}$;

3. $V_1 = V \setminus C_1$;

4. Run the Algorithm A on the instance $(H[V_1], w_0)$ to obtain a cover C_2 for $H[V_1]$;

5. $C = C_1 \cup C_2$;

6. return (C) .

It should be stressed that in order to run the above algorithm, we should be able to find all copies of the hypergraphs from $\mathcal{H}_0$ in H. Clearly, if $\mathcal{H}_0$ contains a fixed number of hypergraphs, this can be done in polynomial time, for example, by exhaustive search.

Define the *local ratio* $lr(\mathcal{H}_0)$ of the family $\mathcal{H}_0$ as $lr(\mathcal{H}_0) = \max_{H_i \in \mathcal{H}_0} lr(H_i)$.

Corollary 4 *(The Local Ratio Corollary)*

$$R_{LOCAL(\mathcal{H}_0)}(H, w) \leq \max\{lr(\mathcal{H}_0), R_A(H[V_1], w_0)\} .$$

The above corollary can be easily proven by induction on the number of iterations of Step 2, using the Local Ratio Theorem (Theorem 3).

Taking another look at the above algorithm, we note that at the end of Step 4, the set V_1 does not contain a copy of any hypergraph from $\mathcal{H}_0$. Thus we may expect that the resulting subhypergraph $H[V_1]$, serving as an input for the algorithm A, is relatively sparse and hence can be efficiently colored by a small number of colors. This observation prompts the use of the approximation algorithm $A2$ for colored hypergraphs, described in Section 3.

The simplest application of the Local Ratio Corollary arises when $\mathcal{H}_0$ consists of one hypergraph, which is just a single edge. In this case, V_1 will clearly span no edge, and there will be no need for algorithm A. The algorithm $LOCAL(\mathcal{H}_0)$ then reduces to the following simple procedure: while H contains an edge $e \in E(H)$ with all vertices of positive weight, choose such an edge e, set $\delta = \min_{v \in e} w(v)$ and update $w(v) = w(v) - \delta$ for all $v \in e$; if all edges contain vertices of zero weight $w(v) = 0$, return the set of all vertices of weight zero as the output. This algorithm runs in $O(|E|)$ steps and has approximation ratio r by the Local Ratio Corollary.

6 Local ratio approach - implementation

In this section we analyze the performance of the algorithm $LOCAL(\mathcal{H}_0)$ for $\mathcal{H}_0 = \{H_0\}$ and for a particular choice of a hypergraph H_0.

We assume that the uniformity number r is fixed throughout the section. Define a hypergraph $H_0 = (V, E)$ as follows. The vertex set $V(H_0)$ consists of $2r - 1$ vertices denoted by $v_1, \dots, v_{2r-1}$. The edge set $E(H_0)$ consists of r edges of type $(v_1, \dots, v_{r-1}, v_i)$, where i ranges from r to $2r - 1$, and one additional edge $(v_r, v_{r+1}, \dots, v_{2r-1})$. One can easily see that a minimal unweighted cover of H_0 has size 2, therefore H_0 has local ratio $lr(H_0) = (2r - 1)/2 = r - 0.5$. Note also that for $r = 2$ the corresponding hypergraph H_0 is simply a triangle.

Now we claim that every H_0-free hypergraph H contains a relatively large independent set and therefore can be colored with a relatively small number of colors, thus providing a platform for using algorithm $A2$.

Lemma 3 *There exists a constant $c = c(r)$ such that if $H = (V, E)$ is an r-uniform H_0-free hypergraph on n vertices, then H contains an independent set of size at least $cn^{1/r}$, which can be found in time polynomial in n.*

Proof. Let $|E| = f$. Averaging implies that there exist $r - 1$ vertices $u_1, \dots, u_{r-1} \in V(H)$ such that H contains at least $f\binom{r}{r-1} \Big/ \binom{n}{r-1} = rf \Big/ \binom{n}{r-1}$ edges passing through $u_1, \dots, u_{r-1}$. Let $e_1, \dots, e_s$ be these edges, then $s \geq rf \Big/ \binom{n}{r-1}$. Denote $U = \bigcup_{i=1}^{s} e_i \setminus \{u_1, \dots, u_{r-1}\}$, then $|U| = s$ and U does not span an edge from E (if such an edge e_0 existed, then e_0 together with the r edges from $e_1, \dots, e_s$, that intersect it, would form a copy of H_0), hence U is an independent set of size at least $rf \Big/ \binom{n}{r-1}$.

On the other hand, choosing each vertex $v \in V$ to belong to a random subset W of V independently and with probability $p = (n/rf)^{1/(r-1)}$ (if $n/rf > 1$ the assertion of the lemma is trivial) and calculating expectations as done in the proof of Theorem 2, we can show that H contains an independent set of size at least $np - fp^r = \frac{r-1}{r}\left(\frac{n^r}{rf}\right)^{\frac{1}{r-1}}$, which can be found in polynomial time using the method of conditional expectations. Therefore, H contains an independent set of size at least

$$\max\left\{\frac{rf}{\binom{n}{r-1}}, \frac{r-1}{r}\left(\frac{n^r}{rf}\right)^{\frac{1}{r-1}}\right\} \geq cn^{\frac{1}{r}}$$

for some constant $c = c(r)$, as claimed. $\square$

An algorithm for finding an independent set is easily converted into a coloring algorithm as follows. At each step, we find an independent set in the current hypergraph, color it by a fresh color and remove it from the hypergraph. The following lemma, used in most of the papers on approximate graph coloring (see, e.g., [16], [26], [12]) provides an upper bound on the number of colors.

Lemma 4 *An iterative application of an algorithm finding an independent set of size $cn^{\frac{1}{r}}$ in a hypergraph H on n vertices, produces a coloring of H with no more than $\frac{r}{c(r-1)}n^{\frac{r-1}{r}}$ colors.*

Proof. Denote by $f(n)$ the guaranteed size of the output of the independent set algorithm applied to an H_0-free hypergraph H on n vertices, then $f(n) \geq cn^{1/r}$. We may assume that f is a continuous, positive and non-decreasing function of its real argument. Let $V_1, \ldots, V_k$ be the resulting coloring of the above described coloring algorithm. We assume that V_1 is the first color used, V_2 is the second one and so on. Enumerate the n vertices of H by the numbers $1, \ldots, n$ in such a way that if $i_1 \in V_{j_1}$ and $i_2 \in V_{j_2}$ and $j_1 < j_2$, then $i_1 > i_2$ (that is, vertices with larger numbers got smaller colors). It can be easily seen that if $i \in V_j$, then $|V_j| \geq f(i)$. Therefore

$$k = \sum_{j=1}^{k} 1 = \sum_{j=1}^{k}\sum_{i \in V_j} \frac{1}{|V_j|} \leq \sum_{i=1}^{n} \frac{1}{f(i)} .$$

This sum can be estimated from above by the integral

$$\int_0^n \frac{dt}{f(t)} \leq \frac{1}{c}\int_0^n \frac{dt}{t^{\frac{1}{r}}} = \frac{r}{c(r-1)}n^{\frac{r-1}{r}} . \qquad \square$$

Now we can define a specific algorithm A that can be substituted in $LOCAL(\mathcal{H}_0)$: given an H_0-free r-uniform hypergraph H on n vertices, color H by $O(n^{(r-1)/r})$ colors as described above, and then use algorithm $A2$. Denote the above described procedure by C and the resulting algorithm by $LOCAL(\mathcal{H}_0) + C$, then we have

Corollary 5

$$\begin{aligned} R_{LOCAL(\mathcal{H}_0)+C} &\leq \max\left\{\frac{2r-1}{2}, r-1, r\left(1-\frac{\Theta(1)}{n^{\frac{r-1}{r}}}\right)\right\} \\ &= r\left(1-\frac{\Theta(1)}{n^{\frac{r-1}{r}}}\right) \end{aligned}$$

for large values of n.

The algorithm $LOCAL(H_0){+}C$ can be viewed as an extension of the algorithm $COVER2$ of [4], based on Wigderson's algorithm [26] for coloring a triangle-free graph on n vertices in $2\sqrt{n}$ colors.

Comparing with the randomized algorithm of Peleg, Schechtman and Wool [23], we note that for the typical case $|E(H)| = \Theta(|V(H)|^r)$, our algorithm has a better approximation ratio.

A possible way of improving the above presented results is to extend the family $\mathcal{H}_0 = \{H_0\}$ to a family of hypergraphs with local ratio strictly less than r and then to show that an $\mathcal{H}_0$-free hypergraph H contains a large independent set. Bar-Yehuda and Even took in their paper [4] $\mathcal{H}_0$ to consist of odd cycles of length at most l, where l is a function of the number n of vertices of the given graph. It is not clear what is the analog of an odd cycle in r-uniform hypergraphs for $r \geq 3$ here (our hypergraph H_0 can be viewed as an analog of a triangle) and how to search in polynomial time for subhypergraphs from a family of size growing with n. In any case, this approach cannot give substantially better results for *any* choice of a *fixed* family $\mathcal{H}_0$, as demonstrated in the next section.

7 Local ratio approach – limitations

The main result of this section shows that for any choice of a fixed family $\mathcal{H}_0$ of excluded r-uniform ($r \geq 3$) hypergraphs the local ratio approach cannot produce an approximation algorithm with approximation ratio asymptotically better than r. A result of a similar flavor has been proven by Boppana and Halldórsson [5] for the case of graphs ($r = 2$).

Let $\mathcal{H}_0 = \{H_1, \ldots, H_t\}$ be a fixed family of r-uniform hypergraphs. If we want to plug this family into our general algorithm $LOCAL(\mathcal{H}_0)$ and to obtain an algorithm with approximation ratio better than r, we should require that $lr(H_i) < r$ for every $H_i \in \mathcal{H}_0$. So assume this is indeed the case and write $lr(\mathcal{H}_0) = r - \epsilon$ for some fixed $0 < \epsilon < r$.

For an r-uniform hypergraph $H = (V, E)$, where $|V| > r$, let

$$\rho(H) = \max_{V' \subseteq V, |V'| > r} \frac{e(V') - 1}{|V'| - r}$$

be the *density* of H. It is easy to check that if H has no isolated vertices, then $\rho(H) \geq |E(H)|/|V(H)|$. Given a family $\mathcal{H}_0 = \{H_1, \ldots, H_t\}$, let $\rho(\mathcal{H}_0) = \min_{1 \leq i \leq t} \rho(H_i)$ denote the *density* of $\mathcal{H}_0$.

The following lemma states that a hypergraph H having a relatively small local ratio should be rather dense.

Lemma 5 *Let $H = (V, E)$ be an r-uniform hypergraph satisfying $lr(H) \leq r - \epsilon$ for some $0 < \epsilon < r$. Then $\rho(H) \geq \frac{2(r-\epsilon)+3}{2r(r-\epsilon)}$.*

Proof. Suppose that the lemma fails, and let $H = (V, E)$ be a hypergraph with the minimal number of vertices, contradicting the lemma statement. Then H has no isolated vertices and therefore $\rho(H) \geq |E(H)|/|V(H)|$, as noted above.

Define the subsets U_1 and U_2 of V by

$$\begin{aligned} U_1 &= \{v \in V : d(v) = 1\}, \\ U_2 &= \{v \in V : d(v) = 2\}, \end{aligned}$$

let also $|V| = n$, $|U_1| = n_1$ and $|U_2| = n_2$. Then clearly

$$n_1 + n_2 \leq n \ . \tag{9}$$

Assume first that H has an edge e_0 contained entirely in U_1. Then if the subset $V \setminus e_0$ spans no edges, then (recall that U_1 consists of vertices of degree 1) H has only one edge e_0, and therefore $lr(H) = r$, contradicting our assumption. Otherwise, we consider the hypergraph $H' = H[V \setminus e_0]$. It has $n - r$ vertices, more than one edge, and its minimal cover has one vertex less than a minimal cover of H. Then

$$lr(H') = \frac{|V(H')|}{\tau(H')} = \frac{n-r}{\tau(H)-1} < \frac{n}{\tau(H)} = lr(H)$$

(in this section $\tau(H)$ denotes the covering number of H for the *unweighted* case). From the definition of $\rho(H)$ we have $\rho(H') \leq \rho(H)$, thus obtaining a contradiction to the minimality of H.

If there exists an edge $e_0 \in E(H)$ with $|e_0 \cap U_1| = r - 1$, then denote $v_0 = e_0 \setminus U_1$ and consider the hypergraph $H' = H[V \setminus e_0]$. Clearly, $|V(H')| = n - r$ and $\tau(H') \geq \tau(H) - 1$ (if C is cover of H', then $C \cup \{v_0\}$ is a cover of H), and we get a contradiction in a similar way as above.

Summarizing the above, we may assume that every edge $e \in E(H)$ has at most $r - 2$ vertices in common with U_1. Now we are going to show that the set $U_1 \cup U_2$ contains a relatively large independent subset in H. For every $2 \leq i \leq r$ define the edge set $E_i \subset E$ by

$$E_i = \{e \in E(H) : |e \cap U_2| = i, \ |e \cap U_1| = r - i\} \ ,$$

let also $a_i = |E_i|$. Then we have

$$2n_2 = \sum_{v \in U_2} d(v) \geq \sum_{i=2}^{r} i a_i \ . \tag{10}$$

If there exist two edges $e_1, e_2 \in E_2$ such that $e_1 \cap e_2 \neq \emptyset$, then let $v_0 \in e_1 \cap e_2 \cap U_2$. Considering the hypergraph $H' = H[V \setminus ((e_1 \setminus U_2) \cup (e_2 \setminus U_2) \cup \{v_0\})]$, we see that $|V(H')| = n - (2r-3)$ and $\tau(H') \geq \tau(H) - 1$ (a cover of H can be obtained by adding v_0 to a cover of H'), and we again get a contradiction. Thus, we may assume that all edges of E_2 are pairwise disjoint, implying

$$2a_2 \leq n_2 \ . \tag{11}$$

Multiplying (10) by $1/24$ and (11) by $1/12$ and adding we get $a_2/4 + a_3/8 + \sum_{i=4}^{r} i a_i/24 \leq n/6$, therefore

$$\sum_{i=2}^{r} \frac{a_i}{2^i} \leq \frac{n}{6} \ . \tag{12}$$

Let V_0 be a random subset of U_2, obtained by choosing each vertex $v \in U_2$ to be in V_0 independently and with probability $1/2$. Then $E[V_0] = n_2/2$ and the expectation of the number of edges spanned by $U_1 \cup V_0$ is $\sum_{i=2}^{r} a_i/2^i \leq n/6$ by (12), and hence there exists a subset $V_0 \subseteq U_2$ with $|V_0| - e(U_1 \cup V_0) \geq n_2/2 - n_2/6 = n_2/3$. Fix such a set V_0, delete one vertex from every edge spanned by $U_1 \cup V_0$, the remaining subset united with U_1 forms an independent set of size at least $n_1 + n_2/3$. Therefore $\tau(H) \leq n - n_1 - n_2/3$. Hence the local ratio $lr(H)$ satisfies $\frac{n}{n-n_1-n_2/3} \leq \frac{|V(H)|}{\tau(H)} = lr(H) \leq r - \epsilon$. We get $n - n_1 - n_2/3 \geq n/(r-\epsilon)$, or

$$n_1 + \frac{n_2}{3} \leq \left(1 - \frac{1}{r-\epsilon}\right) n \ . \tag{13}$$

Multiplying (13) by $3/2$ and (9) by $1/2$ and adding we have

$$2n_1 + n_2 \leq \left(2 - \frac{3}{2(r-\epsilon)}\right) n \ . \tag{14}$$

From the definition of U_1 and U_2 we have the following estimate on the number of edges $|E(H)|$.

$$\begin{aligned} |E(H)| &= \frac{1}{r} \sum_{v \in V} d(v) = \frac{1}{r} \left(\sum_{v \in U_1} d(v) + \sum_{v \in U_2} d(v) + \sum_{v \in V \setminus (U_1 \cup U_2)} d(v) \right) \\ &\geq \frac{1}{r} (|U_1| + 2|U_2| + 3(|V| - |U_1| - |U_2|)) \\ &= \frac{1}{r} (3n - 2n_1 - n_2) \ . \end{aligned}$$

Hence it follows from (14) that

$$\rho(H) \geq \frac{|E(H)|}{|V(H)|} \geq \frac{\frac{1}{r}\left(3n - \left(2 - \frac{3}{2(r-\epsilon)}\right) n\right)}{n} = \frac{1}{r}\left(1 + \frac{3}{2(r-\epsilon)}\right) = \frac{2(r-\epsilon)+3}{2r(r-\epsilon)} \ ,$$

obtaining a contradiction to the assumption about H and thus finishing the proof. $\square$

We deduce immediately

Corollary 6 *If $\mathcal{H}_0 = \{H_1, \ldots, H_t\}$ is a family of r-uniform hypergraphs, satisfying $lr(\mathcal{H}_0) \leq r - \epsilon$, then $\rho(\mathcal{H}_0) \geq \frac{2(r-\epsilon)+3}{2r(r-\epsilon)}$.*

Our next step is to prove the following negative Ramsey-type result.

Lemma 6 *Let $\mathcal{H}_0 = \{H_1, \ldots, H_t\}$ be a fixed family of r-uniform hypergraphs with density $\rho(\mathcal{H}_0)$. Then there exists a constant $c = c(\mathcal{H}_0)$ such that for every sufficiently large integer n there exists an r-uniform hypergraph H_0 on n vertices, having the following properties:*

1. *H_0 does not contain a copy of any hypergraph from $\mathcal{H}_0$;*
2. *H_0 does not contain an independent set of size $\lceil cn^{\frac{1}{(r-1)\rho(\mathcal{H}_0)}}(\ln n)^{\frac{1}{r-1}}\rceil$.*

Proof. This statement can be proven by applying the Lovász local lemma [7], as it was done in [5]. We chose to present a proof based on using large deviation inequalities, as developed in [19].

For every $1 \leq i \leq t$ let H_i' be a subhypergraph of H_i such that $\rho(H_i) = (|E(H_i')| - 1)/(|V(H_i')| - r)$. Setting $\mathcal{H}_0' = \{H_1', \ldots, H_t'\}$, note that if H is $\mathcal{H}_0'$-free, then it is clearly $\mathcal{H}_0$-free, therefore we may assume that $\rho(H_i) = (|E(H_i)| - 1)/(|V(H_i)| - r)$.

For every $1 \leq i \leq t$ set $v_i = |V(H_i)|$, $f_i = |E(H_i)|$. Set also

$$\begin{aligned} f_{min} &= \min\{f_i : 1 \leq i \leq t\}, \\ f_{max} &= \max\{f_i : 1 \leq i \leq t\}. \end{aligned}$$

Clearly we may assume that $\rho(H_0) > 1/(r-1)$, otherwise there is nothing to prove. Consider a random r-uniform hypergraph $H_r(n,p)$ – an r-uniform hypergraph with vertex set V of size $|V| = n$, in which every r-tuple $e \subseteq V$ is chosen to be an edge of H independently and with probability p. We set $p = c_0 n^{-1/\rho(\mathcal{H}_0)}$, where $0 < c_0 < 1$ is a sufficiently small constant.

Let $n_0 = \lceil cn^{\frac{1}{(r-1)\rho(\mathcal{H}_0)}}(\ln n)^{\frac{1}{r-1}}\rceil$. For every subset $V_0 \subset V$ of size $|V_0| = n_0$ let X_{V_0} be the random variable, counting the number of edges of H, spanned by V_0. Also, denote by Y_{V_0} the number of subhypergraphs of H, each isomorphic to one of the hypergraphs from $\mathcal{H}_0$ and having at least one edge inside V_0, and by Z_{V_0} the maximal number of pairwise edge disjoint subhypergraphs of H, each isomorphic to one of the hypergraphs from $\mathcal{H}_0$ and having at least one edge inside V_0. Clearly, $Z_{V_0} \leq Y_{V_0}$. Denote by A_{V_0} the event $X_{V_0} > f_{max} Z_{V_0}$.

Claim 1 *If A_{V_0} holds for every $V_0 \subset V$ of size $|V_0| = n_0$, then H contains a subhypergraph H_0 on n vertices, satisfying the requirements of the lemma.*

Proof. Let $\mathbf{H}$ be a maximal under inclusion family of pairwise edge disjoint subhypergraphs of H, each isomorphic to one of the hypergraphs from $\mathcal{H}_0$. Deleting all edges of all subhypergraphs from $\mathbf{H}$, we clearly obtain an $\mathcal{H}_0$-free hypergraph H_0 on n vertices. For a subset $V_0 \subset V$ of size $|V_0| = n_0$, denote by $\mathbf{H}_{V_0}$ the subfamily of $\mathbf{H}$, consisting of all hypergraphs from $\mathbf{H}$, having at least one edge inside V_0. From the definition of Z_{V_0} it follows that $|\mathbf{H}_{V_0}| \leq Z_{V_0}$. While deleting the edges of the subhypergraphs from $\mathbf{H}$ we delete at most $f_{max}|\mathbf{H}_{V_0}| \leq f_{max} Z_{V_0}$ edges from $E(V_0)$, hence the subhypergraph H_0 has at least one edge in each subset V_0 of size $|V_0| = n_0$. $\Box$

Now our aim is to show that under appropriate choice of constants c_0 and c the inequality $P[\bigwedge_{|V_0|=n_0} A_{V_0}] > 0$ holds for all sufficiently large n. To this end, we show that the random variables X_{V_0} and Z_{V_0} are highly concentrated around their expectations and if, say, $EX_{V_0} > 10 f_{max} EZ_{V_0}$, then the probability $P[\overline{A_{V_0}}]$ is exponentially small, implying in turn that the probability of the existence of a set V_0, for which $\overline{A_{V_0}}$ holds, is less than 1.

The random variable X_{V_0} is binomially distributed with parameters $\binom{n_0}{r}$ and p, therefore well known estimates on the tails of binomial distribution due to Chernoff (see, e.g., [2], Appendix A) assert that for every $0 < \alpha < 1$

$$P[X_{V_0} < (1-\alpha)\binom{n_0}{r}p] < e^{-\alpha^2\binom{n_0}{r}p/2} . \tag{15}$$

Now we turn to bounding the upper tail of Z_{V_0}. The random variable Z_{V_0} is tightly connected with another random variable Y_{V_0}.

Claim 2 $P[Z_{V_0} \geq j] \leq \frac{(EY_{V_0})^j}{j!}$ *for every natural j.*

Proof. This is a particular case of the general result of Erdős and Tetali [8] (see also Lemma 4.1 of Ch. 8 of [2]). $\Box$

In particular, we deduce from the above claim that

$$P[Z_{V_0} \geq 5EY_{V_0}] \leq \left(\frac{e}{5}\right)^{5EY_{V_0}} . \tag{16}$$

Let us write $Y_{V_0} = Y_{V_0,1} + \ldots + Y_{V_0,t}$, where $Y_{V_0,i}$ is the number of copies of H_i, having at least one edge in $E(V_0)$. Representing $Y_{V_0,i}$ as a sum of indicator random variables, we get

$$\binom{n_0}{r}\binom{n-n_0}{v_i-r}p^{f_i} \leq EY_{V_0,i} \leq \binom{n_0}{r}\binom{n-r}{v_i-r}v_i!p^{f_i} .$$

Therefore

$$c_{i,1}\binom{n_0}{r}p\left(n^{\frac{v_i-r}{f_i-1}}\right)^{f_i-1}\le EY_{V_0,i}\le c_{i,2}\binom{n_0}{r}p\left(n^{\frac{v_i-r}{f_i-1}}\right)^{f_i-1} ,$$

where $c_{i,1}$ and $c_{i,2}$ are some positive constants depending only on H_i.

The definitions of $\rho(\mathcal{H}_0)$ and p imply that

$$c_1c_0^{f_{max}-1}\binom{n_0}{r}p\le EY_{V_0}\le c_2c_0^{f_{min}-1}\binom{n_0}{r}p ,$$

where $c_1=c_1(\mathcal{H}_0)$ and $c_2=c_2(\mathcal{H}_0)$ are positive constants.

Comparing EX_{V_0} and EY_{V_0} we observe

$$\frac{1}{c_2c_0^{f_{min}-1}}\le\frac{EX_{V_0}}{EY_{V_0}}\le\frac{1}{c_1c_0^{f_{max}-1}} .$$

Let us choose c_0 so that the expression $c_2c_0^{f_{min}-1}$ will be equal to, say $1/10f_{max}$. Then

$$10f_{max}\le\frac{EX_{V_0}}{EY_{V_0}}\le\frac{c_2}{c_1}c_0^{-f_{max}+f_{min}}10f_{max} .$$

Now, by (15) with $\alpha=1/2$ and (16)

$$\begin{aligned}P[\overline{A_{V_0}}] &= P[X_{V_0}\le f_{max}Z_{V_0}]\le P[X_{V_0}\le\frac{EX_{V_0}}{2}]+P[f_{max}Z_{V_0}\ge\frac{EX_{V_0}}{2}]\\ &\le P[X_{V_0}\le\frac{EX_{V_0}}{2}]+P[Z_{V_0}\ge 5EY_{V_0}]\\ &\le e^{-\binom{n_0}{r}p/8}+e^{-\frac{c_1c_0^{-f_{min}+f_{max}}}{10c_2f_{max}}[5\ln 5-5]\binom{n_0}{r}p}\\ &\le e^{-c_3n_0^rp}\end{aligned}$$

for some constant $c_3>0$. Therefore

$$P[\exists V_0:\ \overline{A_{V_0}}]\le\binom{n}{n_0}e^{-c_3n_0^rp} .$$

Using the inequality $\binom{n}{n_0}\le\left(\frac{en}{n_0}\right)^{n_0}$, we write

$$\binom{n}{n_0}e^{-c_3n_0^rp}\le\left(\frac{en}{n_0}\cdot e^{-c_3n_0^{r-1}p}\right)^{n_0} .$$

Taking c sufficiently large it follows that $P[\bigwedge_{|V_0|=n_0}A_{V_0}]>0$. $\Box$

Combining Lemma 6 and Corollary 6, we have the following result.

Corollary 7 *Let $\mathcal{H}_0=\{H_1,\ldots,H_t\}$ be a family of r-uniform hypergraphs, satisfying $lr(\mathcal{H}_0)\le r-\epsilon$ for some $0<\epsilon<r$. Then for every sufficiently large integer n there exists an $\mathcal{H}_0$-free r-uniform hypergraph H_0 on n vertices, having no independent set of size $n_0=O(n^{\frac{2r(r-\epsilon)}{(r-1)(2(r-\epsilon)+3)}}(\ln n)^{\frac{1}{r-1}})$ (and therefore not colorable by n/n_0 colors).*

The final step is to note that the expression $2r(r-\epsilon)/((r-1)(2(r-\epsilon)+3))$ is always strictly less than 1 for all $r \geq 3$ and all $0 < \epsilon < r$.

How should we interpret the above corollary? Actually, it indicates that the subhypergraph exclusion algorithm $LOCAL(\mathcal{H}_0)$, described in Section 5, can not have approximation ratio better than $r(1-1/n^{\delta})$ for any fixed family $\mathcal{H}_0$, where $\delta = \delta(\mathcal{H}_0)$ (or, more precisely, we cannot show it by our tools of analysis). Some new ideas and algorithms are needed to make a breakthrough in this important problem.

References

[1] R. Aharoni, R. Holzman and M. Krivelevich, *On a theorem of Lovász on covers in r-partite hypergraphs*, Combinatorica 16 (1996), 149–174.

[2] N. Alon and J. H. Spencer, **The probabilistic method**, Wiley, New York, 1992.

[3] R. Bar-Yehuda and S. Even, *A linear time approximation algorithm for the weighted vertex cover problem*, J. Algorithms 2 (1981), 189–203.

[4] R. Bar-Yehuda and S. Even, A local ratio theorem for approximating the weighted vertex cover problem, in *Analysis and design of algorithms for combinatorial problems* (G. Ausiello and M. Lucertini, eds.), Annals of Discrete Math. Vol. 25, Elsevier, Amsterdam, 1985, 27–46.

[5] R. Boppana and M. M. Halldórsson, *Approximating maximum independent sets by excluding subgraphs*, Bit 32 (1992), 180–196.

[6] F. R. K. Chung, Z. Füredi, M. R. Garey and R. L. Graham, *On the fractional covering number of hypergraphs*, SIAM J. Discrete Math. 1 (1988), 45–49.

[7] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in *Infinite and finite sets*, (A. Hajnal et al., eds.), North Holland, Amsterdam, 1975, 609–628.

[8] P. Erdős and P. Tetali, *Representations of integers as the sum of k terms*, Random Struct. Alg. 1 (1990), 245–261.

[9] Z. Füredi, *Matchings and covers in hypergraphs*, Graphs and Combinatorics 4 (1988), 115–206.

[10] M. R. Garey and D. S. Johnson, **Computers and intractability: a guide to the theory of NP-completeness**, W. H. Freeman and Co., San Francisco, 1979.

[11] N. G. Hall and D. S. Hochbaum, *A fast approximation algorithm for the multicovering problem*, Disc. Appl. Math. 15 (1986), 35–40.

[12] M. M. Halldórsson, *A still better performance guarantee for approximate graph coloring*, Inform. Process. Lett. 45 (1993), 19–23.

[13] M. M. Halldórsson and J. Radhakrishnan, *Greed is good: approximating independent sets in sparse and bounded-degree graphs*, Proc. of 26th Ann. ACM Symp. on Theory of Computing, ACM, 1994, 439–448.

[14] D. S. Hochbaum, *Approximation algorithms for the set covering and vertex cover problems*, SIAM J. Computing 11 (1982), 555–556.

[15] D. S. Hochbaum, *Efficient bounds for the stable set, vertex cover and set packing problems*, Disc. Appl. Math. 6 (1983), 243–254.

[16] D. S. Johnson, *Worst case behaviour of graph coloring algorithms*, Proc. 5th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Congr. Numer. 10 (1974), 512–527.

[17] N. Karmarkar, *A new polynomial-time algorithm for linear programming*, Combinatorica 4 (1984), 373–395.

[18] R. Karp, Reducibility among combinatorial problems, in *Complexity of computer computations* (E. Miller and J. W. Thatcher, eds.) Plenum Press, New York, 1972, 85–103.

[19] M. Krivelevich, *Bounding Ramsey numbers through large deviation inequalities*, Random Struct. Alg. 7 (1995), 145–155.

[20] B. Monien and E. Speckenmeyer, *Ramsey numbers and an approximation algorithm for the vertex cover problem*, Acta Informatica 22 (1985), 115–123.

[21] G. L. Nemhauser and L. E. Trotter, *Vertex packing: structural properties and algorithms*, Math. Programming 8 (1975), 232–248.

[22] C. H. Papadimitriou and M. Yannakakis, *Optimization, approximation, and complexity classes*, J. Comput. System Sci. 43 (1991), 425–440.

[23] D. Peleg, G. Schechtman and A. Wool, *Approximating bounded 0-1 integer linear programs*, Proc. of 2nd Israeli Symp. on Theory of Computing and Systems, IEEE Computer Society, 1993, 69–77.

[24] P. Raghavan and C. D. Thompson, *Randomized rounding: a technique for provably good algorithms and algorithmic proofs*, Combinatorica 7 (1987), 365–374.

[25] J. B. Shearer, *A note on the independence number of triangle-free graphs*, Discrete Math. 46 (1983), 83–87.

[26] A. Wigderson, *Improving the performance guarantee for approximate graph coloring*, J. ACM 30 (1983), 729–735.