An improved bound on the minimal number of edges in color-critical graphs

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Abstract

It is proven that for $k \geq 4$ and $n > k$ every $k$-color-critical graph on $n$ vertices has at least
$\left(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)}\right)n$ edges, thus improving a result of Gallai from 1963.

A graph $G$ is $k$-color-critical (or simply $k$-critical) if $\chi(G) = k$ but $\chi(G') < k$ for every proper
subgraph $G'$ of $G$, where $\chi(G)$ denotes the chromatic number of $G$. (See, e.g., [2] for a detailed
account of graph coloring problems). Consider the following problem: given $k$ and $n$, what is the
minimal number of edges in a $k$-critical graph on $n$ vertices? It is easy to see that every vertex of a
$k$-critical graph $G$ has degree at least $k-1$, implying $|E(G)| \geq \frac{k-1}{2}|V(G)|$. Gallai [1] improved this
trivial bound to $|E(G)| \geq \left(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)}\right)|V(G)|$ for every $k$-critical graph $G$ (where $k \geq 4$), which is
not a clique $K_k$ on $k$ vertices. In this note we strengthen Gallai’s result by showing

Theorem 1 Suppose $k \geq 4$, and let $G = (V, E)$ be a $k$-critical graph on more than $k$ vertices. Then

$$|E(G)| \geq \left(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)}\right)|V(G)|.$$}

In the first non-trivial case $k = 4$ we get $|E(G)| \geq \frac{11}{15}|V(G)|$, compared to the estimate $|E(G)| \geq \frac{20}{15}|V(G)|$ of Gallai.

Let us introduce some definitions and notation (we follow the terminology of [4]). If $G = (V, E)$ is
a $k$-critical graph, then the low-vertex subgraph of $G$, denoted by $L(G)$, is the subgraph of $G$, induced
by all vertices of degree $k-1$. The high-vertex subgraph of $G$, which we denote by $H(G)$, is the
subgraph of $G$ induced by all vertices of degree at least $k$ in $G$. Brooks’ theorem implies that if $k \geq 4$
and $G \neq K_k$, then $H(G) \neq \emptyset$. A maximal by inclusion connected subgraph $B$ of a graph $G$ such that
every two edges of $B$ are contained in a cycle of $G$ is called a block of $G$. A connected graph all of
whose blocks are either complete graphs or odd cycles is called a Gallai tree, a Gallai forest is a graph
all of whose connected components are Gallai trees. A $k$-Gallai forest (tree) is a Gallai forest (tree),
in which all vertices have degree at most $k - 1$. 1
Our proof utilizes results of Gallai [1] and Stiebitz [5], describing the structure of color-critical graphs. Gallai proved the following fundamental result.

**Lemma 1** ([1], Satz E.1) *If G is a k-critical graph then its low-vertex subgraph L(G) is a k-Gallai forest (possibly empty).*

Using induction on the number of vertices, it follows from the above statement that

**Lemma 2** ([1], Lemma 4.5) *Let \( k \geq 4 \). Let \( G = (V, E) \neq K_k \) be a k-Gallai forest. Then*

\[
|E(G)| \leq \left( \frac{k-2}{2} + \frac{1}{k-1} \right) |V(G)| - 1.
\]  

(1)

The second ingredient of our proof is the following result of Stiebitz.

**Lemma 3** ([5]) *Let G be a k-critical graph. Then the number of connected components of its high-vertex subgraph H(G) does not exceed the number of connected components of its low-vertex subgraph L(G).*

**Proof of Theorem 1.** Let \( L(G) \) and \( H(G) \) be the low-vertex and the high-vertex subgraphs of \( G \), respectively. Denote \( n_L = |V(L(G))|, n_H = |V(H(G))|, n = |V(G)| = n_L + n_H. \) By Brooks' theorem \( n_H > 0. \)

Let \( r \) be the number of connected components of \( H(G) \), then trivially

\[
|E(H(G))| \geq n_H - r.
\]  

(2)

Also, by Lemma 3, the number of connected components of \( L(G) \) is at least \( r \). Now the crucial observation is that each connected component of \( L(G) \) is itself a k-Gallai tree, therefore the estimate (1) is valid for it too. We infer that

\[
|E(L(G))| \leq \left( \frac{k-2}{2} + \frac{1}{k-1} \right) n_L - r.
\]  

(3)

Indeed, if \( G_1 = (V_1, E_1), \ldots, G_{r'}, = (V_{r'}, E_{r'}) \) are the connected components of \( L(G') \), where \( r' \geq r \), then by Lemma 1

\[
|E_i| \leq \left( \frac{k-2}{2} + \frac{1}{k-1} \right) |V_i| - 1, \quad i = 1, \ldots, r'.
\]

Summing the above inequalities over \( 1 \leq i \leq r' \), we get (3).

Using (2) and (3), the number of edges of \( G \) can be bounded from below as follows:

\[
|E(G)| = \sum_{v \in V(L(G))} d(v) - |E(L(G))| + |E(H(G))|
\geq (k-1)n_L - \left( \frac{k-2}{2} + \frac{1}{k-1} \right) n_L + r + n_H - r
= n + \frac{k^2 - 3k}{2(k-1)} n_L.
\]  

(4)
On the other hand, it follows from the definition of $L(G)$ and $H(G)$ that

$$ |E(G)| = \frac{1}{2} \left( \sum_{v \in V(G)} d(v) \right) = \frac{1}{2} \left( \sum_{v \in V(L(G))} d(v) + \sum_{v \in V(H(G))} d(v) \right) \geq \frac{1}{2}((k-1)n_L + kn_H) = \frac{k}{2}n - \frac{1}{2}n_L .$$

(5)

Multiplying (5) by $(k^2 - 3k)/(k - 1)$ and summing with (4) we get

$$ \left( 1 + \frac{k^2 - 3k}{k - 1} \right) |E(G)| \geq \left( 1 + \frac{k}{2} \frac{k^2 - 3k}{k - 1} \right) n ,$$

or

$$ |E(G)| \geq \left( \frac{k - 1}{2} + \frac{k - 3}{2(k^2 - 2k - 1)} \right) n ,$$

as claimed. □

A more detailed treatment of the problem, containing lower and upper bounds on the minimal number of edges in a $k$-critical graph on $n$ vertices with additional restrictions imposed, and some applications of these bounds to Ramsey-type problems and problems on random graphs, will appear in a forthcoming paper [3].

References


