ON THE EDGE DISTRIBUTION IN TRIANGLE-FREE GRAPHS

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ABSTRACT. Two problems on the edge distribution in triangle-free graphs are considered: 1) Given an $0 < \alpha < 1$. Find the largest $\beta = \beta(\alpha)$ such that for infinitely many n there exists a triangle-free graph G on n vertices in which every αn vertices span at least βn^2 edges. This problem was considered by Erdős, Faudree, Rousseau and Schelp in [4]. Here we extend and improve their results, proving in particular the bound $\beta < 1/36$ for $\alpha = 1/2$; 2) How much does the edge distribution in a triangle-free graph G on n vertices deviate from the uniform edge distribution in a typical (random) graph on n vertices with the same number of edges? We give quantitative expressions for this deviation.

0. NOTATION

All graphs in this paper are finite, undirected, without loops and multiple edges.

Let G be a graph on n vertices with vertex set $V = V(G)$ and edge set $E = E(G)$. For $X, Y \subset V$ we denote by $E(X,Y)$ a set of edges of G with one endpoint in X and the other in Y, $e(X,Y)$ stands for $|E(X,Y)|$. Similarly, $E(X,X)$ or simply $E(X)$ is the edge set spanned by X, and $e(X) = |E(X)|$. Also, $e(G) = |E(G)|$.

The set of vertices adjacent to a vertex $v \in V$ is denoted by $\Gamma(v)$ and the degree of v is $d(v) = |\Gamma(v)|$.

The expectation of a random variable A is denoted by $M(A)$.

For every $0 < \alpha \leq 1$ write

$$\Psi(G, \alpha n) = \min\{e(X) : X \subseteq V, |X| = \alpha n\} ,$$

i.e., the smallest number of edges in an induced graph of G of order αn.

(As from now we shall occasionally disregard integral parts for the sake of simplicity of presentation and use expressions like $|X| = \alpha n$, even when αn may not be integral. Since our results are asymptotical, this will not affect the correctness of the proofs. For the same reason, we shall often omit from our formulations the expression 'for n sufficiently large', where n is the number of vertices of a graph).

1. INTRODUCTION

In this paper we study the edge distribution on triangle-free graphs. We shall consider two main problems.

The first of them concerns a local density condition for triangles. Turán's theorem asserts that every graph G on n vertices with more than $n^2/4$ edges contains a triangle, or, in our notation, if $\Psi(G, n) > n^2/4$, then G contains a triangle. Let us

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try to generalize this result. Suppose that for some fixed $0 < \alpha \leq 1$ every αn vertices of G span more than βn^2 edges. The question is to find the smallest $\beta = \beta(\alpha)$ for which G necessarily contains a triangle.

This problem was raised by Erdős, Faudree, Rousseau and Schelp in [4]. They conjectured that β is determined by a family of extremal triangle-free graphs with no independent set of a given order, i.e., extremal Ramsey graphs for Ramsey numbers $R(3,i)$. In order to formulate their conjecture consider three Ramsey graphs $M_i (i = 1,2,3)$ which are triangle-free with no independent set on $i+1$ vertices. These graphs are defined as follows: $M_1 = K_2, M_2 = C_5, M_3$ is obtained by adding to C_8 all chords of length four. Now blow up each graph M_i to a new graph H_i by replacing every vertex by an independent set of size $n/|V(M_i)|$ and making two vertices from different blown sets adjacent if and only if the corresponding vertices of M_i were adjacent.

A simple calculation shows that

(1) for H_1 if $1/2 \leq \alpha \leq 1$ then $\Psi(H_1, \alpha n) = [(2\alpha - 1)/4]n^2$;
(2) for H_2 if $2/5 \leq \alpha \leq 3/5$ then $\Psi(H_2, \alpha n) = [(5\alpha - 2)/25]n^2$;
(3) for H_3 if $3/8 \leq \alpha \leq 1/2$ then $\Psi(H_3, \alpha n) = [(8\alpha - 3)/64]n^2$.

Note that $\Psi(H_1, \alpha n) \geq \Psi(H_2, \alpha n)$ for $\alpha \geq 17/30$ and $\Psi(H_2, \alpha n) \geq \Psi(H_3, \alpha n)$ for $\alpha \geq 53/120$. These observations motivated the authors of [4] to make the following conjecture:

Conjecture 1. *Let G be a graph of order n and let α be fixed, $53/120 \leq \alpha \leq 1$. Further let*

$$\beta = \begin{cases} (2\alpha - 1)/4 & \textit{when} \quad 17/30 \leq \alpha \leq 1 \\ (5\alpha - 2)/25 & \textit{when} \quad 53/120 \leq \alpha \leq 17/30 . \end{cases}$$

If every αn vertices of G span more than βn^2 edges, then G contains a triangle.

As shown by the above examples this conjecture, if true, would be best possible.

A particularly interesting case of the above conjecture is $\alpha = 1/2$, that is,

Conjecture 2. *If in a graph G of order n every $n/2$ vertices span more than $n^2/50$ edges, then G contains a triangle.*

It should be mentioned that the blown up Petersen graph also gives the same extremal values for β as does H_2.

In [4] Conjecture 1 was proved for $\alpha > 0.647$. For the case $\alpha = 1/2$ the authors obtained the bound $\beta \leq 1/30$ (instead of conjectured $\beta = 1/50$). In this paper we shall improve on these results. A significant part of our efforts will be devoted to improving the bound for the case $\alpha = 1/2$. We shall reduce the bound for this case to $1/36$, namely:

Theorem 1. *If in a graph G of order n every $n/2$ vertices span at least $n^2/36$ edges, then G contains a triangle.*

We shall also prove the following statement, which is asymptotically slightly stronger than Theorem 1:

Theorem 2. *There is a (calculable) constant $\epsilon > 0$ such that if in a graph G of order n every $n/2$ vertices span at least $(1/36 - \epsilon + o(1))n^2$ edges, then G contains a triangle.*

The proofs of Theorems 1 and 2 are presented in Section 3.

It may be worthwhile to mention another extremal problem which is conjectured to have the same optimal examples as Conjecture 2. The question is how many edges should be deleted from a triangle-free graph of order n to make it bipartite. The best results on this problem are due to Erdős, Faudree, Pach and Spencer ([3]) and they are contained in the following two theorems.

Theorem. *Every triangle-free graph G of order n can be made bipartite by deleting at most $n^2/18 + n/2$ edges;*

Theorem. *There is a (calculable) constant $\epsilon > 0$ such that every triangle-free graph G of order n can be made bipartite by deleting at most $(1/18 - \epsilon + o(1))n^2$ edges.*

(Some additional results on this problem were obtained by Erdős, E. Győri and Simonovits in [5]. We shall make use of some ideas from their paper).

These results are rather similar to the results of Theorems 1 and 2, but we succeeded to establish a connection between these two problems only for the case of regular triangle-free graphs. In this case the above two theorems follow from Theorems 1 and 2. To see this, note the following:

Claim. *If in a regular graph G of order n there exists a set of vertices U of size $|U| = n/2$ which spans m_0 edges, then G can be made bipartite by deleting at most $2m_0$ edges.*

The claim follows from the fact that by the regularity of G also $e(V \setminus U) = m_0$ and deleting $E(U) \cup E(V \setminus U)$ we make the graph bipartite.

The assumption about regularity enables us also to prove Conjecture 2 for regular graphs with large vertex degree (Section 4):

Theorem 3. *If in a regular triangle-free graph G of order n with vertex degree $D \geq 2n/5$ every $n/2$ vertices span at least $n^2/50$ edges, then G is a uniformly blown up C_5 (i.e. the graph H_2 described above).*

As mentioned above, in [4] Conjecture 1 was proved for $\alpha > 0.647$. We improve this in Section 5 to $\alpha \geq 0.6$:

Theorem 4. *Let G be a graph of order n and let α be fixed, $\alpha \geq 0.6$. Further let $\beta = (2\alpha - 1)/4$. If every αn vertices of G span more than βn^2 edges, then G contains a triangle.*

The second problem we will treat is how much does the edge distribution in a triangle-free graph deviate from being uniform. A typical (random) graph G on n vertices with cn^2 edges contains $\Theta(n^3)$ triangles, so if G is a triangle-free graph, it deviates strongly from being typical. This common philosophy was developed and discussed by several authors, see, e.g., [2]. In this paper we study quantitatively this deviation for the property of edge distribution. We shall show that the edge distribution in a triangle-free graph G of order n with cn^2 edges is indeed not uniform compared to the edge distribution in a random graph of the same order and size, in which every set of n vertices spans $(1 + o(1))(e(G)/4)$ edges. This is provided by

Theorem 5. *If G is a triangle-free graph on n vertices with $e(G) = cn^2$ edges, then*

$$\Psi(G, n/2) \leq (1/4 - c')e(G) \ ,$$

where $c' = c^2/(4c^2 - 4c + 1)$.

From the other side, the above result is in some sense best possible - the coefficient $c'(c)$ cannot be replaced by an absolute constant $\gamma > 0$ as shown by

Theorem 6. *For every* $\epsilon > 0$ *there exists a constant* $c = c(\epsilon) > 0$ *such that for infinitely many* n *there exists a triangle-free graph* G *of order* n *with* $e(G)$ *edges with* $e(G) > c(\epsilon)n^2$*, for which*

$$\Psi(G, n/2) > (1/4 - \epsilon)e(G) \ .$$

These questions are discussed in Section 6.

2. Two lemmas

Lemma 1. *If every* αn *vertices of a graph* G *on* n *vertices span at least* βn^2 *edges, then every set of vertices* $V_0 \subseteq V$ *of size* $|V_0| > \alpha n$ *spans more than* $(\beta/\alpha^2)|V_0|^2$ *edges.*

Proof. Denote $|V_0| = u_0, e(V_0) = m_0$. Choose randomly and uniformly a set W_0 from among all subsets of V_0 of size αn. Then for every $e \in E(V_0)$

$$Pr(e \in E(W_0) = \frac{\binom{|V_0|-2}{|W_0|-2}}{\binom{|V_0|}{|W_0|}} = \frac{\binom{u_0-2}{\alpha n-2}}{\binom{u_0}{\alpha n}} < \left(\frac{\alpha n}{u_0}\right)^2 \ .$$

So

$$M(e(W_0)) = Pr[e \in E(W_0)]e(V_0) < m_0 \left(\frac{\alpha n}{u_0}\right)^2 \ .$$

We conclude that there exists a set W_0 of size $|W_0| = \alpha n$ such that $E(W_0) < m_0(\alpha n/u_0)^2$. From the conditions of lemma we obtain immediately that $\beta n^2 \leq e(W_0) < m_0 \left(\frac{\alpha n}{u_0}\right)^2$, so

$$m_0 > (\beta/\alpha^2)u_0^2 \ . \qquad \square$$

Lemma 2. *If* G *is a triangle-free graph on* n *vertices with* m *edges then there exist in* G *two disjoint non-empty independent sets of vertices* V_1 *and* V_2 *such that* $|V_1| + |V_2| \geq 4m/n$.

Proof. Consider the following sum

$$\sum_{(v,u) \in E} (d(u) + d(v)) = \sum_{v \in V} d(v)^2 \geq \frac{\left(\sum_{v \in V} d(v)\right)^2}{n} = \frac{4m^2}{n} \ .$$

Hence there exists an edge $e = (v_1, v_2)$ such that $d(v_1) + d(v_2) \geq 4m/n$. Let $V_1 = \Gamma(v_1), V_2 = \Gamma(v_2)$. We claim that V_1 and V_2 are the required sets. Indeed,

(1) $|V_1| + |V_2| = d(v_1) + d(v_2) \geq 4m/n$;
(2) Both V_1 and V_2 contain no edges;
(3) $V_1 \cap V_2 = \emptyset$ (if $u \in V_1 \cap V_2$, then u, v_1, v_2 would form a triangle in G);
(4) $v_1 \in V_1$ and $v_2 \in V_2$, hence V_1, V_2 are both non-empty. $\square$

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let G be a graph on n vertices such that

$$\Psi(G,n/2)\geq n^2/36\ . \tag{1}$$

Denote the vertex set of G by V, the edge set of G by E and let $l=|E|/n^2$. By Lemma 2 there exist two non-empty independent sets of vertices V_1 and V_2 such that $V_1\cap V_2=\emptyset$ and $|V_1|/n+|V_2|/n=4l$. Suppose $|V_1|\geq|V_2|$ and write $|V_1|/n=x\geq 2l$; $|V_2|/n=4l-x$; $U=V\setminus(V_1\cup V_2)$; $|U|/n=1-4l$. Let also $l_1=e(V_1,V_2)/n^2$, $l_2=e(U)/n^2$, $l_3=e(V_1,U)/n^2$, $l_4=e(V_2,U)/n^2$. Our aim is to evaluate the numbers l_1,l_2,l_3,l_4 in terms of n and l and to show that if G is triangle-free and $\Psi(G,n/2)\geq n^2/36$, then $l_1+l_2+l_3+l_4>l$, thus obtaining a contradiction, since in fact $l_1+l_2+l_3+l_4=l$.

Assume first that $l_2>0$ (the case $l_2=0$ will be treated later). Choose randomly and uniformly a set W_1 from among all subsets of U of size $(1/2-x)n$ (so $|W_1\cup V_1|=n/2$). For each $e\in E(U)$

$$Pr[e\in E(W_1)]=\frac{\binom{|U|-2}{|W_1|-2}}{\binom{|U|}{|W_1|}}=\frac{\binom{n-4ln-2}{n/2-xn-2}}{\binom{n-4ln}{n/2-xn}}<\left(\frac{1/2-x}{1-4l}\right)^2\ .$$

For each $e\in E(V_1,U)$

$$Pr[e\in E(V_1,W_1)]=\frac{|W_1|}{|U|}=\frac{1/2-x}{1-4l}\ .$$

Linearity of expectation gives

$$M(e(V_1\cup W_1))<\left(\frac{1/2-x}{1-4l}\right)^2 l_2n^2+\frac{1/2-x}{1-4l}l_3n^2\ .$$

Hence there exists a subset $W_1\subseteq U$, $|W_1|/n=1/2-x$, such that

$$(e(V_1\cup W_1)<\left(\frac{1/2-x}{1-4l}\right)^2 l_2n^2+\frac{1/2-x}{1-4l}l_3n^2\ . \tag{2}$$

Now, since $|V_1\cup W_1|=n/2$, it follows from condition (1) that $e(V_1\cup W_1)\geq n^2/36$, and hence:

$$\left(\frac{1/2-x}{1-4l}\right)^2 l_2+\frac{1/2-x}{1-4l}l_3>1/36\ . \tag{3}$$

Choosing randomly and uniformly a set W_2 from among all subsets of U of size $(1/2-4l+x)n$ (so $|W_2\cup V_2|=n/2$) and evaluating the expected value of $e(V_2\cup W_2)$ in a similar way we obtain the inequality

$$\left(\frac{1/2-4l+x}{1-4l}\right)^2 l_2+\frac{1/2-4l+x}{1-4l}l_4>1/36\ . \tag{4}$$

Dividing (3) by $(1/2-x)/(1-4l)$ and (4) by $(1/2-4l+x)/(1-4l)$ and adding up we obtain

$$l_1+l_2+l_3>\frac{1-4l}{36}\left(\frac{1}{1/2-x}+\frac{1}{1/2-4l+x}\right) .$$

But $2l \le x < 4l$, hence

$$l_1+l_2+l_3>\frac{1-4l}{36}+\frac{4}{1-4l}=\frac{1}{9} . \tag{5}$$

Now we have to evaluate l_1. Consider two following cases: 1) $l \ge 1/8$; 2) $l < 1/8$.

Case 1: $l \ge 1/8$.

So $4l \ge 1/2$ and V_1 and V_2 contain together at least half of the vertices.

Choose randomly and uniformly a set W_3 from among all subsets of V_2 of size $(1/2-x)n$, so W_3 will complement V_1 to $n/2$ vertices. For each $e \in E(V_1, V_2)$

$$Pr[e \in E(V_1, W_3)] = \frac{|W_3|}{|V_2|} = \frac{1/2-x}{4l-x} ,$$

so calculating the expected value of $e(V_1, W_3)$ we conclude that there exists a subset $W_3 \subseteq V_2$ of size $|W_3|/n = 1/2-x$ such that

$$e(V_1, W_3) \le \frac{1/2-x}{4l-x} l_1 n^2 ,$$

and since $|V_1 \cup W_3| = n/2$ we obtain from condition (1)

$$l_1 \ge \frac{4l-x}{1/2-x}\frac{1}{36} .$$

But $x \ge 2l$ and we have

$$l_1 \ge \frac{2l}{1/2-2l}\frac{1}{36} = \frac{l}{9(1-4l)} . \tag{6}$$

From (5) and (6) we obtain the following inequality on l:

$$l = l_1+l_2+l_3+l_4 > \frac{1}{9}+\frac{l}{9(1-4l)} , \tag{7}$$

or

$$36(l-\frac{1}{6})^2 < 0$$

-a contradiction.

Case 2: $l < 1/8$.

Since now $|V_1 \cup V_2| < n/2$ we have to evaluate l_1 in another way.

Choose randomly and uniformly a set W_4 from among all subsets of U of size $(1/2-4l)n$ (so that $|W_4 \cup V_1 \cup V_2| = n/2$). A calculation similar to the above shows that there exists $W_4 \subset U$ such that

$$e(W_4 \cup V_1 \cup V_2) < \left(\frac{1/2-4l}{1-4l}\right)^2 l_2 n^2 + \frac{1/2-4l}{1-4l}(l_3+l_4)n^2 + l_1 n^2 ,$$

so we have

$$\left(\frac{1/2-4l}{1-4l}\right)^2 l_2 + \frac{1/2-4l}{1-4l}(l_3+l_4) + l_1 > 1/36 \ . \tag{8}$$

Let $A = (1-4l)/(1/2-4l)$, then (8) is equivalent to

$$l_2 + A(l_3+l_4) + A^2 l_1 > \frac{A^2}{36} \ .$$

But according to Lemma 1 applied to the set U we have

$$l_2 > \frac{(1-4l)^2}{9} = \frac{A^2}{36(A-1)^2} \ .$$

Joining this with (8) we get

$$A(l_2+l_3+l_4) + A^2 l_1 > \frac{A^2}{36} + (A-1)l_2 > \frac{A^2}{36} + \frac{A^2}{36(A-1)} \ ,$$

and from the last inequality and (5) we obatin

$$A^2(l_1+l_2+l_3+l_4) > \frac{A^2}{36} + \frac{A^2}{36(A-1)} + (A^2-A)\frac{1}{9} \ ,$$

implying:

$$\frac{A_2}{8(A-1)} = l = l_1+l_2+l_3+l_4 > \frac{1}{36} + \frac{1}{36(A-1)} + \frac{A-1}{9A} \ ,$$

or:

$$A^2 + 2A + 8 < 0$$

- a contradiction.

It remains to consider only the case $l_2 = 0$. The only difference is that we have to replace the sign '$<$' by '$\leq$' in (2). Following the proof of the case $l_2 > 0$ we can see that the only case we have to consider (assuming equalities in all non-strict inequalities) is $l = 1/6$; $|V_1|/n = |V_2|/n = 1/3$, $|U|/n = 1/3$. From the proof of Case 1 we also have that equality in (7) is possible only if G is a regular graph with vertex degree $d(v) = n/3$ for all $v \in V$. So suppose that G indeed satisfies the above conditions. Denote $T = V_1 \cup V_2$, so $|T| = 2n/3$, $e(T) = n^2/18$, $e(U,T) = n^2/9$. Pick an arbitrary vertex $u_0 \in U$. Denote $Y = \Gamma(u_0), Z = T \setminus Y, |Y| = |Z| = n/3$. If $e(Z) > 0$, then we are again in the case $l_2 > 0$ with U, Y, Z instead of V_1, V_2, U, respectively, so suppose $e(Z) = 0$. Define two vertex sets $U' = \{u \in U : e(u,Y) \geq n/6\}$ and $U'' = \{u \in U : e(u,Y) < n/6\}$. Obviously, $|U'| + |U''| = n/3$ and $u_0 \in U'$. If $|U'| \geq n/6$, then take $U_0 \subseteq U', |U_0| = n/6, u_0 \in U_0$, and consider a graph $G[U_0 \cup Z]$, for which

$$\begin{aligned} e(U_0,Z) &= \sum_{u\in U_0} e(u,Z) = \sum_{u\in U_0} (n/3 - e(u,Y)) \\ &= n/3 - e(u_1,Y) + \sum_{u\in U_0\setminus u_0} (n/3 - e(u,Y)) \leq 0 + (n/6-1)n/6 < n^2/36 \end{aligned}$$

- a contradiction with condition (1), since $|U_0 \cup Z| = n/2$. In another case, if $|U''| \geq n/6$, then take $U_0 \subseteq U'', |U_0| = n/6$, and consider a graph $G[U_0 \cup Y]$, for which

$$e(U_0, Y) = \sum_{u \in U_0} e(u, Y) < (n/6)(n/6) = n^2/36 \ ,$$

again contradicting the condition (1), since $|U_0 \cup Y| = n/2$, and we have finished the proof! $\square$

Proof of Theorem 2. Suppose indirectly that there is a triangle-free graph G of order n in which every induced subgraph on $n/2$ vertices contains at least $n^2/36 + o(n^2)$ edges. From the proof of Theorem 1 and Lemma 2 we conclude that $e(G) = n^2/6 + o(n^2)$ and $d(v) = n/3 + o(n)$ for all but $o(n)$ vertices which we shall ignore. Fix $v_0 \in V, d(v_0) = n/3 + o(n)$, and let $U = \Gamma(v_0), |U| = n/3 + o(n)$, clearly, U is independent. Set $T = V \setminus U, |T| = 2n/3 + o(n)$. Obviously, $e(U,T) = n^2/9 + o(n^2), e(T) = n^2/18 + o(n^2)$ (otherwise we would have $\epsilon_1 n$ vertices $v \in V$ with degrees $|d(v) - n/3| > \epsilon_2 n$ for some constants $\epsilon_1, \epsilon_2 > 0$). Choose a $u_0 \in U$ with $d(u_0) = n/3 + o(n)$ and set $Y = \Gamma(U_0), |Y| = n/3 + o(n)$, Y is again independent. Let $Z = T \setminus Y, |Z| = n/3 + o(n)$. If $e(Z) = \epsilon n^2$ then to obtain a contradiction we would repeat the argument of Theorem 1 for the case $l_2 > 0$ with sets U, Y, Z instead of V_1, V_2, U, respectively, so suppose $e(Z) = o(n^2)$. Since our graph is almost regular we have $e(U,Y) = n^2/18 + o(n^2), e(U,Z) = n^2/18 + o(n^2)$.

Suppose $|\Gamma(u) \cap Y| \leq n/6 - \epsilon_1 n$ for $\epsilon_2 n$ vertices from U. Take the $|U|/2$ vertices $u \in U$ having the smallest degrees $e(u, Y)$. Denote this set by U_0, then $|U_0| = n/6 + o(n)$ and $e(U_0, Y) \leq n^2/36 - \epsilon_3 n^2 + o(n^2)$ - a contradiction to our assumption. The same argument is valid for Z, so we conclude that for $n/3 + o(n)$ vertices $u \in U$ there holds $|\Gamma(u) \cap Y| = n/6 + o(n)$, $|\Gamma(u) \cap Z| = n/6 + o(n)$. Fix one of such vertices u_1 with degree $d(u_1) = n/3 + o(n)$ and denote $\Gamma(u_1) \cap Y = Y_1$, $Y \setminus Y_1 = Y_2$, $\Gamma(u_1) \cap Z = Z_1$, $Z \setminus Z_1 = Z_2$. Cardinality of all four sets Y_1, Y_2, Z_1, Z_2 is $n/6 + o(n)$. Since $|U \cup Y_1| = n/2 + o(n), |U \cup Z_1| = n/2 + o(n)$, we receive that $e(U, Y_1) = n^2/36 + o(n^2), e(U, Z_1) = n^2/36 + o(n^2), e(U, Y_2) = n^2/36 + o(n^2), e(U, Z_2) = n^2/36 + o(n^2)$. Obviously, $e(Y_1, Z_1) = 0$, because $Y_1 \cup Z_1 = \Gamma(u_1)$, so from almost regularity of G we obtain that $e(Y_1, Z_2) = n^2/36 + o(n^2)$ and $e(Y_2, Z_1) = n^2/36 + o(n^2)$, that is, $Y_1 \times Z_2$ and $Y_2 \times Z_1$ are almost complete bipartite graphs. Then for $n/3 + o(n)$ vertices from U either $|(\Gamma(u) \cap Y) \setminus Y_1| = o(n)$ or $|(\Gamma(u) \cap Y) \setminus Y_2| = o(n)$ (if, for example, there exists a $u \in U$ such that $d(u) = n/3 + o(n)$, $e(u, Y) = n/6 + o(n)$, $e(u, Z) = n/6 + o(n)$, $e(u, Y_1) = \epsilon_1 n$, $e(u, Y_2) = n/6 - \epsilon_1 + o(n)$, and if $e(u, Z_1) = \epsilon_2 n$, then $e(Y, Z_2) \leq n^2/36 - \epsilon_1 \epsilon_2 n^2 + o(n^2)$, if $e(u, Z_1) = \epsilon_2 n$, then $e(Y_2, Z_1) \leq n^2/36 - (n/6 - \epsilon_1 n)\epsilon_2 n + o(n^2)$, contradicting the above assumptions). Let $U_1 = \{u \in U : |(\Gamma(u) \cap Y) \setminus Y_1| = o(n)\}$, $U_2 = \{u \in U : |(\Gamma(u) \cap Y) \setminus Y_2| = o(n)\}$, then $|U_1| = n/6 + o(n), |U_2| = n/6 + o(n)$ (otherwise there would exist ϵn vertices in Y with degrees that differ from $n/3 + o(n)$). We receive that $U_1 \times Y_1$ and $U_2 \times Y_2$ are almost complete bipartite graphs, so $e(U_1, Z_2) = o(n^2)$ (otherwise $U \cup Y_1 \cup Z_2$ would contain a triangle). Now consider a set $U_1 \cup Y_2 \cup Z_2$, we have $|U_1 \cup Y_2 \cup Z_2| = n/2 + o(n)$ and $e(U_1 \cup Y_2 \cup Z_2) = o(n^2)$ - a contradiction to our assumption. $\square$

4. Regular triangle-free graphs

Proof of Theorem 3. Pick an edge $(v_1, v_2) \in E(G)$ and let $V_1 = \Gamma(v_1), V_2 = \Gamma(v_2)$. Obviously, $|V_1| = |V_2| = D$, and V_1 and V_2 are disjoint and independent. Set

$U = V \setminus (V_1 \cup V_2)$, $|U| = n - 2D$. Let $l = |E(G)|/n^2 = D/2n$, $l_1 = e(V_1, V_2)/n^2$, $l_2 = e(U)/n^2$, $l_3 = e(V_1, U)/n^2$, $l_4 = e(V_2, U)/n^2$. The condition of regularity yields $l_3 = l_4 = l - 4l^2 - l_2$, $l_1 = l - (l_2 + l_3 + l_4) = 8l^2 - l + l_2$. Choosing randomly and uniformly a set W from among all subsets of U of size $(1/2 - 2l)n$ we conclude as in the proof of Theorem 1 that there exists such $W \subset U$ for which $(e(V_1 \cup W) + e(V_2 \cup W))/n^2 \leq l - 4l^2 - l_2/2$. But $|V_1 \cup W| = |V_2 \cup W| = n/2$, so $l - 4l^2 - l_2/2 \geq 1/25$ and taking into account the condition $l = D/2n \geq 1/5$ we obtain that $l = 1/5, l_2 = 0, l_1 = 3/25, l_3 = l_4 = 1/25$, so G is a regular graph with vertex degree $2n/5$.

It is easy to see now that for every $u \in U$ one has $e(u, V_1) = e(u, V_2) = n/5$ (otherwise $n/10$ vertices $u \in U$ having the smallest degrees $e(u, V_1)$ would give a set U_0 such that $e(U_0, V_1) < n^2/50$). Pick a $u_1 \in U$ and let $V_{11} = \Gamma(u_1) \cap V_1$, $V_{12} = V_1 \setminus V_{11}$, $V_{21} = \Gamma(u_1) \cap V_2$, $V_{22} = V_2 \setminus V_{21}$, clearly, $|V_{11}| = |V_{12}| = |V_{21}| = |V_{22}| = n/5$. Then $e(V_{11}, V_{21}) = 0$, so it follows from the regularity condition that $e(V_{11}, U) + e(V_{11}, V_{22}) = (2n/5)|V_{11}|$, so $e(V_{11}, U) = e(V_{11}, V_{22}) = n^2/25$, that is, $V_{11} \times U$ and $V_{11} \times V_{22}$ are both complete bipartite graphs. Since $e(V_1, U) = n^2/25$, we obtain $e(V_{12}, U) = 0$, so $e(V_{12}, V_{21}) + e(V_{12}, V_{22}) = (2n/5)|V_{12}|$, and $e(V_{12}, V_{21}) = e(V_{12}, V_{22}) = n^2/25$, that is, $V_{12} \times V_{21}$ and $V_{12} \times V_{22}$ are both complete bipartite graphs. Now the regularity condition completes the picture, giving $e(V_{21}, U) = (2n/5)|V_{21}| - e(V_{12}, V_{21}) = n^2/25$, and $V_{21} \times U$ is also a complete bipartite graph. We have only to write down the order of vertex classes in a blown up C_5: $U, V_{11}, V_{22}, V_{12}, V_{21}$. □

5. Proof of Conjecture 1 for $\alpha \geq 0.6$

In fact, we shall give a proof only for the case $\alpha = 0.6$. The reason is that the proof is rather technical and once written for a general α becomes too cumbersome.

Theorem 4'. *Let G be a graph of order n and let $\alpha = 0.6$. If each αn vertices of G span more than βn^2 edges, where $\beta = (2\alpha - 1)/4$, then G contains a triangle.*

Proof. We outline the proof since the ideas and techniques used are almost the same as in the proofs of Theorems 1 and 2.

Lemma ([4]). *Let G be a graph on n vertices and let $\alpha > 0$. If $\Psi(G, \alpha n) > (2\alpha - 1)/4n^2$ and G contains a vertex independent set of size $(1 - \alpha)n$ then G contains a triangle.* □

Assume that Theorem 4' fails and let G be a triangle-free graph such that $\Psi(G, \alpha n) > \beta n^2$. By the Lemma above we may assume that $d(v) < (1 - \alpha)n$ for every $v \in V(G)$. According to Lemma 1 $e(G) > (\beta/\alpha^2)n^2 = (5/36)n^2$, hence G contains a vertex independent set $U \subset V$ of size $|U| = (5/18)n$. Let $k = 5/18$, $T = V \setminus U$, $|T| = (13/18)n$ and set $l' = e(U, T)/n^2$, $l'' = e(T)/n^2, l = e(G)/n^2 = l' + l''$. Again from Lemma 1 we obtain that

$$l'' > \frac{\beta}{\alpha^2}|T|/n^2 > 0.072 \ . \tag{9}$$

Choosing $W \subset T$ such that $|W \cup U| = \alpha n$, we obtain that

$$\left(\frac{\alpha - k}{1 - k}\right)^2 l'' + \frac{\alpha - k}{1 - k} l' > \beta \ ,$$

that is,

$$\frac{29}{65}l'' + l' > 0.112 \ , \tag{10}$$

so from (9) and (10)

$$e(G) = (l' + l'')n^2 > 0.151n^2 > \frac{\alpha}{4}n^2 \ .$$

Now as in the proof of Theorem 1 fix two disjoint independent sets V_1 and V_2, such that $|V_1|/n + |V_2|/n = 4l > \alpha$ and let $U = V \setminus (V_1 \cup V_2)$. As before, let $l_1 = e(V_1, V_2)/n^2, l_2 = e(U)/n^2, l_3 = e(V_1, U)/n^2, l_4 = e(V_2, U)/n^2$. The same calculations as in Theorem 1 give

$$\frac{2\alpha - 4l}{1 - 4l}l_2 + l_3 + l_4 > 2\beta\frac{1 - 4l}{\alpha - 2l} \ , \tag{11}$$

$$l_1 > \beta\frac{2l}{\alpha - 2l} \ . \tag{12}$$

Summing up the degrees of the vertices from U and taking into account the assumption that $d(v) < (1 - \alpha)n$ for every $v \in V$ we obtain that

$$2l_2 + l_3 + l_4 < (1 - \alpha)(1 - 4l) \ . \tag{13}$$

It follows from (11) and (13) that

$$\left(2 - \frac{2\alpha - 4l}{1 - 4l}\right) l_2 < (1 - \alpha)(1 - 4l) - 2\beta\frac{1 - 4l}{\alpha - 2l} \ , \tag{14}$$

and from (11) and (12) (remembering that $l_1 + l_2 + l_3 + l_4 = l$) that

$$\frac{2\alpha - 1}{1 - 4l}l_2 > 2\beta\frac{1 - 4l}{\alpha - 2l} + \beta\frac{2l}{\alpha - 2l} - l \ . \tag{15}$$

Now, comparing (14) and (15) we obtain a quadratic inequality for l which for $\alpha = 0.6$ is as follows:

$$100l^2 - 32l + 2.6 < 0$$

- a contradiction. $\square$

6. Can the edge distribution in a triangle-free graph be uniform?

In a random graph G on n vertices with $e(G) = cn^2$ edges every $n/2$ vertices span $(1 + o(1))e(G)/4$ edges. From the other side, such a random graph contains $\Theta(n^3)$ triangles. So, if G is a triangle-free graph of order n with cn^2 edges, then it is far from being typical. From this reason we can expect that the edge distribution in a triangle-free graph is not so uniform as in a random graph with the same number of edges. (The reader can find a similar discussion on this subject in a more general context in [2]). This is indeed the case as shown by the following:

Theorem 5. *If G is a triangle-free graph on n vertices with $e(G) = cn^2$ edges, where $c > 0$ is a constant, then*

$$\Psi(G, n/2) \leq (1/4 - c')e(G) \ ,$$

where $c' = c^2/(4c^2 - 4c + 1) > 0$.

Proof. Let v_0 be a vertex of degree $\geq 2cn$ and choose a set $X \subseteq \Gamma(v_0)$ of size $|X| = 2cn$. Then X is independent. Write $Y = V \setminus X, e(Y)/n^2 = l_0$, then $e(X,Y)/n^2 = c - l_0$. We now use the same technique as in the proof of Theorem 1. Choosing randomly and uniformly a set W_1 from among all subsets of Y of size $n/2$, we see that there exists such W_1 for which $e(W_1)/n^2 \leq l_0/4(1-2c)^2 =: z_1(l_0)$. Choosing randomly and uniformly a set W_2 from among all subsets of Y of size $(n/2 - |X|)$ we receive that there exists such W_2 for which

$$\frac{e(W_2 \cup X)}{n^2} \leq \left(\frac{1-4c}{1-2c}\right)^2 + \frac{1-4c}{1-2c}\,\frac{c-l_0}{2} =: z_2(l_0).$$

So

$$\Psi(G, n/2) \leq \min\{z_1(l_0), z_2(l_0)\}n^2 \ .$$

Since $z_1(l_0)$ is an increasing linear function of l_0 and $z_2(l_0)$ is a decreasing linear function of l_0, we obtain that $\Psi(G, n/2) \leq z_1(l_0^*)n^2$, where $z_1(l_0^*) = z_2(l_0^*)$, hence $l_0^* = (1-4c)c$ and

$$\Psi(G, n/2) \leq \frac{1-4c}{(1-2c)^2}\,\frac{cn^2}{4} = \frac{cn^2}{4}\left(1 - \frac{4c^2}{4c^2 - 4c + 1}\right) \ . \qquad \square$$

Remark: we didn't make any attempt to minimize a value of $c'(c)$.

One could be bolder and ask whether there exists an absolute constant $0 < \gamma < 1$ such that for every triangle-free graph G on n vertices with $e(G)$ edges there holds $\Psi(G, n/2) < \gamma e(G)/4$? The following lemma answers this negatively.

Lemma 3. *For every $\epsilon > 0$ there exists $N(\epsilon)$ such that for every $n > N$ there exists a triangle-free graph G of order n in which for every set of vertices U of size $n/2$ there holds:*

$$|e(U) - e(G)/4| < \epsilon e(G) \ .$$

Proof. Let $n^{-1} \ll p \ll n^{-2/3}$ and consider a random graph $G(n,p)$ - a graph on n vertices in which the edges are chosen independently and with probability p. Let $u = n/2$. For any fixed set $U \subset V$ of u vertices a random variable $e(U)$ has a binomial distribution with parameters $\binom{u}{2}$ and p. So (see, e.g., [1], Th. I.7)

$$Pr[|e(U) - p\binom{u}{2}| \geq \frac{\epsilon}{2}p\binom{u}{2}] < \exp\{-\epsilon^2 p\binom{u}{2}/12\} \tag{16}$$

for sufficiently large n. Since we have $\binom{n}{u} < 2^n$ choices for U, the probability that (16) fails for some U is at most $2^n \exp\{-\epsilon^2 p\binom{u}{2}/12\} = o(1)$. A similar evaluation gives

$$Pr[|e(V) - p\binom{n}{2}| \geq \frac{\epsilon}{2}p\binom{n}{2}] = o(1) \ . \tag{17}$$

Denote the number of triangles in G by T. Then

$$M(T) = \binom{n}{3} p^3 = o(n) \ . \tag{18}$$

So by (16), (17), (18) for sufficiently large n there exists a graph G of order n with the following properties:

(1) $e(G) > n$;
(2) For every subset of vertices U of size $|U| = n/2$ there holds: $|e(U) - e(G)/4| < (\epsilon/2)e(G)$;
(3) There are $o(n)$ triangles in G.

Deleting one edge from each triangle in G, we receive a triangle-free graph G' in which for every $U \subset V, |U| = n/2$, one has $|e(U) - e(G')/4| < \epsilon e(G')$. □

The random graph from the above lemma has $o(n^2)$ edges, so is it possible that the situation is different for triangle-free graphs with cn^2 edges? The answer is again negative, as provided by

Theorem 6. *For every $\epsilon > 0$ there exists a constant $c = c(\epsilon) > 0$ such that for infinitely many n there exists a triangle-free graph G of order n with $e(G)$ edges with $e(G) > c(\epsilon)n^2$, for which*

$$\Psi(G, n/2) > (1/4 - \epsilon)e(G) \ .$$

Proof. Take a triangle-free graph G' on k vertices $a_1, \ldots, a_k$ with $e(G') = l_0 > k$ edges from the proof of Lemma 3 and for sufficiently large even number t define a new graph $H = G'[t, \ldots, t]$ as follows: each vertex a_i from G' is replaced by t new independent vertices, forming a set A_i, two vertices $v_1 \in A_i$ and $v_2 \in A_j$ are joined in H if and only if $(a_i, a_j) \in E(G')$. Set $n = kt$. Obviously, $|V(H)| = n, e(H) = t^2 e(G') = \Theta(n^2)$, and $\Psi(H, n/2)/e(H) \leq \Psi(G', k/2)/e(G')$, so to prove a theorem it is sufficient to show that $\Psi(H, n/2)/e(H) = \Psi(G', k/2)/e(G')$.

For every set $U \subset V$ define $s(U) := |\{1 \leq i \leq n : 0 < |A_i \cap U| < t\}|$ - the number of substitution classes split by U. If $s(U) = 0$ then U will be called *integral*. Let us prove that there exists an integral set $U \subset V$ of size $|U| = n/2$ such that $e(U) = \Psi(H, n/2)$ (from this result it will follow that $\Psi(H, n/2) = t^2 \Psi(G', k/2)$). Take on optimal set U of size $n/2$, $e(U) = \Psi(H, n/2)$, with the smallest value of $s(U)$. If $s(U) = 0$, then we are done, so suppose to the contrary that $s(U) > 0$. Obviously, then $s(U) \geq 2$. Consider two split classes A_1 and A_2. If $v, u \in A_1$, then $e(v, U) = e(u, U)$, so set $e(v, U) = D_i$ for $v \in A_i$, $i = 1, 2$, and suppose $D_1 \leq D_2$.

Pick two vertices $v_1 \in A_1 \setminus U$, $v_2 \in A_2 \cap U$ and consider a set $U' = U - v_2 + v_1$. Obviously, $e(U') \leq e(U) - e(v_2, U) + e(v_1, U) \leq e(U)$, so U' is also an optimal set. We can proceed with this procedure of changing vertices from $A_2 \cap U$ to those of $A_1 \cap U$ until $A_1 \setminus U$ will be empty or $A_2 \cap U$ will be empty. In both cases we shall receive an optimal set U^* with $s(U^*) < s(U)$ - a contradiction with the choice of U. □

Remark: The idea of taking a random graph with $o(n^2)$ edges and blowing it up to receive a graph with $\Theta(n^2)$ edges with desired properties was used by Erdős, E. Győri and Simonovits in [5].

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