A lower bound for irredundant Ramsey numbers

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April 20, 2001

Abstract

Given a graph $G = (V, E)$, a vertex subset $U \subseteq V$ is called irredundant if every vertex $v \in U$ either has no neighbours in U or there exists a vertex $w \in V \setminus U$ such that v is the only neighbour of w in U. The irredundant Ramsey number $s(m, n)$ is the smallest N such that any red-blue edge colouring of K^N yields either an m-element irredundant subset in the blue graph or an n-element irredundant subset in the red graph. Using probabilistic methods we show that

$$s(m,n) > c_m \left(\frac{n}{\log n}\right)^{\frac{m^2-m-1}{2(m-1)}} .$$

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A subset of vertices $U \subseteq V$ is called *irredundant* if every vertex $v \in U$ either has

*This research forms part of a Ph. D. thesis written by the author under the supervision of Professor Noga Alon.

no neighbours in U or there exists a vertex $w \in V \setminus U$ such that v is the only neighbour of w in U (in this case w is called a *private neighbour* of v). For every pair of integers $m, n \geq 2$ the *irredundant Ramsey number* $s(m, n)$ is the smallest integer N such that in any red-blue colouring of the edges of a complete graph K^N on N vertices either the blue graph contains an m-element irredundant subset or the red graph contains an n-element irredundant subset. Define also the *mixed Ramsey number* $t(m, n)$ as the smallest N such that any red-blue colouring of the edges of K^N yields an m-element irredundant subset in the blue graph or an n-element independent subset in the red graph. Recall that the *Ramsey number* $r(m, n)$ is the smallest N such that any red-blue colouring of the edges of K^N contains either a blue copy of K^m or a red copy of K^n. Since an independent set is clearly irredundant, the above definitions imply that

$$s(m, n) \leq t(m, n) \leq r(m, n) \ , \tag{1}$$

for all admissible m, n.

Irredundant Ramsey numbers were introduced in [2]. In [4] asymptotic lower bounds for the diagonal irredundant Ramsey numbers $s(n, n)$ and off-diagonal mixed Ramsey numbers $t(m, n), m < n$, were obtained. In particular, it was shown that

$$t(m, n) > c_m \left(\frac{n}{\log n} \right)^{\frac{m^2 - m - 1}{2(m-1)}} \ , \tag{2}$$

this result was obtained by using the so-called *probabilistic method* (see [1] as a general reference). It was also shown in [4] that

$$t(3,n) \le \frac{\sqrt{10}}{2} n^{3/2} \ .$$

The exact values for some small irredundant Ramsey numbers are known and can be found, e.g., in [4].

The purpose of this note is to establish an asymptotic lower bound for the off-diagonal irredundant Ramsey numbers $s(m,n)$, where m is fixed and n tends to infinity. We prove the following result.

Theorem 1 *For every $m \ge 3$ there exists a positive constant c_m such that*

$$s(m,n) > c_m \left(\frac{n}{\log n} \right)^{\frac{m^2-m-1}{2(m-1)}} \ .$$

This result matches (up to a constant factor) the bound (2) for the mixed Ramsey numbers $t(m,n)$ and thus implies (2) in view of (1).

Our proof is also based on the probabilistic method and applies *large deviation inequalities.* A similar approach has already been used in [7] for obtaining asymptotic lower bounds for various Ramsey-type numbers. We discuss this approach in Section 2. In Section 3 the proof of the main result is presented.

We end this section with some notation used in the sequel. We denote by $[N]$ the set $\{1,\ldots,N\}$. The complete graph on $[N]$ is denoted by K^N. For every two disjoint vertex subsets $S,T \subseteq V(G)$ let $E(S)$ be the edge set of the subgraph of G spanned by S, $E(S,T)$ is the set of all edges between S

and T, $e(S) = |E(S)|$ and $e(S,T) = |E(S,T)|$. A red-blue colouring of the edges of K^N induces the red graph $< R >$ and the blue graph $< B >$. We denote by $< U >_R$ ($< U >_B$, resp.) the induced subgraph of $< R >$ ($< B >$, resp.) on U.

For a fixed graph H we define

$$\rho(H) = \max_{H' \subseteq H,\ |H'|>2} \frac{e(H')-1}{|H'|-2}$$

(in order to avoid trivialities throughout the paper we always assume that $e(H) \geq 2$). For a finite family of fixed graphs $\mathcal{H} = \{H_1, \ldots, H_l\}$ the *density* of the family $\rho(\mathcal{H})$ is

$$\rho(\mathcal{H}) = \min\{\rho(H_i) : 1 \leq i \leq l\} \ .$$

Given a family $\mathcal{H} = \{H_1, \ldots, H_l\}$, a graph G is called *$\mathcal{H}$-free* if it does not contain a copy of H_i for every $1 \leq i \leq l$.

2 Large deviation inequalities

Roughly speaking, large deviation inequalities assert that under certain conditions a random variable X is highly concentrated near its mean and its tail probabilities are exponentially small.

The simplest example of a large deviation inequality is the bound on the tail of a binomial distribution, essentially due to Chernoff ([3]). If X is the sum of n mutually independent indicator random variables each of which taking the value 1 with probability p and the value 0 with probability $1-p$, then the expectation of X equals np and for every constant $0 < \epsilon < 1$ the following inequalities hold:

$$Pr[X < (1-\epsilon)np] \quad < \quad e^{-\epsilon^2 np/2} \,, \tag{3}$$

$$Pr[X > (1+\epsilon)np] \quad < \quad e^{-\epsilon^2 (1-\epsilon) np/2} \,. \tag{4}$$

When X is the sum of many 'rarely dependent' indicator random variables, it is also possible in certain cases to obtain exponential bounds on the tails of X. Let us describe a general scheme first presented in [6].

Suppose Q is a finite universal set (in our instances Q is the edge set of a complete graph on N vertices). Let $\{J_i : i \in Q\}$ be a set of independent indicator random variables, $Pr[J_i = 1] = p_i$ for every $i \in Q$ ($J_i = 1$ if the corresponding edge belongs to $E(G)$, where G is a random graph on N vertices in which each edge is chosen independently with probability p). Let $\{Q(\alpha)\}_{\alpha \in I}$ be a family of subsets of Q, where I is a finite index set. Define $X_\alpha = \prod_{i \in Q(\alpha)} J_i$ (then $X_\alpha = 1$ if and only if all the edges of $Q(\alpha)$ belong to $E(G)$). Now define

$$X = \sum_{\alpha \in I} X_\alpha \,,$$

(in our instances X counts the number of subgraphs of G having some specified properties).

We shall make use of the bound on the upper tail of another random variable X_0 which is tightly connected to X and is defined as

$$X_0 = \max\{ \quad r : \exists \text{ distinct } \alpha_1, \ldots, \alpha_r \in I \text{ with } X_{\alpha_i} = 1$$

$$\text{and } Q(\alpha_i) \cap Q(\alpha_j) = \emptyset,\ i \neq j\}$$

Clearly, $X_0 \leq X$. Let $\mu = EX$ be the expectation of X, then the following holds (see [5]):

Claim 1

$$Pr[X_0 \geq k] \leq \frac{\mu^k}{k!}$$

for every natural k.

For the sake of completeness we repeat the short proof.

Proof.

$$\begin{aligned} Pr[X_0 \geq k] &\leq \sum^1 Pr[(X_{\alpha_1} = 1) \wedge \ldots \wedge (X_{\alpha_k} = 1)] \\ &= \frac{1}{k!} \sum^2 Pr[(X_{\alpha_1} = 1) \wedge \ldots \wedge (X_{\alpha_k} = 1)] \\ &= \frac{1}{k!} \sum^2 Pr[X_{\alpha_1} = 1] \ldots Pr[X_{\alpha_k} = 1] \\ &\leq \frac{1}{k!} \sum^3 Pr[X_{\alpha_1} = 1] \ldots Pr[X_{\alpha_k} = 1] = \frac{\mu^k}{k!}\ , \end{aligned}$$

where $\sum^1$ is over sets of k mutually independent events $X_{\alpha_i} = 1$, while $\sum^2$ is over ordered k-tuples of mutually independent events and $\sum^3$ is over all ordered k-tuples of events. $\square$

In particular, we deduce from the above Claim that

$$Pr[X_0 \geq 5\mu] < \left(\frac{e}{5}\right)^{5\mu}\ . \tag{5}$$

(It is worth noting that in certain cases one can also obtain exponential bounds on the lower tail of X_0, see, e.g., [7]. However, the above cited simple bound will suffice for our purposes here).

3 Asymptotic lower bounds for $s(m,n)$

Recall that we are treating the off-diagonal irredundant Ramsey numbers $s(m,n)$, that is, m is fixed while n tends to infinity.

The proof of the main result is a simple consequence of the following

Lemma 1 *Let $\mathcal{H} = \{H_1, \ldots, H_l\}$ be a family of fixed graphs with density $\rho(\mathcal{H}) > 0$. Then there exists a constant $c = c(\mathcal{H})$ such that for every sufficiently large integer N there exists a graph G_0 on N vertices having the following properties:*

1. *G_0 is $\mathcal{H}$-free;*
2. *G_0 has no independent set of size $n = \lceil cN^{1/\rho(\mathcal{H})} \ln N \rceil$;*
3. *for every two disjoint subsets of vertices $S, T \subseteq V(G_0)$ of size $|S| = |T| = n$ one has $e(S,T) > n$.*

Proof. For every $1 \le i \le l$ let H_i' be a subgraph of H_i such that $\rho(H_i') = \rho(H_i)$. Denoting $\mathcal{H}' = \{H_1', \ldots, H_l'\}$, note that if G_0 is $\mathcal{H}'$-free then it is clearly $\mathcal{H}$-free, therefore we may assume that $\rho(H_i) = (e(H_i) - 1)/(|H_i| - 2)$, $1 \le i \le l$. For every $1 \le i \le l$ set $v_i = |H_i|$, $e_i = e(H_i)$. Set also

$$
\begin{aligned}
e_{min} &= \min\{e_i : 1 \le i \le l\} , \\
e_{max} &= \max\{e_i : 1 \le i \le l\} .
\end{aligned}
$$

Consider a random graph $G(N, p)$ - a graph on N labelled vertices in which all edges are chosen independently with probability p. We set with foresight $p = c_0 N^{-1/\rho(\mathcal{H})}$, where $0 < c_0 < 1$ is a sufficiently small constant.

For every two disjoint subsets $S, T \subseteq V(G)$ of size $|S| = |T| = n$ we define the following random variables. First, let $X_S = e(S), X_{S,T} = e(S, T)$. Also, denote by Y_S the number of subgraphs, each isomorphic to one of the graphs from $\mathcal{H}$ and having at least one edge inside S, and by Z_S the maximal number of pairwise edge disjoint subgraphs, each isomorphic to one of the graphs from $\mathcal{H}$ and having at least one edge inside S. Let $Y_{S,T}$ denote the number of subgraphs, each isomorphic to one of the graphs from $\mathcal{H}$ and having at least one edge in $E(S, T)$, and let $Z_{S,T}$ denote the maximal number of pairwise edge disjoint subgraphs, each isomorphic to one of the graphs from $\mathcal{H}$ and having at least one edge in $E(S, T)$. Clearly, $Y_S \geq Z_S$ and $Y_{S,T} \geq Z_{S,T}$. Denote by A_S the event $X_S > e_{max} Z_S$ and by $A_{S,T}$ the event $X_{S,T} > e_{max} Z_{S,T} + n$.

Claim 2 *If A_S holds for every $S \subseteq V$ of size $|S| = n$ and $A_{S,T}$ holds for every pair of disjoint subsets $S, T \subseteq V(G)$ of size $|S| = |T| = n$, then G contains a subgraph G_0 on N vertices, satisfying the requirements of the lemma.*

Proof. Let $\mathbf{H}$ be a maximal (under inclusion) family of pairwise edge disjoint subgraphs of G, each isomorphic to one of the graphs from $\mathcal{H}$. Deleting all edges of all graphs from $\mathbf{H}$ we clearly obtain an $\mathcal{H}$-free graph G_0 on N

vertices. Denote by $\mathbf{H}_S$, $|S| = n$, the subfamily of $\mathbf{H}$, consisting of all subgraphs from $\mathbf{H}$, having at least one edge in $E(S)$, and by $\mathbf{H}_{S,T}$, $|S| = |T| = n$, the subfamily of $\mathbf{H}$, consisting of all subgraphs from $\mathbf{H}$, sharing at least one edge with $E(S,T)$. Obviously, $|\mathbf{H}_S| \leq Z_S$, $|\mathbf{H}_{S,T}| \leq Z_{S,T}$. While deleting the edges of subgraphs from $\mathbf{H}$, we delete at most $e_{max}|\mathbf{H}_S| \leq e_{max}Z_S$ edges from $E(S)$ and at most $e_{max}|\mathbf{H}_{S,T}| \leq e_{max}Z_{S,T}$ edges from $E(S,T)$, hence the subgraph G_0 satisfies also the conditions 2) and 3) of the lemma. $\Box$

Now our aim is to show that under appropriate choice of the constants c_0 and c the inequality $Pr[\bigwedge_{|S|=n} A_S \wedge \bigwedge_{|S|=|T|=n} A_{S,T}] > 0$ holds. To this end, we show that the random variables $X_S, Z_S, X_{S,T}, Z_{S,T}$ are highly concentrated around their means and hence if, say, $EX_S > 10e_{max}EZ_S$ and $EX_{S,T} > 10e_{max}EZ_{S,T}$, then both probabilities $Pr[\overline{A}_S]$ and $Pr[\overline{A}_{S,T}]$ are exponentially small. This will imply that the probability that either there exists some set S for which $\overline{A}_S$ holds or there exists a pair S,T for which $\overline{A}_{S,T}$ holds is less than 1.

The random variable X_S is clearly binomially distributed with parameters $\binom{n}{2}$ and p, therefore from (3) we obtain for every $0 < \epsilon < 1$

$$Pr[X_S < (1-\epsilon)\binom{n}{2}p] < e^{-\epsilon^2\binom{n}{2}p/2} \ . \tag{6}$$

Similarly, $X_{S,T}$ is binomially distributed with parameters n^2 and p, and hence (3) implies that

$$Pr[X_{S,T} < (1-\epsilon)n^2p] < e^{-\epsilon^2 n^2 p/2} \ . \tag{7}$$

Now we bound the upper tail of Z_S by using Claim 1. To this end, we estimate EY_S. Note that

$$Y_S = Y_{S,1} + \dots Y_{S,l} \ ,$$

where $Y_{S,i}$ is the number of copies of H_i, having at least one edge in $E(S)$. Representing $Y_{S,i}$ as a sum of indicator random variables we can write

$$\binom{n}{2}\binom{N-n}{v_i-2}p^{e_i} \le EY_{S,i} \le \binom{n}{2}\binom{N-2}{v_i-2}v_i!p^{e_i} \ ,$$

therefore (recalling that $n = o(N)$ and hence $\binom{N-n}{v_i-2} = \Theta(N^{v_i-2})$) we have

$$c_{i,1}\binom{n}{2}p\left(N^{\frac{v_i-2}{e_i-1}}p\right)^{e_i-1} \le EY_{S,i} \le c_{i,2}\binom{n}{2}p\left(N^{\frac{v_i-2}{e_i-1}}p\right)^{e_i-1} \ ,$$

where $c_{i,1}$ and $c_{i,2}$ are some positive constants depending only on H_i.

The definitions of $\rho(\mathcal{H})$ and p imply that

$$c_1 c_0^{e_{max}-1}\binom{n}{2}p \le EY_S \le c_2 c_0^{e_{min}-1}\binom{n}{2}p \ ,$$

where $c_1 = c_1(\mathcal{H})$ and $c_2 = c_2(\mathcal{H})$ are positive constants.

Substituting in (5) EY_S and Z_S instead of μ, X_0, respectively, we conclude that

$$Pr[Z_S \ge 5EY_S] \le e^{-5(\ln 5-1)EY_S} \ . \tag{8}$$

Turning to estimating the upper tail of $Z_{S,T}$ we act in a quite similar manner. Taking c_1 sufficiently small and $c_2 > 1$ sufficiently large we can write

$$c_1 c_0^{e_{max}-1}n^2 p \le EY_{S,T} \le c_2 c_0^{e_{min}-1}n^2 p \ ,$$

Also, (5) implies that

$$Pr[Z_{S,T} \geq 5EY_{S,T}] \leq e^{-5(\ln 5-1)EY_{S,T}} \ . \tag{9}$$

Comparing EX_S and EY_S, $EX_{S,T}$ and $EY_{S,T}$ we observe

$$\frac{1}{c_2 c_0^{e_{min}-1}} \leq \frac{EX_S}{EY_S}, \frac{EX_{S,T}}{EY_{S,T}} \leq \frac{1}{c_1 c_0^{e_{max}-1}} \ .$$

Let us choose c_0 so that the expression $c_2 c_0^{e_{min}-1}$ will be equal to, say, $1/10e_{max}$. Then

$$10e_{max} \leq \frac{EX_S}{EY_S}, \frac{EX_{S,T}}{EY_{S,T}} \leq \frac{c_2}{c_1} c_0^{-e_{max}+e_{min}} 10e_{max} \ . \tag{10}$$

Now, by (6) with $\epsilon = 1/2$, (8) and (10)

$$\begin{aligned}
Pr[\overline{A}_S] &= Pr[X_S \leq e_{max} Z_S] \leq Pr[X_S \leq \frac{EX_S}{2}] + Pr[e_{max} Z_S \geq \frac{EX_S}{2}] \\
&\leq Pr[X_S \leq \frac{EX_S}{2}] + Pr[Z_S \geq 5EY_S] \\
&\leq e^{-\binom{n}{2}p/8} + e^{-\frac{c_1 c_0^{-e_{min}+e_{max}}}{10 c_2 e_{max}}[5\ln 5-5]\binom{n}{2}p} \leq 2e^{-c_3 n^2 p}
\end{aligned}$$

for some constant $c_3 > 0$.

Also, (6) with $\epsilon = 1/3$, (9) and (10) imply for sufficiently large N (noting that $EX_{S,T}/n \to \infty$)

$$\begin{aligned}
Pr[\overline{A}_{S,T}] &= Pr[X_{S,T} \leq e_{max} Z_{S,T} + n] \\
&\leq Pr[X_{S,T} \leq \frac{EX_{S,T}}{2} + n] + Pr[e_{max} Z_{S,T} \geq \frac{EX_{S,T}}{2}] \\
&\leq Pr[X_{S,T} \leq \frac{2EX_{S,T}}{3}] + Pr[Z_{S,T} \geq 5EY_{S,T}] \\
&\leq e^{-n^2 p/18} + e^{-\frac{c_1 c_0^{-e_{min}+e_{max}}}{10 c_2 e_{max}}[5\ln 5-5]n^2 p} \leq 2e^{-c_3 n^2 p}
\end{aligned}$$

(taking c_3 small enough).

Therefore

$$\begin{aligned} Pr[\exists S : \overline{A}_S] &\leq \binom{N}{n} 2e^{-c_3 n^2 p} , \\ Pr[\exists S, T : \overline{A}_{S,T}] &\leq \binom{N}{n}^2 2e^{-c_3 n^2 p} . \end{aligned}$$

Using the inequality $\binom{N}{n} \leq \left(\frac{eN}{n}\right)^n$, we write

$$\binom{N}{n} 2e^{-c_3 n^2 p} < \binom{N}{n}^2 2e^{-c_3 n^2 p} < \left(\frac{eN}{n} 2e^{-c_3 np/2}\right)^{2n} .$$

Taking c sufficiently large it follows that

$$Pr[\bigwedge_{|S|=n} A_S \wedge \bigwedge_{|S|=|T|=n} A_{S,T}] > 0 . \quad \Box$$

Returning to the proof of the main result, we modify its formulation slightly for the sake of convenience and prove that

$$s(m, 2n-2) > c'_m \left(\frac{n}{\log n}\right)^{\frac{m^2-m-1}{2(m-1)}} ,$$

where $c'_m > 0$ is a constant depending only on m. Denote $H_0 = K^m$, $H_i = K^{m-i} + (K^{i,i} - iK^2)$, $2 \leq i \leq m$, where $G_1 + G_2$ denotes the join of G_1 and G_2 and $K^{i,i} - iK^2$ is obtained from the complete bipartite graph $K^{i,i}$ by removing a perfect matching. Denote $\mathcal{H} = \{H_0, H_2, \ldots, H_m\}$. The density of $\mathcal{H}$ is easily computable and equals to $(m^2 - m - 1)/(2(m-1))$. Consider an $\mathcal{H}$-free graph G_0 on N vertices $[N]$ having the properties stated in the preceding lemma. We colour the edges of G_0 red and the edges of $\overline{G}_0$ blue.

Now we claim that $< B >$ does not contain an irredundant set of size m and $< R >$ does not contain an irredundant set of size $2n-2$. As observed in [4], if $< B >$ contains an m-element irredundant subset U, then either U is independent in $< B >$ (in this case $< U >_R = K^m = H_0$) or for some $2 \leq i \leq m$ U contains a subset U_0 of size $|U_0| = i$ such that $U \setminus U_0$ is independent in $< B >$ and every vertex v of U_0 has a private neighbour w relative to U (in this case, denoting by W_0 the set of the private neighbours of the vertices from U_0, we can easily see that $< U \cup W_0 >_R$ contains a copy of H_i), hence $< R >$ contains one of the graphs from $\mathcal{H}$. Therefore, since G_0 is $\mathcal{H}$-free, $< B >$ indeed does not contain any irredundant set of size m.

Consider now a set $U \subseteq [N]$ of size $|U| = 2n-2$. Since $\alpha(G_0) < n$, there are at least n non-isolated vertices in $< U >_R$. Fix a subset $U_0 \subseteq U$ of size $|U_0| = n$, whose members are non-isolated vertices in $< U >_R$. If U is irredundant, then clearly there is a subset W_0 of size $|W_0| = n$ such that the induced bipartite graph $< U_0, W_0 >_R$ consists of a matching of size n and thus contains exactly n edges - a contradiction with the properties of G_0. Therefore $< R >$ does not contain any irredundant set of size $2n-2$. $\Box$

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