A lower bound for irredundant Ramsey numbers

Michael Krivelevich *
Department of Mathematics,
Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel-Aviv University, Tel-Aviv, Israel

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Abstract

Given a graph \( G = (V, E) \), a vertex subset \( U \subseteq V \) is called irredundant if every vertex \( v \in U \) either has no neighbours in \( U \) or there exists a vertex \( w \in V \setminus U \) such that \( v \) is the only neighbour of \( w \) in \( U \). The irredundant Ramsey number \( s(m, n) \) is the smallest \( N \) such that any red-blue edge colouring of \( K^N \) yields either an \( m \)-element irredundant subset in the blue graph or an \( n \)-element irredundant subset in the red graph. Using probabilistic methods we show that

\[
s(m, n) > c_m \left( \frac{n}{\log n} \right)^{\frac{m^2 - m - 1}{2(m-1)}}.
\]

1 Introduction

Let \( G = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \). A subset of vertices \( U \subseteq V \) is called irredundant if every vertex \( v \in U \) either has

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no neighbours in \( U \) or there exists a vertex \( w \in V \setminus U \) such that \( v \) is the only neighbour of \( w \) in \( U \) (in this case \( w \) is called a \textit{private neighbour} of \( v \)). For every pair of integers \( m, n \geq 2 \) the \textit{irredundant Ramsey number} \( s(m, n) \) is the smallest integer \( N \) such that in any red-blue colouring of the edges of a complete graph \( K^N \) on \( N \) vertices either the blue graph contains an \( m \)-element irredundant subset or the red graph contains an \( n \)-element irredundant subset. Define also the \textit{mixed Ramsey number} \( t(m, n) \) as the smallest \( N \) such that any red-blue colouring of the edges of \( K^N \) yields an \( m \)-element irredundant subset in the blue graph or an \( n \)-element independent subset in the red graph. Recall that the \textit{Ramsey number} \( r(m, n) \) is the smallest \( N \) such that any red-blue colouring of the edges of \( K^N \) contains either a blue copy of \( K^m \) or a red copy of \( K^n \). Since an independent set is clearly irredundant, the above definitions imply that

\[
s(m, n) \leq t(m, n) \leq r(m, n) ,
\]

for all admissible \( m, n \).

Irredundant Ramsey numbers were introduced in [2]. In [4] asymptotic lower bounds for the diagonal irredundant Ramsey numbers \( s(n, n) \) and off-diagonal mixed Ramsey numbers \( t(m, n) \), \( m < n \), were obtained. In particular, it was shown that

\[
t(m, n) > c_m \left( \frac{n}{\log n} \right)^{\frac{m^2-m-1}{2(m-1)}} ,
\]

this result was obtained by using the so-called \textit{probabilistic method} (see [1]
as a general reference). It was also shown in [4] that

$$t(3, n) \leq \frac{\sqrt{10}}{2} n^{3/2}.$$  

The exact values for some small irredudant Ramsey numbers are known and can be found, e.g., in [4].

The purpose of this note is to establish an asymptotic lower bound for the off-diagonal irredudant Ramsey numbers $s(m, n)$, where $m$ is fixed and $n$ tends to infinity. We prove the following result.

**Theorem 1** For every $m \geq 3$ there exists a positive constant $c_m$ such that

$$s(m, n) > c_m \left( \frac{n}{\log n} \right)^{\frac{m^2 - m - 1}{2(m-1)}}.$$  

This result matches (up to a constant factor) the bound (2) for the mixed Ramsey numbers $t(m, n)$ and thus implies (2) in view of (1).

Our proof is also based on the probabilistic method and applies *large deviation inequalities*. A similar approach has already been used in [7] for obtaining asymptotic lower bounds for various Ramsey-type numbers. We discuss this approach in Section 2. In Section 3 the proof of the main result is presented.

We end this section with some notation used in the sequel. We denote by $[N]$ the set $\{1, \ldots, N\}$. The complete graph on $[N]$ is denoted by $K^N$. For every two disjoint vertex subsets $S, T \subseteq V(G)$ let $E(S)$ be the edge set of the subgraph of $G$ spanned by $S$, $E(S, T)$ is the set of all edges between $S$
and $T$, $e(S) = |E(S)|$ and $e(S, T) = |E(S, T)|$. A red-blue colouring of the
elect of $K^N$ induces the red graph $< R >$ and the blue graph $< B >$. We
denote by $< U >_R$ ($< U >_B$, resp.) the induced subgraph of $< R >$ ($< B >$,
resp.) on $U$.

For a fixed graph $H$ we define

$$
\rho(H) = \max_{H' \subseteq H, |H'| \geq 2} \frac{e(H') - 1}{|H'| - 2}
$$

(in order to avoid trivialities throughout the paper we always assume that
$e(H) \geq 2$). For a finite family of fixed graphs $\mathcal{H} = \{H_1, \ldots, H_l\}$ the density
of the family $\rho(\mathcal{H})$ is

$$
\rho(\mathcal{H}) = \min \{\rho(H_i) : 1 \leq i \leq l\}.
$$

Given a family $\mathcal{H} = \{H_1, \ldots, H_l\}$, a graph $G$ is called $\mathcal{H}$-free if it does not
contain a copy of $H_i$ for every $1 \leq i \leq l$.

## 2 Large deviation inequalities

Roughly speaking, large deviation inequalities assert that under certain con-
tions a random variable $X$ is highly concentrated near its mean and its tail
probabilities are exponentially small.

The simplest example of a large deviation inequality is the bound on the
tail of a binomial distribution, essentially due to Chernoff ([3]). If $X$ is the
sum of $n$ mutually independent indicator random variables each of which
taking the value 1 with probability $p$ and the value 0 with probability $1 - p$,
then the expectation of $X$ equals $np$ and for every constant $0 < \epsilon < 1$ the following inequalities hold:

\[
Pr[X < (1 - \epsilon)np] < e^{-\epsilon^2 np / 2}, \quad (3)
\]
\[
Pr[X > (1 + \epsilon)np] < e^{-\epsilon^2 (1-\epsilon)np / 2}. \quad (4)
\]

When $X$ is the sum of many ‘rarely dependent’ indicator random variables, it is also possible in certain cases to obtain exponential bounds on the tails of $X$. Let us describe a general scheme first presented in [6].

Suppose $Q$ is a finite universal set (in our instances $Q$ is the edge set of a complete graph on $N$ vertices). Let \( \{ J_i : i \in Q \} \) be a set of independent indicator random variables, \( Pr[J_i = 1] = p_i \) for every \( i \in Q \) (\( J_i = 1 \) if the corresponding edge belongs to \( E(G) \), where $G$ is a random graph on $N$ vertices in which each edge is chosen independently with probability $p$). Let \( \{ Q(\alpha) \}_{\alpha \in I} \) be a family of subsets of $Q$, where $I$ is a finite index set. Define \( X_\alpha = \prod_{i \in Q(\alpha)} J_i \) (then \( X_\alpha = 1 \) if and only if all the edges of $Q(\alpha)$ belong to $E(G)$). Now define

\[
X = \sum_{\alpha \in I} X_\alpha,
\]

(in our instances $X$ counts the number of subgraphs of $G$ having some specified properties).

We shall make use of the bound on the upper tail of another random variable \( X_0 \) which is tightly connected to $X$ and is defined as

\[
X_0 = \max \{ r : \exists \text{ distinct } \alpha_1, \ldots, \alpha_r \in I \text{ with } X_{\alpha_i} = 1 \}
\]
and \( Q(\alpha_i) \cap Q(\alpha_j) = \emptyset, \ i \neq j \) \}

Clearly, \( X_0 \leq X \). Let \( \mu = E X \) be the expectation of \( X \), then the following holds (see [5]):

**Claim 1**

\[
Pr[X_0 \geq k] \leq \frac{\mu^k}{k!}
\]

for every natural \( k \).

For the sake of completeness we repeat the short proof.

**Proof.**

\[
Pr[X_0 \geq k] \leq \sum_{i=1}^{k} Pr[(X_{\alpha_i} = 1) \land \ldots \land (X_{\alpha_k} = 1)]
\]

\[
= \frac{1}{k!} \sum_{i=1}^{k} Pr[(X_{\alpha_i} = 1) \land \ldots \land (X_{\alpha_k} = 1)]
\]

\[
= \frac{1}{k!} \sum_{i=1}^{k} Pr[X_{\alpha_1} = 1] \ldots Pr[X_{\alpha_k} = 1]
\]

\[
\leq \frac{1}{k!} \sum_{i=1}^{k} Pr[X_{\alpha_1} = 1] \ldots Pr[X_{\alpha_k} = 1] = \frac{\mu^k}{k!},
\]

where \( \sum^1 \) is over sets of \( k \) mutually independent events \( X_{\alpha_i} = 1 \), while \( \sum^2 \) is over ordered \( k \)-tuples of mutually independent events and \( \sum^3 \) is over all ordered \( k \)-tuples of events. \( \Box \)

In particular, we deduce from the above Claim that

\[
Pr[X_0 \geq 5\mu] < \left( \frac{e}{5} \right)^{5\mu}.
\]

(5)

(It is worth noting that in certain cases one can also obtain exponential bounds on the lower tail of \( X_0 \), see, e.g., [7]. However, the above cited simple bound will suffice for our purposes here).
3 Asymptotic lower bounds for $s(m, n)$

Recall that we are treating the off-diagonal irredundant Ramsey numbers $s(m, n)$, that is, $m$ is fixed while $n$ tends to infinity.

The proof of the main result is a simple consequence of the following

**Lemma 1** Let $\mathcal{H} = \{H_1, \ldots, H_l\}$ be a family of fixed graphs with density $\rho(\mathcal{H}) > 0$. Then there exists a constant $c = c(\mathcal{H})$ such that for every sufficiently large integer $N$ there exists a graph $G_0$ on $N$ vertices having the following properties:

1. $G_0$ is $\mathcal{H}$-free;

2. $G_0$ has no independent set of size $n = \lceil cN^{1/\rho(\mathcal{H})}\ln N \rceil$;

3. for every two disjoint subsets of vertices $S, T \subseteq V(G_0)$ of size $|S| = |T| = n$ one has $e(S, T) > n$.

**Proof.** For every $1 \leq i \leq l$ let $H_i'$ be a subgraph of $H_i$ such that $\rho(H_i') = \rho(H_i)$. Denoting $\mathcal{H}' = \{H_1', \ldots, H_l'\}$, note that if $G_0$ is $\mathcal{H}'$-free then it is clearly $\mathcal{H}$-free, therefore we may assume that $\rho(H_i) = (e(H_i) - 1)/(|H_i| - 2)$, $1 \leq i \leq l$. For every $1 \leq i \leq l$ set $v_i = |H_i|$, $e_i = e(H_i)$. Set also

\[ e_{\min} = \min\{e_i : 1 \leq i \leq l\}, \]

\[ e_{\max} = \max\{e_i : 1 \leq i \leq l\}. \]
Consider a random graph $G(N, p)$ - a graph on $N$ labelled vertices in which all edges are chosen independently with probability $p$. We set with foresight $p = c_0 N^{-1/\rho(H)}$, where $0 < c_0 < 1$ is a sufficiently small constant.

For every two disjoint subsets $S, T \subseteq V(G)$ of size $|S| = |T| = n$ we define the following random variables. First, let $X_S = e(S), X_{S,T} = e(S, T)$. Also, denote by $Y_S$ the number of subgraphs, each isomorphic to one of the graphs from $\mathcal{H}$ and having at least one edge inside $S$, and by $Z_S$ the maximal number of pairwise edge disjoint subgraphs, each isomorphic to one of the graphs from $\mathcal{H}$ and having at least one edge inside $S$. Let $Y_{S,T}$ denote the number of subgraphs, each isomorphic to one of the graphs from $\mathcal{H}$ and having at least one edge in $E(S, T)$, and let $Z_{S,T}$ denote the maximal number of pairwise edge disjoint subgraphs, each isomorphic to one of the graphs from $\mathcal{H}$ and having at least one edge in $E(S, T)$. Clearly, $Y_S \geq Z_S$ and $Y_{S,T} \geq Z_{S,T}$. Denote by $A_S$ the event $X_S > e_{\max} Z_S$ and by $A_{S,T}$ the event $X_{S,T} > e_{\max} Z_{S,T} + n$.

**Claim 2** If $A_S$ holds for every $S \subseteq V$ of size $|S| = n$ and $A_{S,T}$ holds for every pair of disjoint subsets $S, T \subseteq V(G)$ of size $|S| = |T| = n$, then $G$ contains a subgraph $G_0$ on $N$ vertices, satisfying the requirements of the lemma.

**Proof.** Let $H$ be a maximal (under inclusion) family of pairwise edge disjoint subgraphs of $G$, each isomorphic to one of the graphs from $\mathcal{H}$. Deleting all edges of all graphs from $H$ we clearly obtain an $\mathcal{H}$-free graph $G_0$ on $N$
vertices. Denote by $H_S$, $|S| = n$, the subfamily of $H$, consisting of all subgraphs from $H$, having at least one edge in $E(S)$, and by $H_{S,T}$, $|S| = |T| = n$, the subfamily of $H$, consisting of all subgraphs from $H$, sharing at least one edge with $E(S, T)$. Obviously, $|H_S| \leq Z_S, |H_{S,T}| \leq Z_{S,T}$. While deleting the edges of subgraphs from $H$, we delete at most $e_{max}|H_S| \leq e_{max}Z_S$ edges from $E(S)$ and at most $e_{max}|H_{S,T}| \leq e_{max}Z_{S,T}$ edges from $E(S, T)$, hence the subgraph $G_0$ satisfies also the conditions 2) and 3) of the lemma. □

Now our aim is to show that under appropriate choice of the constants $c_0$ and $c$ the inequality $Pr[\bigwedge_{|S|=n} A_S \land \bigwedge_{|T|=n} A_{S,T}] > 0$ holds. To this end, we show that the random variables $X_S, Z_S, X_{S,T}, Z_{S,T}$ are highly concentrated around their means and hence if, say, $EX_S > 10e_{max}EZ_S$ and $EX_{S,T} > 10e_{max}EZ_{S,T}$, then both probabilities $Pr[\overline{A_S}]$ and $Pr[\overline{A_{S,T}}]$ are exponentially small. This will imply that the probability that either there exists some set $S$ for which $\overline{A_S}$ holds or there exists a pair $S, T$ for which $\overline{A_{S,T}}$ holds is less than 1.

The random variable $X_S$ is clearly binomially distributed with parameters $\left(\begin{array}{c} n \\ 2 \end{array}\right)$ and $p$, therefore from (3) we obtain for every $0 < \epsilon < 1$

$$Pr[X_S < (1 - \epsilon)\left(\begin{array}{c} n \\ 2 \end{array}\right)p] < e^{-\epsilon^2\left(\begin{array}{c} n \\ 2 \end{array}\right)p/2}.$$  \hspace{1cm} (6)

Similarly, $X_{S,T}$ is binomially distributed with parameters $n^2$ and $p$, and hence (3) implies that

$$Pr[X_{S,T} < (1 - \epsilon)n^2p] < e^{-\epsilon^2n^2p/2}.$$  \hspace{1cm} (7)
Now we bound the upper tail of $Z_S$ by using Claim 1. To this end, we estimate $EY_S$. Note that
\[
Y_S = Y_{S,1} + \cdots + Y_{S,i},
\]
where $Y_{S,i}$ is the number of copies of $H_i$, having at least one edge in $E(S)$. Representing $Y_{S,i}$ as a sum of indicator random variables we can write
\[
\left( \frac{n}{2} \right) \binom{N - n}{v_i - 2} p^{v_i} \leq EY_{S,i} \leq \left( \frac{n}{2} \right) \binom{N - 2}{v_i - 2} v_i p^{v_i},
\]
therefore (recalling that $n = o(N)$ and hence $\binom{N - n}{v_i - 2} = \Theta(N^{v_i - 2})$) we have
\[
c_i,1 \left( \frac{n}{2} \right) p \left( \frac{N^{v_i - 2}}{C_i} \right)^{v_i - 1} \leq EY_{S,i} \leq c_i,2 \left( \frac{n}{2} \right) p \left( \frac{N^{v_i - 2}}{C_i} \right)^{v_i - 1},
\]
where $c_i,1$ and $c_i,2$ are some positive constants depending only on $H_i$.

The definitions of $\rho(H)$ and $p$ imply that
\[
c_1 c_0^{e_{\max} - 1} \left( \frac{n}{2} \right) p \leq EY_S \leq c_2 c_0^{e_{\min} - 1} \left( \frac{n}{2} \right) p,
\]
where $c_1 = c_1(H)$ and $c_2 = c_2(H)$ are positive constants.

Substituting in (5) $EY_S$ and $Z_S$ instead of $\mu, X_0$, respectively, we conclude that
\[
Pr[Z_S \geq 5EY_S] \leq e^{-5(n^{5 - 1})EY_S}.
\]

Turning to estimating the upper tail of $Z_{S,T}$ we act in a quite similar manner. Taking $c_1$ sufficiently small and $c_2 > 1$ sufficiently large we can write
\[
c_1 c_0^{e_{\max} - 1} n^2 p \leq EY_{S,T} \leq c_2 c_0^{e_{\min} - 1} n^2 p,
\]
Also, (5) implies that

$$Pr[Z_{S,T} \geq 5EY_{S,T}] \leq e^{-5(\ln 5-1)EY_{S,T}}. \quad (9)$$

Comparing $EX_S$ and $EY_S$, $EX_{S,T}$ and $EY_{S,T}$ we observe

$$\frac{1}{c_2c_0^{e_{\min}-1}} \leq \frac{EX_S}{EY_S}, \quad \frac{EX_{S,T}}{EY_{S,T}} \leq \frac{1}{c_1c_0^{e_{\max}-1}}.$$  

Let us choose $c_0$ so that the expression $c_2c_0^{e_{\min}-1}$ will be equal to, say, $1/10e_{\max}$. Then

$$10e_{\max} \leq \frac{EX_S}{EY_S}, \quad \frac{EX_{S,T}}{EY_{S,T}} \leq \frac{c_2}{c_1}c_0^{-e_{\max}+e_{\min}}10e_{\max}. \quad (10)$$

Now, by (6) with $\epsilon = 1/2$, (8) and (10)

$$Pr[\overline{A}_S] = Pr[X_S \leq e_{\max}Z_S] \leq Pr[X_S \leq \frac{EX_S}{2}] + Pr[e_{\max}Z_S \geq \frac{EX_S}{2}]$$

$$\leq Pr[X_S \leq \frac{EX_S}{2}] + Pr[Z_S \geq 5EY_S]$$

$$\leq e^{-\left(\frac{n}{2}\right)p/8} + e^{-\frac{c_1c_0^{-e_{\min}+e_{\max}}}{16c_2e_{\max}}[5\ln 5-5](\frac{n}{2})p} \leq 2e^{-c_3n^2p}$$

for some constant $c_3 > 0$.

Also, (6) with $\epsilon = 1/3$, (9) and (10) imply for sufficiently large $N$ (noting that $EX_{S,T}/n \to \infty$)

$$Pr[\overline{A}_{S,T}] = Pr[X_{S,T} \leq e_{\max}Z_{S,T} + n]$$

$$\leq Pr[X_{S,T} \leq \frac{EX_{S,T}}{2} + n] + Pr[e_{\max}Z_{S,T} \geq \frac{EX_{S,T}}{2}]$$

$$\leq Pr[X_{S,T} \leq \frac{2EX_{S,T}}{3}] + Pr[Z_{S,T} \geq 5EY_{S,T}]$$

$$\leq e^{-n^2p/18} + e^{-\frac{c_1c_0^{-e_{\min}+e_{\max}}}{16c_2e_{\max}}[5\ln 5-5]n^2p} \leq 2e^{-c_3n^2p}$$
(taking $c_3$ small enough).

Therefore

$$\Pr[\exists S : \overline{A_S}] \leq \left(\frac{N}{n}\right)^2 2e^{-c_3 n^2 p},$$

$$\Pr[\exists S, T : \overline{A_{S,T}}] \leq \left(\frac{N}{n}\right)^2 2e^{-c_3 n^2 p}.$$  

Using the inequality $\left(\frac{N}{n}\right)^2 \leq \left(\frac{eN}{n}\right)^n$, we write

$$\left(\frac{N}{n}\right)^2 2e^{-c_3 n^2 p} < \left(\frac{N}{n}\right)^2 2e^{-c_3 n^2 p} < \left(\frac{eN}{n} 2e^{-c_3 np/2}\right)^{2n}.$$  

Taking $c$ sufficiently large it follows that

$$\Pr[\bigwedge_{|S|=n} A_S \wedge \bigwedge_{|S|=|T|=n} A_{S,T}] > 0. \quad \square$$

Returning to the proof of the main result, we modify its formulation slightly for the sake of convenience and prove that

$$s(m, 2n - 2) > c'_m \left(\frac{n}{\log n}\right)^{\frac{m^2-m-1}{2(m-1)}},$$

where $c'_m > 0$ is a constant depending only on $m$. Denote $H_0 = K^m$, $H_i = K^{m-i} + (K^{i,i} - iK^2)$, $2 \leq i \leq m$, where $G_1 + G_2$ denotes the join of $G_1$ and $G_2$ and $K^{i,i} - iK^2$ is obtained from the complete bipartite graph $K^{i,i}$ by removing a perfect matching. Denote $\mathcal{H} = \{H_0, H_1, \ldots, H_m\}$. The density of $\mathcal{H}$ is easily computable and equals to $(m^2 - m - 1)/(2(m - 1))$. Consider an $\mathcal{H}$-free graph $G_0$ on $N$ vertices $[N]$ having the properties stated in the preceding lemma. We colour the edges of $G_0$ red and the edges of $\overline{G_0}$ blue.
Now we claim that \(< B >\) does not contain an irredundant set of size \(m\) and \(< R >\) does not contain an irredundant set of size \(2n - 2\). As observed in [4], if \(< B >\) contains an \(m\)-element irredundant subset \(U\), then either \(U\) is independent in \(< B >\) (in this case \(< U >_R = K^m = H_0\)) or for some \(2 \leq i \leq m\) \(U\) contains a subset \(U_0\) of size \(|U_0| = i\) such that \(U \setminus U_0\) is independent in \(< B >\) and every vertex \(v\) of \(U_0\) has a private neighbour \(w\) relative to \(U\) (in this case, denoting by \(W_0\) the set of the private neighbours of the vertices from \(U_0\), we can easily see that \(< U \cup W_0 >_R\) contains a copy of \(H_i\)), hence \(< R >\) contains one of the graphs from \(\mathcal{H}\). Therefore, since \(G_0\) is \(\mathcal{H}\)-free, \(< B >\) indeed does not contain any irredundant set of size \(m\).

Consider now a set \(U \subseteq [N]\) of size \(|U| = 2n - 2\). Since \(\alpha(G_0) < n\), there are at least \(n\) non-isolated vertices in \(< U >_R\). Fix a subset \(U_0 \subseteq U\) of size \(|U_0| = n\), whose members are non-isolated vertices in \(< U >_R\). If \(U\) is irredundant, then clearly there is a subset \(W_0\) of size \(|W_0| = n\) such that the induced bipartite graph \(< U_0, W_0 >_R\) consists of a matching of size \(n\) and thus contains exactly \(n\) edges - a contradiction with the properties of \(G_0\). Therefore \(< R >\) does not contain any irredundant set of size \(2n - 2\). \(\Box\)

References


