On the minimal number of edges in color-critical graphs

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Abstract

A graph G is k-critical if it has chromatic number k, but every proper subgraph of it is $(k-1)$-colorable. This paper is devoted to investigating the following question: for given k and n, what is the minimal number of edges in a k-critical graph on n vertices, with possibly some additional restrictions imposed? Our main result is that for every $k \geq 4$ and $n > k$ this number is at least $\left(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)}\right) n$, thus improving a result of Gallai from 1963. We discuss also the upper bounds on the minimal number of edges in k-critical graphs and provide some constructions of sparse k-critical graphs. A few applications of the results to Ramsey-type problems and problems about random graphs are described.

1 Introduction

All graphs considered in this paper are finite, non-empty (having at least one vertex), undirected, without loops and multiple edges.

For a graph $G = (V, E)$, let $\chi(G)$ denote the chromatic number of G. A graph G is called *k-critical*, if $\chi(G) = k$ and $\chi(G') < k$ for every proper subgraph G' of G. In this paper we study the following basic problem and some variants of it. Given a pair of integers k and n, how small can the number of edges in a k-critical graph on n vertices be? For $k = 1, 2$ and 3 the structure of k-critical graphs is completely known, they are K^1, K^2 and odd cycles, respectively, where K^i denotes a complete graph (a clique) on i vertices. However, for $k \geq 4$ such a simple description of k-critical graphs does not exist, and the above mentioned problem (as well as many other interesting problems about critical graphs) remains open. The famous theorem of Brooks [4] can be reformulated using the notion of critical graphs in the following way: in a k-critical graph G all vertices have degree at least $k-1$; moreover, if all vertices have degree $k-1$, then G is either the complete graph K^k, or $k = 3$ and G

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is on odd cycle. Relying on Brooks' theorem, we will always assume in the sequel that the number of vertices in a k-critical graph G is greater than k.

Let us define the following two functions. For a pair of integers $2 \le s \le k$ let

$$f_{k,s}(n) = \min\{|E(G)| : G = (V,E) \text{ is } k\text{-critical}, |V| = n,$$
$$G \text{ does not contain a clique of size } s+1\} \ .$$

Also, for a pair of integers $k \ge 1$, $l \ge 1$ define

$$g_{k,l}(n) = \min\{|E(G)| : G = (V,E) \text{ is } k\text{-critical }, |V| = n,$$
$$G \text{ does not contain an odd cycle of length less than } 2l+1\} \ .$$

(An additional motivation for considering lengths of odd cycles in the definition of $g_{k,l}(n)$ will be provided later). If a k-critical graph, satisfying the restrictions in the definition of $f_{k,s}(n)$ or $g_{k,l}(n)$, does not exist, we put $f_{k,s}(n) = \infty$ or $g_{k,l}(n) = \infty$. Using this notation, Brooks' theorem gives $f_{k,k}(n) \ge (k-1)n/2$, and also $f_{k,k-1}(n) > (k-1)n/2$ for $k \ge 4$.

The problem of estimating the minimal number of edges in k-critical graphs has been considered by Dirac in 1957 [5], who proved $f_{k,k}(n) \ge ((k-1)n + k - 3)/2$. Gallai showed in 1963 in his fundamental paper [10] on color-critical graphs that for $k \ge 4$ every k-critical graph $G = (V,E) \ne K^k$ satisfies $|E(G)| \ge ((k-1)/2 + (k-3)/(2k^2-6))|V(G)|$, or in our notation

$$f_{k,k-1}(n) \ge \left(\frac{k-1}{2} + \frac{k-3}{2(k^2-3)}\right) n \ , \tag{1}$$

thus improving the trivial bound $f_{k,k-1}(n)/n \ge (k-1)/2$. We note that his argument can be used also to produce lower bounds for $f_{k,s}(n)$ and $g_{k,l}(n)$ for various values of k, l and s. (Actually, we will exploit this idea in our proofs). Since Gallai's paper, no further progress in finding lower bounds for $f_{k,s}(n)$ and $g_{k,l}(n)$ has been achieved.

In the present paper we improve the result of Gallai. We prove the following theorems.

Theorem 1 *Let k and s be a pair of integers, satisfying $3 \le s < k$. Let $G = (V,E)$ be a k-critical graph not containing a clique of size $s+1$. Then*

1. *If $s \le \frac{2k}{3}$, then $|E(G)| \ge \left(\frac{k}{2} - \frac{k-2}{2(2k-s-3)}\right)|V(G)|$;*
2. *If $s \ge \frac{2k}{3}$, then $|E(G)| \ge \left(\frac{k}{2} - \frac{(k-2)s}{2(2ks-2k-s^2)}\right)|V(G)|$.*

Corollary 1 $f_{k,k-1}(n) \ge \left(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)}\right) n$.

In particular, for the first non-trivial case $k = 4$, we get $f_{4,3}(n) \ge \frac{11}{7} n$ (compared with the $\frac{20}{13} n$ lower bound of Gallai).

Theorem 2 *Let $k \geq 4$ and $l \geq 1$ be integers. Let $G = (V, E)$ be a k-critical graph not containing an odd cycle of length less than $2l+1$. Then*

1. *If $k = 4$, then $|E(G)| \geq \left(2 - \frac{2l+1}{6l+1}\right)|V(G)|$;*

2. *If $k \geq 5$, then $|E(G)| \geq \left(\frac{k}{2} - \frac{(k-2)l}{2(2kl-5l-1)}\right)|V(G)|$.*

The rest of the paper is organized as follows. In Section 2 we introduce some definitions and notation to be used in the sequel. In Section 3 we cite basic tools needed for our proof. Proofs of Theorems 1 and 2 are presented in Section 4. In Section 5 we discuss upper bounds on the minimal number of edges in a k-critical graph and provide constructions of k-critical graphs. Finally (Section 6), applications of Theorems 1 and 2 to some Ramsey-type problems and problems on random graphs are given. A short note, containing only the proof of Corollary 1, will be published separately.

2 Definitions and notation

Our notation follows mainly the notation of two survey papers [18] and [19] of Sachs and Stiebitz. All concepts not defined here can be found in any textbook on graph theory.

Given a graph $G = (V, E)$, a *k-coloring* of G is a mapping $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for every edge $e = (u, v) \in E(G)$. A graph that admits a k-coloring is *k-colorable*. The *chromatic number* $\chi(G)$ of a graph G is the minimal k for which G is k-colorable. A graph G is called k-color-critical (or briefly *k-critical*), if $\chi(G) = k$ and $\chi(G') < k$ for every proper subgraph G' of G.

We denote by $w(G)$ the clique number of G, that is, the maximal size of a clique (complete graph) contained in G. Also, let $oddgirth(G)$ be the minimal length of an odd cycle in G (if G is bipartite, we set $oddgirth(G) = \infty$). Using this notation, we can rewrite the definitions of $f_{k,s}(n)$ and $g_{k,l}(n)$ so:

$$f_{k,s}(n) = \min\{|E(G)| : G = (V, E) \text{ is } k\text{-critical}, |V| = n, w(G) \leq s\} \; ; \tag{2}$$

$$g_{k,l}(n) = \min\{|E(G)| : G = (V, E) \text{ is } k\text{-critical}, |V| = n, oddgirth(G) \geq 2l+1\} \; . \tag{3}$$

It is easy to see that if $G = (V, E)$ is k-critical, then the degree $d(v)$ of every vertex $v \in V(G)$ is at least $k-1$. If $d(v) = k-1$, we call v a *low vertex*, otherwise v is a *high vertex*. The subgraphs of G induced by the set of low vertices and the set of high vertices are called the *low-vertex subgraph* $L(G)$ and the *high-vertex subgraph* $H(G)$, respectively. We can reformulate Brooks' theorem in the following way: a k-critical graph G has no high vertices if and only if either G is K^k, or $k = 3$ and G is an odd cycle.

3 Basic tools

Our proof is based on two main ingredients: the theorem of Gallai [10], describing the structure of the low-vertex subgraph of a critical graph, and the theorem of Stiebitz [21] about the number of connected components of the low-vertex and the high-vertex subgraphs of a critical graph.

A maximal connected subgraph B of a graph G such that any two edges of B are contained in a cycle of G is called a *block* of G. An *endblock* of G is a block which contains at most one cut vertex of G. It is well known that every graph with at least one edge contains an endblock with at least two vertices.

A connected graph all of whose blocks are either complete graphs or odd cycles is called a *Gallai tree*, a *Gallai forest* is a graph all of whose connected components are Gallai trees. A *k-Gallai tree (forest)* is a Gallai tree (forest) in which all vertices have degree at most $k-1$. Gallai [10] proved the following deep structural theorem about critical graphs.

Theorem 3.1 *If G is a k-critical graph, then its low-vertex subgraph $L(G)$ is a k-Gallai forest (possibly empty).*

(For a somewhat simpler proof, see [19], pp. 216–217.) This theorem shows the relevance of the restriction on the length of odd cycles in the definition of $g_{k,l}(n)$.

The second theorem we will use is the following result of Stiebitz [21].

Theorem 3.2 *Let G be a k-critical graph. Then the number of connected components of its high-vertex subgraph $H(G)$ does not exceed the number of connected components of its low-vertex subgraph $L(G)$.*

Now we can roughly describe the core idea of our proof. Let $G=(V,E)$ be a k-critical graph with low- and high-vertex subgraphs $L(G)$ and $H(G)$. Denote $n_L=|V(L(G))|$, $n_H=|V(H(G))|$, $n=n_L+n_H$, and assume $n>k$. If $r\geq 1$ denotes the number of connected components of $H(G)$, then trivially $|E(H(G))|\geq n_H-r$. Also, by Stiebitz's theorem the number of connected components of $L(G)$ is at least r. Now, using Gallai's theorem, we show that the number of edges in $L(G)$ as a function of k, n_L and r is small and therefore the number of edges between $V(L(G))$ and $V(H(G))$ is large, implying in turn that the total number of edges in G is large. We present a detailed proof in the next section.

4 Proofs of Theorems 1 and 2

4.1 Number of edges in a k-Gallai forest

The first step of our proof is to bound from above the number of edges in a k-Gallai forest with additional restrictions imposed.

Lemma 4.1 *Let $k \geq 4$, $s \geq 3$. Let $G = (V, E)$ be a k-Gallai forest, not containing a clique of size $s+1$. Then $|E(G)| \leq \frac{s}{2}|V(G)| - \frac{s}{2}$.*

Proof. By joint induction on the number of blocks and the number of vertices in G.

Assume first that G has only one block. According to Theorem 3.1 this block is either a clique K^j for $1 \leq j \leq s$ or an odd cycle of length n. In both cases the validity of the lemma is easily checked.

Assume now that G has more than one block. If G has no edges, then the lemma holds trivially, therefore assume $E(G) \neq \emptyset$. Then G has an endblock B containing at least two vertices. Define $v \in V(B)$ as follows: if B contains a separating vertex, let v be this vertex, otherwise choose an arbitrary vertex of B as v. The number of blocks of the graph $G' = G[V \setminus (V(B) - v)]$ does not exceed the number of blocks of G, while $|V(G')| < |V(G)|$. Denote $|V(G)| = n$, $|E(G)| = e$, $|V(G')| = n'$, $|E(G')| = e'$. By the induction hypothesis

$$e' \leq \frac{s}{2}n' - \frac{s}{2} \ . \tag{4}$$

If B is a clique K^j, where $2 \leq j \leq s$, we have $n' = n - j + 1$, $e' = e - \binom{j}{2}$, and thus from (4)

$$\begin{aligned} e &= e' + \binom{j}{2} \leq \frac{s}{2}n' - \frac{s}{2} + \binom{j}{2} = \frac{s}{2}(n - j + 1) - \frac{s}{2} + \binom{j}{2} \\ &= \frac{s}{2}n - \frac{s}{2} + \left(\binom{j}{2} - \frac{s}{2}(j-1)\right) \leq \frac{s}{2}n - \frac{s}{2} \ . \end{aligned}$$

If B is an odd cycle of length j, then $n' = n - j + 1$ and $e' = e - j$, and using (4) we get

$$\begin{aligned} e &= e' + j \leq \frac{s}{2}n' - \frac{s}{2} + j = \frac{s}{2}(n - j + 1) - \frac{s}{2} + j \\ &= \frac{s}{2}n - \frac{s}{2} + \left(j - \frac{s}{2}(j-1)\right) \leq \frac{s}{2}n - \frac{s}{2} \ . \qquad \square \end{aligned}$$

Lemma 4.2 *Let $k \geq 4$, $s \geq 3$. Let $G = (V, E)$ be a k-Gallai forest not containing a clique of size $s+1$. Then*

$$|E(G)| \leq \frac{s^2 - 3s + 2k}{2s}|V(G)| - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\} \ .$$

Proof. The proof is quite similar to the proof of the previous lemma, but this time, while considering an endblock B, we delete the whole block in the case $B = K^s$.

We proceed by induction on the number of vertices of $G = (V, E)$. Denote $|V(G)| = n$, $|E(G)| = e$. Let first $n \leq s$. It is easy to see that the only case to be checked is when G is a clique K^n. In this case we need to verify

$$\binom{n}{2} \leq \frac{s^2 - 3s + 2k}{2s} n - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\}$$

for every $1 \leq n \leq s$, which is true (equality holds when either $n = 1$ and $s \leq 2k/3$ or $n = s$ and $s \geq 2k/3$).

Assume now $n \geq s$. Clearly we may assume also $E(G) \neq \emptyset$. Let B be an endblock of G with at least two vertices. If B has a separating vertex, let v be this vertex, otherwise let v be an arbitrary vertex of B. Consider first the case $B = K^j$, $2 \leq j \leq s - 1$. In this case we define $G' = G[V \setminus (V(B) - v)]$. Let $|V(G')| = n'$, $|E(G')| = e'$. By the induction hypothesis

$$e' \leq \frac{s^2 - 3s + 2k}{2s} n' - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\}, \tag{5}$$

and also $n' = n - j + 1$, $e' = e - \binom{j}{2}$. Therefore

$$\begin{aligned} e &= e' + \binom{j}{2} \leq \frac{s^2 - 3s + 2k}{2s} n' - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\} + \binom{j}{2} \\ &\leq \frac{s^2 - 3s + 2k}{2s} n - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\} + \\ &\quad \left(\binom{j}{2} - \frac{s^2 - 3s + 2k}{2s}(j - 1)\right) \\ &\leq \frac{s^2 - 3s + 2k}{2s} n - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\} \end{aligned}$$

(note that by the definition of a k-Gallai forest we have $s \leq k$).

Now let $B = K^s$. In this case we define $G' = G[V \setminus (V(B))]$, $n' = |V(G')|$, $e' = |E(G')|$. The degree $d(v)$ satisfies $d(v) \leq k - 1$, therefore we have $n' = n - s$ and $e' \geq e - \binom{s}{2} - ((k-1) - (s-1)) = e - \binom{s}{2} - (k - s)$. Assuming the induction hypothesis (5), we derive

$$\begin{aligned} e &\leq e' + \binom{s}{2} + k - s \leq \frac{s^2 - 3s + 2k}{2s} n' - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\} + \binom{s}{2} + k - s \\ &= \frac{s^2 - 3s + 2k}{2s} n - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\}. \end{aligned}$$

Finally, if B is an odd cycle of length $j \geq 5$ (the case $j = 3$ has already been covered), we consider the graph $G' = G[V \setminus (V(B) - v)]$. Denoting $n' = |V(G')|$, $e' = |E(G')|$, we have $n' = n - j + 1$,

$e' = e - j$, and again assuming the induction hypothesis (5) we obtain

$$\begin{aligned} e &= e' + j \le \frac{s^2 - 3s + 2k}{2s} n' - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\} + j \\ &\le \frac{s^2 - 3s + 2k}{2s} n - \min\left\{k - s, \frac{s^2 - 3s + 2k}{2s}\right\} , \end{aligned}$$

finishing the proof of the lemma. $\square$

Note that for $3 \le s \le k - 1$ one has $\frac{s^2 - 3s + 2k}{2s} > 1$, therefore the results of Lemmas 4.1 and 4.2 can be rewritten in the following unified form.

Corollary 4.1 *Let $3 \le s \le k - 1$. Let $G = (V, E)$ be a k-Gallai forest with $w(G) \le s$. Then*

$$|E(G)| \le \min\left\{\frac{s}{2}, \frac{s^2 - 3s + 2k}{2s}\right\} |V(G)| - 1 .$$

One can easily check that $s/2 \le (s^2 - 3s + 2k)/2s$ if and only if $s \le 2k/3$. In the case $s = k - 1$ the statement of Corollary 4.1 coincides with that of Lemma 4.5 of [10].

The crucial observation here is the result of Corollary 4.1 is valid for *each* connected component of a k-Gallai forest G, therefore we get the following

Corollary 4.2 *Let $3 \le s \le k - 1$. Let $G = (V, E)$ be a k-Gallai forest with r connected components, not containing a clique of size $s + 1$. Then*

$$|E(G)| \le \min\left\{\frac{s}{2}, \frac{s^2 - 3s + 2k}{2s}\right\} |V(G)| - r .$$

Proof. Let $G_1 = (V_1, E_1), \ldots, G_r = (V_r, E_r)$ be the connected components of G. By Corollary 4.1,

$$|E_i| \le \min\left\{\frac{s}{2}, \frac{s^2 - 3s + 2k}{2s}\right\} |V_i| - 1 , \qquad i = 1, \ldots, r .$$

Summing the above inequalities over $1 \le i \le r$ we get the desired result. $\square$

Now we treat the case when $oddgirth(G)$ is bounded from below.

Lemma 4.3 *Let $k \ge 4$, $l \ge 1$. Let $G = (V, E)$ be a k-Gallai forest not containing an odd cycle of length less than $2l + 1$. Then*

1. *If $k = 4$ then $|E(G)| \le \frac{2l+2}{2l+1}|V(G)| - 1$;*
2. *if $k \ge 5$ then $|E(G)| \le \frac{2l+1}{2l}|V(G)| - \frac{2l+1}{2l}$.*

Proof. The proof utilizes the same idea. The difference between the cases $k=4$ and $k\geq 5$ follows from the fact that when $k\geq 5$ two cycles of G may have a vertex in common.

Denote $|V(G)|=n$, $|E(G)|=e$. We prove the assertion of the lemma by induction on the number of vertices of G. If $n\leq 2l+1$ then it follows from the definition of a Gallai forest that G is either a forest or an odd cycle of length $2l+1$, in both cases the lemma is trivially true.

Now let $n>2l+1$. Let first $k=4$. If $E(G)=\emptyset$ we are done, therefore assume $E(G)\neq\emptyset$. Let B be an endblock of G with at least two vertices. If B has a separating vertex, let v be this vertex, otherwise let v be an arbitrary vertex of B. If B is a clique K^2 (an edge), consider the graph $G'=G[V\setminus(V(B)-v)]$. Then clearly $n'=|V(G')|=n-1$, $e'=|E(G')|=e-1$ and by the induction hypothesis

$$e=e'+1\leq\frac{2l+2}{2l+1}n'=\frac{2l+2}{2l+1}(n-1)<\frac{2l+2}{2l+1}n-1\ .$$

If B is an odd cycle of length $j\geq 2l+1$, we define $G'=G[V\setminus V(B)]$. (Clearly, we may assume that G contains blocks other than B, hence $V(G')\neq\emptyset$). Let $n'=|V(G')|$, $e'=|E(G')|$. Note that $d_G(v)\leq k-1=3$, therefore $n'=n-j$ and $e'\geq e-j-1$ and by induction we have

$$e\leq e'+j+1\leq\frac{2l+2}{2l+1}n'-1+j+1\leq\frac{2l+2}{2l+1}(n-j)+j\leq\frac{2l+2}{2l+1}n-1\ ,$$

thus establishing the case $k=4$.

Now assume $k\geq 5$. We may assume $E(G)\neq\emptyset$. Let B be an endblock of G with $|V(B)|\geq 2$. Define a vertex $v\in V(B)$ as in the case $k=4$ and consider $G'=G[V\setminus(V(B)-v)]$. Denote $n'=|V(G')|$, $e'=|E(G')|$. By the induction hypothesis $e'\leq\frac{2l+1}{2l}n'-\frac{2l+1}{2l}$. If $B=K^2$, then $n'=n-1$ and $e'=e-1$ and thus

$$e=e'+1\leq\frac{2l+1}{2l}n'-\frac{2l+1}{2l}+1<\frac{2l+1}{2l}n-\frac{2l+1}{2l}\ .$$

Finally, if B is an odd cycle of length $j\geq 2l+1$, we have $n'=n-j+1$, $e'=e-j$ and

$$e=e'+j\leq\frac{2l+1}{2l}n'-\frac{2l+1}{2l}+j\leq\frac{2l+1}{2l}n-\frac{2l+1}{2l}\ . \qquad\square$$

Corollary 4.3 *Let $k\geq 4$, $l\geq 1$. Let $G=(V,E)$ be a k-Gallai forest with r connected components, not containing an odd cycle of length less than $2l+1$. Then*

1. *If $k=4$, then $|E(G)|\leq\frac{2l+2}{2l+1}|V(G)|-r$;*
2. *If $k\geq 5$, then $|E(G)|\leq\frac{2l+1}{2l}|V(G)|-r$.*

4.2 Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $L(G)$ and $H(G)$ be the low-vertex and the high-vertex subgraphs of G, respectively. Denote $n_L = |V(L(G))|$, $n_H = |V(H(G))|$, $n = n_L + n_H$. By Brooks' theorem $n_H > 0$. Let r be the number of connected components of $H(G)$, then $|E(H(G))| \geq n_H - r$. By Theorem 3.2 the number of connected components of $L(G)$ is at least r. Hence Corollary 4.2 implies

$$\begin{aligned}
|E(G)| &= \sum_{v\in V(L(G))} d(v) - |E(L(G))| + |E(H(G))| \\
&\geq (k-1)n_L - \min\left\{\frac{s}{2}, \frac{s^2-3s+2k}{2s}\right\} n_L + r + n_H - r \\
&= n + \left(k - 2 - \min\left\{\frac{s}{2}, \frac{s^2-3s+2k}{2s}\right\}\right) n_L \ . \qquad (6)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|E(G)| &= \frac{1}{2}\sum_{v\in V(G)} d(v) = \frac{1}{2}\left(\sum_{v\in V(L(G))} d(v) + \sum_{v\in V(H(G))} d(v)\right) \\
&\geq \frac{1}{2}((k-1)n_L + kn_H) = \frac{k}{2}n - \frac{1}{2}n_L \ . \qquad (7)
\end{aligned}$$

We distinguish between two cases.

1) $s \leq 2k/3$.

Then (6) can be rewritten as

$$|E(G)| \geq n + \left(k - 2 - \frac{s}{2}\right) n_L \ . \qquad (8)$$

Multiplying (7) by $2k - s - 4$ and adding the resulting inequality with (8) we get

$$(2k - s - 3)|E(G)| \geq \left(1 + \frac{(2k-s-4)k}{2}\right) n \ ,$$

or

$$|E(G)| \geq \frac{(2k-s-4)k+2}{2(2k-s-3)} n = \left(\frac{k}{2} - \frac{k-2}{2(2k-s-3)}\right) n \ ;$$

2) $s \geq 2k/3$.

In this case (6) looks as

$$|E(G)| \geq n + \left(k - 2 - \frac{s^2-3s+2k}{2s}\right) n_L = n + \frac{2ks-2k-s^2-s}{2s} n_L \ . \qquad (9)$$

Multiplying (7) by $(2ks - 2k - s^2 - s)/s$ and adding with (9), we get

$$\left(1 + \frac{2ks-2k-s^2-s}{s}\right)|E(G)| \geq \left(1 + \frac{(2ks-2k-s^2-s)k}{2s}\right) n \ ,$$

or

$$|E(G)| \geq \frac{2s+(2ks-2k-s^2-s)k}{2(2ks-2k-s^2)} n = \left(\frac{k}{2} - \frac{(k-2)s}{2(2ks-2k-s^2)}\right) n \ . \qquad \square$$

Proof of Theorem 2. We use the notation introduced in the previous proof. Note that the estimate (7) remains valid. Also,

$$\begin{aligned} |E(G)| &= \sum_{v \in V(L(G))} d(v) - |E(L(G))| + |E(H(G))| \\ &\geq (k-1)n_L - |E(L(G))| + n_H - r \end{aligned} \tag{10}$$

Consider first the case $k=4$. In this case we have in (7)

$$|E(G)| \geq 2n - \frac{1}{2} n_L \ , \tag{11}$$

and from (10) and Corollary 4.3

$$\begin{aligned} |E(G)| &\geq 3n_L - \frac{2l+2}{2l+1} n_L + r + n_H - r \\ &= n + \left(2 - \frac{2l+2}{2l+1}\right) n_L = n + \frac{2l}{2l+1} n_L \ . \end{aligned} \tag{12}$$

Multiplying (11) by $4l/(2l+1)$ and adding with (12), we get

$$\left(1 + \frac{4l}{2l+1}\right) |E(G)| \geq \left(1 + \frac{8l}{2l+1}\right) n \ ,$$

or

$$|E(G)| \geq \frac{10l+1}{6l+1} n = \left(2 - \frac{2l+1}{6l+1}\right) n \ ,$$

as claimed.

Assume now that $k \geq 5$. Then (10) and Corollary 4.3 yield

$$\begin{aligned} |E(G)| &\geq (k-1)n_L - \frac{2l+1}{2l} n_L + r + n_H - r \\ &= n + \left(k - 2 - \frac{2l+1}{2l}\right) n_L = n + \frac{2kl-6l-1}{2l} n_L \ . \end{aligned} \tag{13}$$

Adding (7), multiplied by $(2kl-6l-1)/l$, and (13) we have

$$\left(1 + \frac{2kl-6l-1}{l}\right) |E(G)| \geq \left(1 + \frac{(2kl-6l-1)k}{2l}\right) n \ ,$$

or

$$|E(G)| \geq \left(\frac{k}{2} - \frac{(k-2)l}{2(2kl-5l-1)}\right) n \ . \qquad \square$$

5 Constructions of color-critical graphs

In this section we obtain upper bounds on the minimal number of edges in k-critical graphs. We discuss upper bounds mainly for the function $f_{k,s}(n)$ defined in Section 1. Some systematic ways of constructing k-critical graphs with restricted clique number are presented. The reader is referred to the survey [19] for additional information about constructive methods in the theory of color-critical graphs, relevant bibliographic references can be found therein.

5.1 The Hajos construction

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be k-critical vertex disjoint graphs. Let $e_1 = (u_1, v_1) \in E_1$, $e_2 = (u_2, v_2) \in E_2$. Denote by G the graph obtained from G_1 and G_2 by applying the following operations:

1. delete e_1, e_2;

2. identify u_1 and u_2;

3. join v_1 and v_2 by an edge.

Then G is k-critical.

Note that Hajos' construction does not increase the clique number, that is, if G is obtained from G_1 and G_2 by means of Hajos' construction, then $w(G) \leq \max\{w(G_1), w(G_2)\}$. This important observation leads us to the following consequences.

Claim 5.1 *For $k \geq 3$ and all m, n and s one has*

$$f_{k,s}(n+m) \leq f_{k,s}(n) + f_{k,s}(m+1) - 1 \ .$$

Claim 5.2 *For every pair k, s the limit (finite or infinite) $\lim_{n\to\infty} \frac{f_{k,s}(n)}{n}$ exists.*

Proof. Follows by a standard application of Fekete's lemma [9]. (The proof for the case $s = k - 1$ is presented in the monograph [11] of Jensen and Toft, pp. 100–101). Indeed, for a fixed pair k, s denote $h(n) = f_{k,s}(n+1)$, then from Claim 5.1

$$h(n+m) = f_{k,s}(n+m+1) \leq f_{k,s}(n+1) + f_{k,s}(m+1) = h(n) + h(m) \ .$$

Thus $h(n)$ is submodular and by Fekete's lemma the limit $\lim_{n\to\infty} h(n)/n = \lim_{n\to\infty} f_{k,s}(n)/n$ exists. $\square$

We denote $\varphi(k, s) = \lim_{n\to\infty} f_{k,s}(n)/n$.

Claim 5.3 *Let $2 \le s \le k$. If $G=(V,E)$ is a k-critical K^{s+1}-free graph, then $\varphi(k,s) \le (|E|-1)/(|V|-1)$.*

Proof. For every $i \ge 0$ apply Hajos' construction to G and G_i to get G_{i+1}, starting from $G_0 = G$. The ratio $|E(G_i)|/|V(G_i)|$ tends to $(|E|-1)/(|V|-1)$. $\square$

We call a graph $G=(V,E)$ *almost K^s-free* if there is an edge $e \in E$ such that every copy of K^s in G (if exists) contains e.

Claim 5.4 *If $G=(V,E)$ is a k-critical almost K^{s+1}-free graph, then $\varphi(k,s) \le (|E|-1)/(|V|-1)$.*

Proof. Let e be an edge of G participating in all copies of K^{s+1} in G. Denote by G_0 the result of Hajos' construction applied to two copies of G, where in each copy the edge e is chosen to be deleted. Also, for every $i \ge 0$ define G_{i+1} as the result of Hajos' construction applied to G and G_i, where in G the edge e is chosen again to be deleted. At each step $i \ge 1$ the number of edges increases by $|E|-1$, while the number of vertices increases by $|V|-1$, therefore $\lim_{i\to\infty} \frac{|E(G_i)|}{|V(G_i)|} = \frac{|E|-1}{|V|-1}$. $\square$

Corollary 5.1 $\varphi(k,k-1) \le \frac{k}{2} - \frac{1}{k-1}$.

Proof. Set $G = K^k$ and apply the above claim.

5.2 The Gallai construction

This construction invented by Gallai in [10] generalizes the construction of Hajos. Actually, we will use some particular case of it, suitable for our purposes.

Let $k \ge 4$, $q \ge 2$ and $k \ge 2q-1$. Let $G_1,\dots,G_q$ be k-critical vertex disjoint graphs such that for every $1 \le i \le q$ the graph G_i contains a clique $A_i = \{v_i^1,\dots,v_i^q\}$ of size q. A graph G is obtained from $G_1,\dots,G_q$ by performing the following operations.

1. for $1 \le i \le q$, delete all edges between v_i^q and v_i^j, where $1 \le j \le q-1$;
2. for $1 \le j \le q-1$, identify the vertices $v_1^j,\dots,v_q^j$;
3. for all $1 \le i < i' \le q$ join v_i^q and $v_{i'}^q$ by an edge.

Then G is k-critical.

The Hajos construction arises when $q=2$.

How can we use Gallai's construction for obtaining k-critical K^s-free graphs? Let $2 \le q < s < k$ and let G be a k-critical graph, in which there exists a q-clique A such that every copy of K^s in

G contains an edge from A. Our aim is to construct a k-critical graph G', in which there exists a $(q-1)$-clique A' such that every copy of K^s in G' contains an edge from A' (thus, is $q=2$ the graph G' will be K^s-free). To perform this task, we take q vertex disjoint copies of G and apply Gallai's construction with $G_i = G$ and $A_i = A$. (For $q=2$ we have already used this idea in Claim 5.3). This process can be performed iteratively to obtain a K^s-free graph.

To initiate the above described iterative construction we need graphs to start with. Assume that $k \geq 4$ and $s+1 \leq k \leq 2s-3$ and denote $q = k-s+1$. Define now a graph $G = G_{k,s}$ as follows. To build the low-vertex subgraph $L(G)$, take a clique $A = \{v_1, \ldots, v_{q+1}\}$ of size $q+1$. Now take $q+1$ vertex disjoint $(s-2)$-cliques $A_1, \ldots, A_{q+1}$ and for every $1 \leq i \leq q+1$ join v_i completely to $V(A_i)$. The high-vertex subgraph $H(G)$ is a clique K^q. In addition we join every vertex of $H(G)$ with every vertex of $\bigcup_{i=1}^{q+1} V(A_i)$. The graph G can be checked to be k-critical (for example, G can be obtained from $L(G)$ using the construction K of Sachs and Stiebitz - see, e.g., [19], and thus is k-critical) and every copy of K^s in G contains an edge from $E(H(G))$.

This construction combined with Claim 5.4 can be used to get upper bounds for $\varphi(k,s)$ for some values of k and s. In particular, it gives $\varphi(k,k-2) \leq \frac{k}{2} + \frac{k-6}{3k-5}$ for $k \geq 5$. In the next section we present a different construction, using graphs obtained by means of Gallai's construction as the starting point and enabling in particular to improve the above bound for $\varphi(k,k-2)$.

5.3 The Toft construction

This rather simple construction, which is a particular case of a much more general construction of Toft ([22], Th. 2) aims to build a k-critical graph from $(k-1)$-critical and k-critical graphs. It is particularly suitable for maintaining (almost) K^s-freeness.

Let $k \geq 3$ and let $G_1 = (V_1, E_1)$ be a k-critical graph and $G_2 = (V_2, E_2)$ be a $(k-1)$-critical graph, and let $e^* = (u_1, u_2)$ be an edge of G_1. Denote by $G = (V, E)$ the graph obtained from G_1 and G_2 by applying the following operations.

1. delete e^*;

2. partition V_2 into two non-empty parts U_1 and U_2;

3. join the vertex u_1 completely to U_1 and the vertex u_2 completely to U_2.

Claim 5.5 *G is k-critical.*

Proof. The proof can be found (in an implicit form) in the paper of Toft [22]. We present it here mainly for the sake of completeness of presentation.

Let us first prove that $\chi(G) \geq k$. Suppose to the contrary that G has a $(k-1)$-coloring $c : V \to \{1, \ldots, k-1\}$. Then clearly $c(u_1) = c(u_2)$ (otherwise G_1 would be $(k-1)$-colorable). Without loss of generality assume $c(u_1) = c(u_2) = k-1$. Every vertex of V_2 is connected to either u_1 or u_2, hence $c(v) \neq k-1$ for every $v \in V_2$. Thus the colors of the vertices of V_2 in c lie in $\{1, \ldots, k-2\}$, implying that G_2 is $(k-2)$-colorable - a contradiction.

To establish the claim we need to prove that G is k-critical. It suffices to show that $\chi(G-e) \leq k-1$ for every $e \in E(G)$.

Case 1: $e \in E(G_1)$.

The graph $G_1 - e$ is $(k-1)$-colorable. Let $c_1 : V_1 \to \{1, \ldots, k-1\}$ be a $(k-1)$-coloring of $G_1 - e$. Then obviously $c_1(u_1) \neq c_1(u_2)$. Without loss of generality assume $c_1(u_1) = k-1$ and $c_1(u_2) = k-2$. The graph G_2 is $(k-1)$-critical, then G_2 is connected and, in particular, there exists an edge $e_1 = (v_1, v_2)$ such that $v_1 \in U_1$ and $v_2 \in U_2$. Let $c_2 : V_2 \to \{1, \ldots, k-2\}$ be a $(k-2)$-coloring of $G_2 - e_1$ such that $c_2(v_1) = c_2(v_2) = k-2$ (such a coloring exists due to criticality of G_2). Define now a mapping $c : V \to \{1, \ldots, k-1\}$ as follows:

$$c(v) = \begin{cases} c_1(v), & v \in V_1, \\ c_2(v), & \text{either } v \in U_1 \text{ or } v \in U_2 \text{ and } c_2(v) \neq k-2, \\ k-1, & v \in U_2 \text{ and } c_2(v) = k-2 \ . \end{cases}$$

Then $c(v_1) = k-2, c(v_2) = k-1$, and c can be easily seen to be a $(k-1)$-coloring of the whole graph G.

Case 2: e connects a vertex of V_2 with u_1 or u_2.

Assume that $e = (v_1, u_1)$, where $v_1 \in U_1$ (the case $e = (v_1, u_2)$ with $v_1 \in U_2$ is treated quite similarly). Let $c_1 : V_1 \to \{1, \ldots, k-1\}$ be a $(k-1)$-coloring of $G_1 - e^*$ satisfying $c_1(u_1) = c_1(u_2) = k-1$. The graph G_2 is $(k-1)$-critical, hence there exists a $(k-2)$-coloring $c_2 : V_2 \setminus \{v_1\} \to \{1, \ldots, k-2\}$ of $G_2[V_2 - v_1]$. Define now $c : V \to \{1, \ldots, k-1\}$ by

$$c(v) = \begin{cases} c_1(v), & v \in V_1, \\ c_2(v), & v \in V_2 \setminus \{v_1\}, \\ k-1, & v = v_1 \ . \end{cases}$$

Then c is a $(k-1)$-coloring of G.

Case 3: $e = (v_1, v_2)$, where $v_1, v_2 \in V_2$.

Let $c_1 : V_1 \to \{1, \ldots, k-1\}$ be a $(k-1)$-coloring of $G_1 - e^*$ with $c_1(u_1) = c_1(u_2) = k-1$. There exists a $(k-2)$-coloring $c_2 : V_2 \to \{1, \ldots, k-2\}$ of $G_1 - e$. Define $c : V \to \{1, \ldots, k-1\}$ by

$$c(v) = \begin{cases} c_1(v), & v \in V_1, \\ c_2(v), & v \in V_2 \ , \end{cases}$$

then c is a $(k-1)$-coloring of G. $\square$

The following observation plays an important role in our arguments.

Claim 5.6 *Suppose that G_1 is almost K^s-free and e^* participates in all copies of K^s in G_1. Suppose further that G_2 is almost K^s-free and $G_2[U_1]$, $G_2[U_2]$ are K^{s-1}-free. Then G is almost K^s-free.*

Proof. Indeed, the deletion of e^* destroys all copies of K^s in G_1. Also, since $G_2[U_1]$ and $G_2[U_2]$ are K^{s-1}-free, the subgraphs $G[U_1 \cup \{u_1\}]$ and $G[U_2 \cup \{u_2\}]$ are K^s-free, implying that G is almost K^s-free. $\square$

We remark that a partition $V_2 = U_1 \cup U_2$, for which $G_2[U_1]$ and $G_2[U_2]$ are K^{s-1}-free, exists in particular if $\lceil (k-1)/2 \rceil < s-1$, in this case this partition can be built by partitioning the color classes of a $(k-1)$-coloring of G_2 into two subsets of colors, one consisted of $\lceil (k-1)/2 \rceil$ colors, and the other one consisted of the remaining $\lfloor (k-1)/2 \rfloor$ colors.

Claim 5.7 *Let $k \geq 3$, $s \geq 3$. If $G = (V, E)$ is a $(k-1)$-critical almost K^s-free graph such that there exists a partition $V = U_1 \cup U_2$ for which $G[U_1]$ and $G[U_2]$ are K^{s-1}-free, then*

$$\varphi(k, s-1) \leq \frac{|E| + |V| - 1}{|V|} \ .$$

Proof. Take H_0 to be any k-critical almost K^s-free graph and for every $i \geq 0$ apply the above described construction with $G_1 = H_i$ and $G_2 = G$ to get H_{i+1}, each time preserving almost K^s-freeness, increasing the number of edges by $|E| + |V| - 1$ and the number of vertices by $|V|$. Finally, use Claim 5.4. $\square$

To apply Claim 5.7, we can use the construction of Gallai to get a $(k-1)$-critical almost K^s-free graph, as described in the previous subsection. This approach yields in particular the following results.

Theorem 3 *1. $\varphi(k, k-2) \leq \frac{k}{2} - \frac{1}{k-1}$ for $k \geq 4$;*

2. $\varphi(k, k-3) \leq \frac{k}{2} + \frac{3k-14}{6k-12}$ for $k \geq 6$;

3. $\varphi(k, k-4) \leq \frac{k}{2} + \frac{31k-164}{24k-86}$ for $k \geq 8$.

Proof. 1) For $k \geq 5$ take $G = K^{k-1}$ in Claim 5.7. For $k = 4$ we should be a bit more careful. In this case it is easy to see that $G = K^3$ can be used to run the iterative procedure of the proof of Claim 5.7, this is due to the following argument: if G_1 is a four-critical almost K^3-free graph with e^* being an edge participating in all copies of K^3 in G_1 and $G_2 = K^3$, then for any non-trivial partition $V(G_2) = U_1 \cup U_2$ the application of the construction of this subsection produces again a four-critical almost K^3-free graph.

2) Let $G_1 = K^{k-1}$ and let G_2 be the result of Hajos' construction applied to K^{k-2} and K^{k-2}. Choose any edge $e^* = (u_1, u_2) \in E(G_1)$ and apply the construction of this subsection to G_1 and G_2,

using a partition $V(G_2) = U_1 \cup U_2$ for which $G_2[U_1]$ and $G_2[U_2]$ do not span a copy of K^{k-3} (this can be done because G_2 is $(k-2)$-colorable). This way we get a $(k-1)$-critical graph G, which can be easily seen to be almost K^{k-2}-free (any edge of G_1 with both endpoints different from u_1, u_2 participates in all copies of K^{k-2} in G). The number of vertices of G is $3k-6$, its number of edges is $3k(k-3)/2$, and the application of Claim 5.7 implies the desired result.

3) Start with the graph $G_{k-1,k-3}$ described in the previous subsection, it has $4k-13$ vertices and $2k^2-6k-11$ edges. Apply the Gallai construction to three disjoint copies of G as suggested in the same subsection, this gives a $k-1$-critical almost K^{k-3}-free graph G' with $12k-43$ vertices and $6k^2-18k-38$ edges. Now apply Claim 5.7. $\square$

In particular case $k=10$, $s=8$ Schonheim conjectured (see, e.g., [11], p. 100) that every ten-critical K^9-free graph G on n vertices has more than $5n-11$ edges. According to our result $\varphi(10,8) \leq 44/9$, thus disproving the conjecture of Schonheim.

6 Applications

In this section we present some applications of the lower bounds on the number of edges in color-critical graphs, obtained in Theorems 1 and 2. It is important to stress that we do not attempt here to get the most comprehensive results, instead, our aim is more instructive. We are pretty sure that many further applications of the results of this paper are still to be found.

All three problems we are going to treat in this section have a probabilistic flavor, thus either a question itself is formulated in probabilistic terms or we tract it using random spaces of discrete objects. Actually, in all three problems we will make use of the concept of a *random graph*. The reader is referred to the book [2] of Bollobás, devoted entirely to this topic, and also to the monograph [1] of Alon and Spencer, covering the probabilistic method and its applications to combinatorics.

Throughout the section we will omit in many places floor and ceiling signs for the sake of simplicity of presentation. Symbols $c_1, c_2, \ldots$ are used as generic names of constants, taking possibly different values in different places.

6.1 A lemma about random graphs

The first two applications we will present rely on the lemma about random graphs proven in this subsection. In order to formulate the lemma we introduce some notation.

As usually, $G(n,p)$ denotes a random graph on n labeled vertices, in which all edges are chosen independently and with probability p, where p may depend on n. For a graph $H=(V,E)$ with at least three vertices the *density* $\rho(H)$ is $\rho(H) = \min_{H'=(V,E),H'\subseteq H}(|E|-1)/(|V|-2)$. Given a family

$\mathcal{H}_0 = \{H_1, \ldots, H_t\}$ of fixed graphs, each having at least three vertices, let $\rho(\mathcal{H}_0) = \min_{1\le i\le t} \rho(H_i)$ denote the *density* of $\mathcal{H}_0$.

Lemma 6.1 *Let $\mathcal{H}_0 = \{H_1, \ldots, H_t\}$ be a family of fixed graphs, satisfying $|V(H_i)| \ge 3$ for every $1 \le i \le t$. Let $\rho(\mathcal{H}_0)$ denote the density of $\mathcal{H}_0$ and assume $\rho(\mathcal{H}_0) > 1$. Then there exist positive constants $C = C(\mathcal{H}_0)$ and $c = c(\mathcal{H}_0)$ such that if probability p satisfies $\ln^2 n/n \le p \le cn^{-1/\rho(\mathcal{H}_0)}$, then a random graph $G = G(n,p)$* **whp**[1] *has the following property: if* $\mathbf{H}$ *is any maximal by inclusion family of pairwise edge disjoint subgraphs of G, each isomorphic to one of the graphs from $\mathcal{H}_0$, then after the deletion of all edges of all subgraphs from* $\mathbf{H}$*, the resulting subgraph G_0 on n vertices does not contain an independent set of size $\lceil C \ln n/p \rceil$.*

Note that after the deletion of the edges of subgraphs from $\mathbf{H}$, the graph G_0 is $\mathcal{H}_0$-free (that is, G_0 does not contain a copy of any graph from $\mathcal{H}_0$). Thus, as pointed in [14], where the lemma is proven implicitly for the case $t = 1$, this lemma yields the best known lower bounds for the off-diagonal Ramsey numbers $R(r,k)$ for every fixed $r \ge 4$.

Proof of the lemma. The lemma is proven using the approach based on large deviation inequalities.

For every $1 \le i \le t$ set $v_i = |V(H_i)|$, $f_i = |E(H_i)|$. Set also

$$\begin{aligned} f_{min} &= \min\{f_i : 1 \le i \le t\}\,, \\ f_{max} &= \max\{f_i : 1 \le i \le t\}\,. \end{aligned}$$

Let $n_0 = \lceil C \ln n/p \rceil$. For every subset $V_0 \subset V$ of size $|V_0| = n_0$ let X_{V_0} be the random variable, counting the number of edges of G, spanned by V_0. Also, denote by Y_{V_0} the number of subgraphs of G, each isomorphic to one of the graphs from $\mathcal{H}_0$ and having at least one edge inside V_0, and by Z_{V_0} the maximal number of pairwise edge disjoint subgraphs of G, each isomorphic to one of the graphs from $\mathcal{H}_0$ and having at least one edge inside V_0. Clearly, $Z_{V_0} \le Y_{V_0}$. Denote by A_{V_0} the event $X_{V_0} > f_{max} Z_{V_0}$.

Claim 6.1 *If A_{V_0} holds for every $V_0 \subset V$ of size $|V_0| = n_0$, then G has the property stated in the formulation of the lemma.*

Proof. Let $\mathbf{H}$ be a maximal under inclusion family of pairwise edge disjoint subgraphs of G, each isomorphic to one of the graphs from $\mathcal{H}_0$. Deleting all edges of all subgraphs from $\mathbf{H}$, we clearly obtain an $\mathcal{H}_0$-free graph G_0 on n vertices. For a subset $V_0 \subset V$ of size $|V_0| = n_0$, denote by $\mathbf{H}_{V_0}$ the subfamily of $\mathbf{H}$, consisting of all subgraphs from $\mathbf{H}$, having at least one edge inside V_0. From the definition of Z_{V_0} it follows that $|\mathbf{H}_{V_0}| \le Z_{V_0}$. While deleting the edges of the subgraphs from $\mathbf{H}$ we delete at

[1]An event $\mathcal{E}_n$ happens **whp** (with high probability) in $G(n,p)$ if the probability of $\mathcal{E}_n$ tends to 1 as n tends to infinity.

most $f_{max}|\mathbf{H}_{V_0}| \leq f_{max} Z_{V_0}$ edges from $E(G[V_0])$, hence the subgraph G_0 has at least one edge in each subset V_0 of size $|V_0| = n_0$. $\square$

Now our aim is to show that under appropriate choice of constants C and c one has $P[\bigwedge_{|V_0|=n_0} A_{V_0}] \to 1$ as $n \to \infty$. To this end, we show that the random variables X_{V_0} and Z_{V_0} are highly concentrated around their expectations and if the ratio $EX_{V_0}/f_{max}EZ_{V_0}$ is sufficiently large, then the probability $P[\overline{A_{V_0}}]$ is exponentially small, implying in turn that the probability of the existence of a set V_0, for which $\overline{A_{V_0}}$ holds, tends to zero.

The random variable X_{V_0} is binomially distributed with parameters $\binom{n_0}{2}$ and p, therefore $EX_{V_0} = \binom{n_0}{2}p$, and well known estimates on the tails of binomial distribution due to Chernoff (see, e.g., [1], Appendix A) assert that for every $0 < \alpha < 1$

$$P[X_{V_0} < (1-\alpha)\binom{n_0}{2}p] < e^{-\alpha^2\binom{n_0}{2}p/2} . \tag{14}$$

Now we turn to bounding the upper tail of Z_{V_0}. The random variable Z_{V_0} is tightly connected with another random variable Y_{V_0}.

Claim 6.2 *$P[Z_{V_0} \geq j] \leq \frac{(EY_{V_0})^j}{j!} \leq \left(\frac{3EY_{V_0}}{j}\right)^j$ for every natural j.*

Proof. This is a particular case of the general result of Erdős and Tetali [8] (see also Lemma 4.1 of Ch. 8 of [1]). $\square$

Let us write $Y_{V_0} = Y_{V_0,1} + \ldots + Y_{V_0,t}$, where $Y_{V_0,i}$ is the number of copies of H_i in G, having at least one edge in $E(G[V_0])$. Representing $Y_{V_0,i}$ as a sum of indicator random variables, we get

$$\begin{aligned} EY_{V_0,i} &\leq \binom{n_0}{2}\binom{n-2}{v_i-2} v_i! p^{f_i} \\ &\leq c_{i,1}\binom{n_0}{2} p \left(n^{\frac{v_i-2}{f_i-1}} p\right)^{f_i-1} \\ &\leq c_{i,1}\binom{n_o}{2} p \left(n^{1/\rho(\mathcal{H}_0)} p\right)^{f_i-1} , \end{aligned}$$

where $c_{i,1} > 0$ is a constant depending only on H_i. Hence, assuming that $c < 1$ we have $EY_{V_0} \leq c_1\binom{n_0}{2}p\left(n^{1/\rho(\mathcal{H}_0)}p\right)^{f_{min}-1}$ for some constant $c_1 = c_1(\mathcal{H}_0) > 0$ and thus

$$\frac{EY_{V_0}}{EX_{V_0}} \leq c_1 \left(n^{1/\rho(\mathcal{H}_0)} p\right)^{f_{min}-1} . \tag{15}$$

The above ratio can be made smaller than any fixed constant by choosing c to be sufficiently small.

Now, by (14) with $\alpha = 1/2$ and Claim 6.2

$$
\begin{aligned}
P[\overline{A_{V_0}}] &= P[X_{V_0} \le f_{max} Z_{V_0}] \le P[X_{V_0} \le \frac{EX_{V_0}}{2}] + P[f_{max} Z_{V_0} \ge \frac{EX_{V_0}}{2}] \\
&\le e^{-\binom{n_0}{2}p/8} + \left(\frac{3EY_{V_0}}{\frac{EX_{V_0}}{2f_{max}}} \right)^{\frac{EX_{V_0}}{2f_{max}}} \\
&\le \exp\left\{ -\binom{n_0}{2} p/8 \right\} + \exp\left\{ \frac{\ln(6 f_{max} \frac{EY_{V_0}}{EX_{V_0}})}{2 f_{max}} \binom{n_0}{2} p \right\} \\
&= e^{-\frac{n_0^2 p}{16}(1+o(1))}
\end{aligned}
$$

for c sufficiently small (see (15)). Then using the inequality $\binom{n}{n_0} \le \left(\frac{en}{n_0}\right)^{n_0}$ we have

$$
\begin{aligned}
P[\exists V_0 : \overline{A_{V_0}}] \le \binom{n}{n_0} P[\overline{A_{V_0}}] &\le \left(\frac{en}{n_0}\right)^{n_0} e^{-\frac{n_0^2 p}{16}(1+o(1))} \\
&= \left(\frac{en}{n_0} e^{-\frac{n_0 p}{16}(1+o(1))} \right)^{n_0} = \left(\frac{c_1 e n^{1-1/\rho(\mathcal{H}_0)}}{C \ln n} e^{-\frac{C \ln n}{16}(1+o(1))} \right)^{n_0} .
\end{aligned}
$$

Choosing a constant C to be sufficiently large we ensure that $P[\bigwedge_{|V_0|=n_0} A_{V_0}] \to 1$ as $n \to \infty$. $\Box$

6.2 Locally three-colorable graphs with high chromatic number

Following Erdős [6], define a function $f(m,k,n)$ as the largest possible chromatic number of a graph G on n vertices in which every induced subgraph on m vertices is k-colorable. The most studied case is when $k = 2$, in this case $f(m,2,n)$ is the maximal chromatic number of an n-vertex graph G with $oddgirth(G) > m$. Kierstead, Szemerédi and Trotter have proven in [12] that for every fixed l there exists a constant c_l so that if G is a graph on n vertices with $oddgirth(G) \ge c_l n^{1/l}$, then $\chi(G) \le l+1$, thus showing $f(c_l n^{1/l}, 2, n) \le l+1$. They also noticed that this bound is tight up to a constant factor, that is, for every fixed l there exists a graph G on n vertices with $oddgirth(G) \ge n^{1/l}$, but $\chi(G) = l+2$, they referred in their paper to the constructions of Gallai, Lovász and Schrijver. Recently, Youngs [23] gave a construction of a graph G on n vertices with $oddgirth(G) \ge 2\sqrt{n}$ and $\chi(G) = 4$.

However, for every fixed $k \ge 3$ the problem of estimating $f(m,k,n)$ remains more or less completely open. Here, aiming to illustrate the applicability of our results, we treat the particular case $m = \sqrt{n}$, $k = 3$. We obtain a lower bound for $f(\sqrt{n}, 3, n)$ by proving the following

Theorem 4 *Let $\epsilon > 0$ be a constant. For every sufficiently large n there exists a graph G_0 on n vertices, in which every induced subgraph on at most $\sqrt{n}$ vertices is three-colorable, but $\chi(G_0) \ge n^{6/31-\epsilon}$.*

Corollary 2 *For every fixed $\epsilon > 0$ and sufficiently large n one has $f(\sqrt{n}, 3, n) \ge n^{6/31-\epsilon}$.*

Proof. Fix a constant $0 < \epsilon' < \epsilon$ and consider a random graph $G(n,p)$, where $p = n^{-25/31-\epsilon'}$. Our program is as follows: we pick any maximal by inclusion family $\mathbf{H}$ of pairwise edge disjoint cycles of length three or five and delete the edges of subgraphs composing it, thus obtaining a graph G_0 with $oddgirth(G_0) \geq 7$. Every $t \leq \sqrt{n}$ vertices of G (and therefore of G_0) span almost surely a small number of edges, and thus by applying Theorem 2 we will prove that every subgraph of G_0 on $\sqrt{n}$ vertices is three-colorable. Also, by Lemma 6.1 the deletion of the edges of $\mathbf{H}$ **whp** does not increase drastically the independence number of G, implying that $\chi(G_0)$ is still high.

Let us first prove that **whp** for every $4 \leq t \leq \sqrt{n}$ every subset $V_0 \subset V(G)$ of size $|V_0| = t$ spans less than $31t/19$ edges. Indeed, the probability that this is not so can be bounded from above by

$$\sum_{t=4}^{\sqrt{n}} \binom{n}{t} \binom{\binom{t}{2}}{\frac{31t}{19}} p^{\frac{31t}{19}} \leq \sum_{t=4}^{\sqrt{n}} \left(\frac{en}{t}\right)^t \left(\frac{e\frac{t^2}{2}}{\frac{31t}{19}}\right)^{\frac{31t}{19}} p^{\frac{31t}{19}} \leq \sum_{t=4}^{\sqrt{n}} \left(c_1 n t^{\frac{12}{19}} p^{\frac{31}{19}}\right)^t$$
$$= \sum_{t=4}^{\sqrt{n}} \left(c_1 n^{1-\frac{31}{19}\left(\frac{25}{31}+\epsilon'\right)} t^{\frac{12}{19}}\right)^t \leq \sum_{t=4}^{\sqrt{n}} \left(c_1 n^{-\frac{31}{19}\epsilon'}\right)^t = o(1)\,.$$

Now we define $\mathcal{H}_0 = \{C^3, C^5\}$, where C^i is a cycle on i vertices, clearly $\rho(\mathcal{H}_0) = 4/3$. According to Lemma 6.1 with $p = n^{-25/31-\epsilon'}$ and $\mathcal{H}_0$ as defined above, the graph G_0, obtained from G by deleting any maximal by inclusion family $\mathbf{H}$ of pairwise edge disjoint copies of C^3 and C^5, is $\mathcal{H}_0$-free and **whp** contains no independent set of size $\lceil C \ln n/p \rceil = \lceil C n^{25/31+\epsilon'} \ln n \rceil < n^{25/31+\epsilon}$, implying trivially that $\chi(G_0) \geq n^{6/31-\epsilon}$. On the other hand, G_0 does not contain cycles of length three and five and **whp** for every t satisfying $4 \leq t \leq \sqrt{n}$ every t vertices of it still span less than $31t/19$ edges. Fix one such graph G_0. Then by Theorem 2 G_0 does not contain a four-critical graph on at most $\sqrt{n}$ vertices, hence every subgraph of G_0 on $\sqrt{n}$ vertices is three-colorable, establishing the theorem. $\square$

It is worth noting that applying the same method combined with the trivial bound on the minimal number of edges in a four-critical graph ($|E(G)| \geq 3|V(G)|/2$) yields the bound $f(\sqrt{n}, 3, n) \geq n^{1/6-\epsilon}$ for every fixed $\epsilon > 0$.

6.3 Local and global independence numbers

Our second application addresses the relation between local and global independence numbers.

Let m, n, r, s be integers satisfying $r \leq m \leq n$ and $s \leq n$. Using the arrow notation introduced by Rado, we write $(m, r) \to (n, s)$ if every graph G on n vertices, in which every m vertices span an independent set of size r, contains an independent set of size s. The question is, for given m, r and s, to determine the minimal possible value of n for which $(m, r) \to (n, s)$ holds. One should note that the above defined problem covers many well known problems in Ramsey theory. For example, it is easy to see that if $r = 2$ then there holds $(m, 2) \to (R(s, m), s)$, where $R(s, m)$ is the corresponding

Ramsey number, thus for this case the determination of the minimal n, for which $(m,2) \to (n,s)$ holds, reduces actually to the determination of the off-diagonal Ramsey numbers — a well known and seemingly difficult task.

The case of fixed r and s and growing m has been considered by Erdős and Rogers [7], Bollobás and Hind [3], and Krivelevich [13],[14]. Here we concentrate on the case where both m and r are fixed and s grows. This case has been studied by Linial and Rabinovich in [15]. In order to obtain negative type results (that is, results of the form $(m,r) \not\to (n,s)$) they used the Lovász local lemma together with the Turán theorem.

Here we present a somewhat different approach, combining our Lemma 6.1 and Theorem 1. To illustrate it, we chose the case $m=20$, $r=5$. For this case Linial and Rabinovich showed $(20,5) \not\to \left(C\left(\frac{s}{\log s}\right)^{\frac{39}{18}}, s\right)$ for some absolute constant $C>0$. Our improvement is based on the following

Claim 6.3 *If $H=(V,E)$ is a graph on 20 vertices that does not contain an independent set of size five, then H contains a subgraph H_0 for which $\rho(H_0) \geq 33/14$.*

Proof. If H contains a copy of K^4 then we are done, therefore assume that H is K^4-free. Since H does not contain an independent set of size five, it satisfies $\chi(H) \geq 5$. Let $U \subseteq V$, $|U| = m_0$, be such that $H_0 = H[U]$ is five-critical, then by Theorem 1 $|E(H_0)| \geq \left\lceil \frac{17}{8} m_0 \right\rceil$, implying

$$\rho(H_0) \geq \frac{\left\lceil \frac{17}{8} m_0 \right\rceil - 1}{m_0 - 2} \ .$$

A routine checking of all possible cases shows that the above expression is minimal when $m_0 = 16$, in this case $\rho(H_0) \geq 33/14$. $\square$

Note that a direct application of Turán's theorem, as was done in [15], gives only $\rho(H) \geq 39/18$.

Now let $\mathcal{H}_0$ be a family of all graphs H_0 on at most 20 vertices satisfying $\rho(H_0) \geq 33/14$, then clearly $\rho(\mathcal{H}_0) \geq 33/14$. Applying Lemma 6.1 with $\mathcal{H}_0$ as defined above and $p = cn^{-14/33}$ for sufficiently small constant $c>0$, we show that there exists a graph G_0 on n vertices in which every subgraph on at most 20 vertices has density less than 33/14 (and thus by Claim 6.3 every subgraph on 20 vertices contains an independent set of size five), but G_0 does not contain an independent set of size $\lceil Cn^{14/33} \log n \rceil$ for some absolute constant $C>0$. Therefore, we have proven

Theorem 5 *There exists an absolute constant $C>0$ such that*

$$(20,5) \not\to \left(C\left(\frac{s}{\log s}\right)^{\frac{33}{14}}, s\right) \ .$$

6.4 Sharp concentration of the chromatic number of $G(n,p)$

Our last application has a purely probabilistic formulation. Fix a function $p = p(n)$ for probability p and consider a random graph $G(n,p)$. Let X be the random variable equal to the chromatic number of a graph G, drawn from $G(n,p)$. Our aim is to establish results about the *concentration* of X, that is, to show that there exist functions $u(n)$ and $\delta(n)$ such that

$$P[u(n) \leq \chi(G) \leq u(n) + \delta(n)] = 1 - o(1) \ ,$$

as n tends to infinity, of course, our goal is to find $\delta(n)$ as small as possible. If the above inequality holds for some $\delta(n)$, we say that $\chi(G)$ is *concentrated in width* $\delta(n)$ for this value of probability p.

A breakthrough in this problem has been achieved by Shamir and Spencer [20]. Their approach is based on using *martingales*. In particular, they succeeded to prove ([20], Theorem 2(ii)) that for every fixed $0.5 < \alpha < 1$ and $p = n^{-\alpha}$, $\chi(G)$ is concentrated in width k where k is the minimal integer greater than $(2\alpha+1)/(2\alpha-1)$, implying that for every fixed $0.5 < \alpha < 1$ $\chi(G)$ is concentrated in some fixed number of values. One should note that their approach yields actually a slightly better result, namely, for every fixed $0.5 < \alpha < 1$ and $p = n^{-\alpha}$, $\chi(G)$ is concentrated in width k, where k is the minimal integer satisfying $k > 2/(2\alpha - 1)$. (See, e.g., Theorem 3.3 of Chapter 7 of [1]). In particular, if $\alpha > 5/6$, then $\chi(G)$ is concentrated in four values ($k = 3$). A further step was done by Łuczak [16] who showed that if $\alpha > 5/6$, then $\chi(G)$ **whp** takes one of two consecutive values. The main idea of Łuczak's proof can be used to show that if $\alpha > 3/4$, then $\chi(G)$ is concentrated in three values, but unfortunately his method seems to break for every $\alpha > 3/4$.

Here we improve the results of Shamir and Spencer, combining their approach with results of Theorem 1. We give quantitative bounds on the width of the concentration interval for every fixed $0.5 < \alpha < 1$ and $p = n^{-\alpha}$.

Lemma 6.2 *Let α, c be fixed. Consider a random graph $G(n,p)$, where $p = n^{-\alpha}$. Then*

1. *if $\alpha > 9/11$, then* **whp** *every subgraph of G on $c\sqrt{n}$ vertices is three-colorable;*
2. *if $\alpha > 25/34$, then* **whp** *every subgraph of G on $c\sqrt{n}$ vertices is four-colorable;*
3. *for every $k \geq 5$, if $\alpha > \frac{1}{2} + \frac{2k-5}{2k^2-4k-4}$, then* **whp** *every subgraph of G on $c\sqrt{n}$ vertices is k-colorable.*

Proof. The difference between our argument and the argument of [20] is that we exploit the fact that a random graph $G(n,p)$ **whp** does not contain large cliques.

1) $\alpha > 9/11$.

The expectation of the number of copies of K^4 in G is $O(n^4p^6) = o(1)$, hence by Markov's inequality G **whp** does not contain K^4.

Suppose to the contrary that there exists a set $U \subset V(G)$ of $c\sqrt{n}$ vertices spanning a subgraph which is not three-colorable. Consider a four-critical subgraph G_0 of G with veetex set V_0, where $V_0 \subseteq U$. Denoting $|V_0| = t$ and assuming that G is K^4-free, we have from Theorem 1 that V_0 spans at least $11t/7$ edges. The probability of the existence of such a subset can be bounded from above by

$$\sum_{t=4}^{c\sqrt{n}} \binom{n}{t} \binom{\binom{t}{2}}{\frac{11t}{7}} p^{\frac{11t}{7}} \le \sum_{t=4}^{c\sqrt{n}} \left(\frac{en}{t}\right)^t \left(\frac{e\frac{t^2}{2}}{\frac{11t}{7}}\right)^{\frac{11t}{7}} p^{\frac{11t}{7}}$$
$$\sum_{t=4}^{c\sqrt{n}} \left(c_1 n^{1-\frac{11\alpha}{7}} t^{\frac{4}{7}}\right)^t \le \sum_{t=4}^{c\sqrt{n}} \left(c_2 n^{1-\frac{11\alpha}{7}} n^{\frac{2}{7}}\right)^t = o(1) \ ,$$

as claimed;

2) $\alpha > 25/34$.

Note that again **whp** $G(n,p)$ is K^4-free. Applying the same argument as above and using the estimate $f_{5,3}(t) \ge 17t/8$ from Theorem 1, we get that the probability of the existence of a subgraph on $c\sqrt{n}$ vertices which is not four-colorable is at most

$$\sum_{t=5}^{c\sqrt{n}} \binom{n}{t} \binom{\binom{t}{2}}{\frac{17t}{8}} p^{\frac{17t}{8}} \le \sum_{t=5}^{c\sqrt{n}} \left(c_1 n^{1-\frac{17\alpha}{8}} t^{\frac{9}{8}}\right)^t$$
$$\le \sum_{t=6}^{c\sqrt{n}} \left(c_2 n^{1-\frac{17\alpha}{8}} n^{\frac{9}{16}}\right)^t = o(1) \ ;$$

3) $\alpha > \frac{1}{2} + \frac{2k-5}{2k^2-4k-4}$.

In this case $G(n,p)$ **whp** is K^5-free, therefore using the estimate on $f_{k+1,4}(t)$ from Theorem 1, we obtain that the probability that there exists a subset of $c\sqrt{n}$ vertices violating the lemma assertion is at most

$$\sum_{t=k+1}^{c\sqrt{n}} \binom{n}{t} \binom{\binom{t}{2}}{\left(\frac{k+1}{2} - \frac{k-1}{4k-10}\right) t} p^{\left(\frac{k+1}{2} - \frac{k-1}{4k-10}\right)t}$$
$$\le \sum_{t=k+1}^{c\sqrt{n}} \left(c_1 n^{1-\left(\frac{k+1}{2}-\frac{k-1}{4k-10}\right)\alpha} t^{\frac{k+1}{2}-\frac{k-1}{4k-10}-1}\right)^t$$
$$\le \sum_{t=k+1}^{c\sqrt{n}} \left(c_2 n^{1-\left(\frac{k+1}{2}-\frac{k-1}{4k-10}\right)\alpha} n^{\frac{1}{2}\left(\frac{k+1}{2}-\frac{k-1}{4k-10}-\right)}\right)^t = o(1) \ . \qquad \square$$

Theorem 6 *Let $p = n^{-\alpha}$, where $\alpha > 1/2$ is fixed. Let $G = G(n,p)$. There exists a function $u = u(n,p)$ such that* **whp**

$$u \leq \chi(G) \leq \chi(G) + k\,,$$

where

1. *$k = 3$ for $\alpha > 9/11$;*
2. *$k = 4$ for $\alpha > 25/34$;*
3. *for $\alpha \leq 25/34$, k is the least integer satisfying $\alpha > \frac{1}{2} + \frac{2k-5}{2k^2-4k-4}$.*

Proof. We exploit a key idea of the proof of Łuczak [16], who attributes it to Frieze (see also [1], Ch. 7, Th. 3.3).

Let $\epsilon > 0$ be arbitrarily small and let $u = u(n,p,\epsilon)$ be the least integer for which

$$P[\chi(G) \leq u] > \epsilon\ .$$

Now define a random variable $Y = Y(G)$ to be the minimal number of vertices that should be deleted in order to make G u-colorable. It is easy to see that Y satisfies the vertex Lipschitz condition (that is, if G and G' differ at only one vertex, then $|Y(G) - Y(G')| \leq 1$), therefore we can apply the vertex exposure martingale (see, e.g., [17], or [1], Ch. 7, for information about martingales and their applications to random graphs). Denoting $\mu = EY$, we get for every $\lambda > 0$

$$\begin{aligned} P[Y \leq \mu - \lambda\sqrt{n}] &< e^{-\lambda^2/2}\,, & (16)\\ P[Y \geq \mu + \lambda\sqrt{n}] &< e^{-\lambda^2/2}\,. & (17) \end{aligned}$$

Let λ satisfy $e^{-\lambda^2/2} = \epsilon$, then these tail events have probability less than ϵ. On the other hand, from the definition of u we have $P[Y = 0] > \epsilon$. Hence (16) implies $\mu \leq \lambda\sqrt{n}$. Now, applying (17) we get

$$P[Y \leq 2\lambda\sqrt{n}] \leq P[Y \leq \mu + \lambda\sqrt{n}] < \epsilon\ .$$

Thus, with probability at least $1-\epsilon$ there is a u-coloring of all but at most $2\lambda\sqrt{n}$ vertices of G. By Lemma 6.2, with probability at least $1-\epsilon$ these $\lambda\sqrt{n}$ vertices can be colored by at most k new colors, giving a $u+k$-coloring of G. The minimality of u guarantees that with probability at least $1-\epsilon$ at least u colors are needed to color G. Summing the above, we obtain

$$P[u \leq \chi(G) \leq u + k] \geq 1 - 3\epsilon\ .$$

Taking ϵ to be arbitrarily small we get the desired result. $\square$

Remark. Very recently, N. Alon and the author proved that for every constant $\alpha > 0.5$ the chromatic number of $G(n, n^{-\alpha})$ **whp** takes one of two consecutive values, thus extending the result of Łuczak [16]. The proof uses ideas different from those presented above.

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