Sharp Thresholds for Certain Ramsey Properties of Random Graphs.

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Abstract

In a series of papers culminating in [11] Rödl, Ruciński and others study the thresholds of Ramsey properties of random graphs i.e. for a given graph H, when does a random graph almost surely have the property that for every coloring of its vertices (edges) in r colors there exists a monochromatic copy of H. They prove in many cases the existence of a function $p(n, H)$ and two constants $c(H)$ and $C(H)$ such that a random graph with edge probability at most cp almost surely does not have this Ramsey property, whereas when the edge probability is at least Cp it almost surely has this property.

We complement their results by showing that in certain cases, the multiplicative gap between upper and lower bound can be closed: There exists a function $p(n)$ such that for every ε, a random graph with edge probability less than $(1-\varepsilon)p$ almost surely does not have the Ramsey property, whereas when the edge probability is at least $(1+\varepsilon)p$ it almost surely has this property.
However, this is an existence result only, since our method yields no information about the value of the function $p(n)$.

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1 Introduction

Let H and G be two graphs. The notation $G \to (H)^e_r$ ($G \to (H)^v_r$), which is commonly used in Ramsey theory, means that for every coloring of the edges (vertices) of G by r colors there exists a monochromatic copy of H.

The basic theme studied in Ramsey theory is, given a fixed H, when is G "rich" enough for $G \to H$? Here richness can be interpreted as the size of G, or the density, i.e. the ratio of edges to vertices. When studying random graphs a natural question is: Given H, find a threshold function $p(n)$ such that $G(n,p) \to H$ ("G has the Ramsey property") almost surely when $p(n) = o(p)$ and $G(n,p) \not\to H$ almost surely when $p = o(p(n))$. (The existence of such a threshold function is guaranteed by a general result of Bollobás and Thomason [3].) In a series of papers ([6], [7], [8], [9], [10], [11]) the asymptotic value of the threshold function $p(n)$ is calculated for most graphs H and in the vertex coloring case for most hypergraphs. For the edge coloring case in hypergraphs such a theorem is proven where H is the complete 3-uniform hypergraph on 4 vertices. We will bring the exact formulation of these theorems shortly.

Using these results, and a technique from a recent paper by the first author we will demonstrate in certain cases a sharper concentration result as to the critical value of the edge probability for the Ramsey property of a random graph. We will show the existence of a function $p(n)$ such that for every fixed $\varepsilon > 0$, $G(n,(1-\varepsilon)p)$ almost surely does not have the Ramsey property, whereas $G(n,(1+\varepsilon)p)$ almost surely has the Ramsey property. This too is formulated in Theorems 1.3 and 1.4 below. Here is some notation we will use: Whenever we say a property holds in $G(n,p)$ almost surely we mean that the probability of it holding tends to 1 as n tends to infinity. For a graph H let e_H be the number of edges of H, v_H be the number of vertices. For any graph G define $m^1_G = \max_{H \subseteq G, v_H \geq 2} \frac{e_H}{v_H - 1}$ and $m^2_G = \max_{H \subseteq G, v_H \geq 3} \frac{e_H - 1}{v_H - 2}$.
Let us start by quoting a result of Łuczak, Ruciński and Voigt:

Theorem 1.1 ([7]) *For every integer r, $r \geq 2$ and for every graph H, which in case $r = 2$ is not a matching, there exist constants c and C such that*

$$\lim_{n \to \infty} Pr[G(n,p) \to (H)^v_r] = \begin{cases} 1 & \text{if } p > Cn^{-1/m^1_H} \\ 0 & \text{if } p < cn^{-1/m^1_H} \end{cases} \tag{1}$$

This theorem was later generalized to the vertex coloring case in hypergraphs by Rödl and Ruciński [11], who also proved the following:

Theorem 1.2 (**[8]**,**[10]**) *For every integer r, $r \geq 2$, and for every graph H which is not a star forest there exist constants c and C such that*

$$\lim_{n\to\infty} Pr[G(n,p) \to (H)^e_r] = \begin{cases} 1 & \textit{if } p > Cn^{-1/m^2_H} \\ 0 & \textit{if } p < cn^{-1/m^2_H} \end{cases} \tag{2}$$

It should be pointed out that Rödl and Ruciński overlooked an exception to Theorem 1.2: for $G = P_4$, the path consisting of 3 edges the theorem does not hold with $r = 2$. We will elaborate on this case later. The hypergraph analog to Theorem 1.1 is an open conjecture, proven only for the case where G is the complete 3-uniform hypergraph on 4 vertices, see [11].

A graph G is called balanced if the average degree of G is no smaller than that of any subgraph. It is called strictly balanced if its average degree is strictly larger than that of any proper subgraph. A lesser known but even stronger property is the property sometimes called in the literature *strongly strictly balanced* or *strictly K_1 balanced*: we say a graph G with at least three vertices is strongly strictly balanced, (henceforth SSB), if for any subgraph H such that $v_G > v_H > 1$ one has $\frac{e_G}{v_G-1} < \frac{e_G-e_H}{v_G-v_H}$. First note that a simple calculation shows that this indeed implies being strictly balanced. Trees are an example of a graph that is strictly balanced but not SSB. Cliques are examples of SSB graphs. Note that for a SSB graph $m_1(G) = \frac{e_G}{v_G-1}$. The idea behind this definition is as follows:

Let G be SSB, H be a non-empty subgraph of G, and $p \sim n^{-1/m^1_G}$. Now, given H_0, a fixed copy of H that appears in $G(n,p)$, one can compute the expected number of copies of G that appear in $G(n,p)$ and contain H_0 as a subgraph (conditioned on the appearance of H_0.) It turns out that this number is a constant if H consists of a single vertex, but is $o(1)$ for any other H. For example if G is a triangle and $p = n^{-2/3}$ then the expected number of triangles containing a given vertex in $G(n,p)$ is constant, whereas the expected number of triangles containing a given edge, even conditioning on the appearance of that edge in $G(n,p)$, is $o(1)$.

The main results in this paper are as follows:

Theorem 1.3 *For every graph H that is SSB and for every integer r, $r \geq 2$, there exists a function $p(n)$ such that for every $\varepsilon > 0$*

$$\lim_{n\to\infty} Pr[G(n,p) \to (H)^v_r] = \begin{cases} 1 & \textit{if } p > (1+\varepsilon)p(n) \\ 0 & \textit{if } p < (1-\varepsilon)p(n) \end{cases} \tag{3}$$

Theorem 1.4 *For every integer r, $r \geq 2$, and for every tree T, which is not a star and in the case of $r = 2$ not P_4 (a path of three edges), there exists a function $p(n)$ such that* *for every* $\varepsilon > 0$

$$\lim_{n\to\infty} Pr[G(n,p) \to (T)^e_r] = \begin{cases} 1 & \textit{if } p > (1+\varepsilon)p(n) \\ 0 & \textit{if } p < (1-\varepsilon)p(n) \end{cases} \tag{4}$$

Naturally in both these theorems the value of $p(n)$ is bounded from above and below by the values found in Theorems 1.1 and 1.2.

2 Sharp Thresholds of Monotone Graph Properties

In this section we wish to introduce the notion of a *sharp threshold* of a graph property, and introduce the technique we use to prove sharpness of the thresholds for the Ramsey properties. Consider a graph property $\mathcal{R}$ of graphs on n vertices. We will restrict ourselves to properties which are monotone (i.e. preserved under addition of edges) and invariant under graph automorphisms. For such a property let

$$\mu(p) = Pr[G(n,p) \in \mathcal{R}].$$

Thus $\mu(p)$ is a monotone function of $p \in [0,1]$. Fix a small $\varepsilon > 0$, and let p_0 and p_1 be such that

$$\mu(p_0) = \epsilon,$$
$$\mu(p_1) = 1 - \epsilon.$$

We will call $[p_0, p_1]$ the threshold interval. Define the threshold length $\delta(\epsilon) = p_1 - p_0$. We will denote by p_c a certain critical value of p in this interval, which when not further specified may be considered to be an arbitrary point. Bollobás and Thomason proved the existence of threshold functions for all monotone set properties ([3]), and in particular showed that $\delta = O(p_c)$ for all graph properties. In a recent paper ([5]) the first author gives a necessary and sufficient condition for $\delta(\epsilon) = o(p_c)$ for all ϵ, in which case we say that the property in question has a *sharp threshold*, otherwise we say it has a *coarse threshold*. This condition will be the central tool used in this paper. Of course, for the asymptotic notation to make sense one must deal with a property that is meaningful for all large enough n. In our case Theorems 1.3 and 1.4 mean exactly that the Ramsey property has a sharp threshold. Roughly stated, the condition presented in [5] for a property to have a sharp threshold is that the property is not usually determined by a simple local reason. For example, the property of having a triangle as a subgraph is obviously "local", and indeed has a coarse threshold (both the critical probability and the length of the threshold interval are of order $1/n$), whereas it seems obvious that connectivity is a non-local property, and, indeed, it is well known that the property of connectivity has a sharp threshold.

Let us now present the conditions for a coarse threshold. Theorem 2.1 below gives the precise conditions, however the version we will be using in this paper is Corollary 2.3. First note that for the threshold interval to be large there must be a point in the threshold interval, let us denote this point by $p_c(n)$, such that the slope of the function $\mu(p)$ is small there. Properly normalized this translates to the fact that there is a constant K such that $\frac{d\mu}{dp}|_{p=p_c} \cdot p_c < K$, for infinitely many values of n. For a graph H let $Exp(H)$ denote the expected number of copies of H in $G(n,p)$. Recall that a balanced graph is one where the average degree is no less than that of any subgraph. Let $(G(n,p) \cup G)$ denote the graph obtained by adding a fixed copy of G to $G(n,p)$ (adding the edges to vertices $v_1, \ldots, v_{v_G}$.) The condition characterizing coarse thresholds is as follows:

Theorem 2.1 ([5]) *Let $\alpha > 0$. There exist functions $0 < b_1(\epsilon, K) < b_2(\epsilon, K)$ and $B(\epsilon, K)$ such that for all $K > 0$, all n and p and any monotone symmetric family of graphs $\mathcal{R}$ on n vertices such that $p \cdot \frac{d\mu(p)}{dp} \leq K$ and $\alpha < \mu_p(\mathcal{R}) < 1 - \alpha$, for every $\epsilon > 0$ there exists a graph G^* with the following properties:*

- *G^* is balanced*
- $|V(G^*)| < B$
- $b_1 < Exp(G^*) < b_2$
- $Pr\left[(G(n,p) \cup G^*) \in \mathcal{R}\right] > 1 - \epsilon.$

Implicit in [5] is the analogous statement for hypergraphs.

Remark 2.2 *Note that by symmetry the copy of G^* may either be placed on a fixed set of vertices or on a random set. The resulting probability space is isomorphic. In this paper we will use both possibilities.*

For a function $p = p(n)$ we will define a graph G to be *modest* with respect to p if it is balanced and there exist constants $0 < b_1 < b_2$ such that $b_1 < Exp(G) < b_2$ for all values of n.

The following corollary translates Theorem 2.1 to a condition we will use in this paper:

Corollary 2.3 *If the property $G(n,p) \to H$ has a coarse threshold, then there exist a function $p_c(n)$, a graph G^* modest for $p_c(n)$, and a constant $\epsilon > 0$ so that for infinitely many values of n one has:*

1. $\epsilon < Pr[G(n,p_c) \to H] < 1-\epsilon$;

2. $Pr[G(n,p_c) \cup G^* \to H] - Pr[G(n,p_c) \cup G(n,\epsilon p_c) \to H] \geq \epsilon$.

Let us explain some of the notation and the meaning of corollary 2.3:
$G(n,p_c) \cup G(n,\epsilon p_c)$ denotes the union of two random graphs taken on the same set of vertices.

The graph $G(n,p_c) \cup G^*$ is formed as follows: we will fix ordering of the vertices of G^*, choose at random an ordered set of $|V(G^*)|$ vertices in $G(n,p_c)$ and add the corresponding edges. The probability statement regarding the outcome of this procedure (i.e. $Pr[G(n,p_c) \cup G^* \to H]$) is with respect to the corresponding probability space.

The first point in the above corollary means that we are looking at the threshold interval, whereas the second combines the fact that there is a coarse threshold, (and hence adding $G(n,\epsilon p_c)$ has small effect for small ϵ) with the existence of the special graph guaranteed by Theorem 2.1.
Our strategy in proving a sharp threshold will be to compare the effect of adding to $G(n,p)$ one fixed copy of the modest graph G^* (guaranteed to exist by Corollary 2.3 in the case of a coarse threshold) to the effect of adding a large set of random edges, thus showing the conclusions of the corollary cannot hold.

3 The Vertex Coloring Case.

This section is dedicated to the proof of Theorem 1.3. The case where the graph H consists of a single edge is the case of usual proper coloring of a graph. This case was treated in [1], where it is shown that for any constant $r > 2$ the property of being non-r-colorable has a sharp threshold.

We will use Corollary 2.3 to prove Theorem 1.3. The idea is as follows: for large enough n Corollary 2.3 implies the existence of a single graph G_0, typical in an appropriate sense in $G(n,p_c)$, such that G_0 is not Ramsey with respect to H (i.e. $G_0 \not\to H_r^v$) and such that adding a random copy of G^* to G_0 has significantly more impact on the property of the resulting graph being Ramsey with respect to H than adding the edge set of $G(n,\epsilon p_c)$. Then we show that this leads to a contradiction.
Remark: To remove a doubt which may have already risen in the mind of the astute reader: part of the proof must show that G^* itself does not have the Ramsey property, or else no contradiction will arise. Something even stronger than this is proven in Lemma 3.7. This is a typical state of affairs when proving that a property has a sharp threshold. We want to show that the property is not "local". It is relatively simple to rule out trivial local reasons (e.g. the existence of a modest graph which is Ramsey) and most

of the effort is in proving that there do not exist subtle local effects, i.e. small local pertubations resulting in a global change of the graph's properties.

We now need a long list of definitions, notions and notations. Let $r \geq 2$ be a fixed integer throughout this section, $0 < \epsilon < 1$ be a fixed constant and H be a fixed SSB graph. We will say a graph G has the Ramsey property (or "G is Ramsey") if $G \to H_r^v$. A coloring of the vertices of a graph with no monochromatic copy of H will be called a *proper coloring.*

Assume that Theorem 1.3 fails for a fixed H. Then we can apply Corollary 2.3. By the above mentioned result of Łuczak, Ruciński and Voigt [7] we know that the critical probability $p = p_c(n)$ from Corollary 2.3 has order $p \sim n^{-1/m_H^1}$. Let G^* be the "influential" graph whose existence is guaranteed by Corollary 2.3. Throughout this section we denote $|V(H)| = k$, $|V(G^*)| = j$. We also fix arbitrary orderings of the vertices of H and G^*.

For a fixed graph G we will say that an ordered j-tuple of vertices $(v_{i_1}, \ldots, v_{i_j})$ is *bad* if ($G \cup$ a copy of G^* added on $(v_{i_1}, \ldots, v_{i_j})) \to H$. If $G \to H$ then every j-tuple is bad. We denote by $\mathcal{B} = \mathcal{B}(G)$ the family of all bad j-tuples.
An ordered j-tuple $(A_1, \ldots, A_j)$ of vertex disjoint (henceforth v.d. for brevity) subsets of $V(G)$ is called *$\mathcal{B}$-complete* if for every choice of $v_i \in A_i$, $1 \leq i \leq j$, the resulting ordered j-tuple $(v_1, \ldots, v_j)$ belongs to $\mathcal{B}$. We say that a family of v.d. k-sets $\{A_1, \ldots, A_C\}$ *hits* $\mathcal{B}$ if there exists an ordered subset of indices $i_1, \ldots, i_j \in \{1, \ldots, C\}$ such that $(A_{i_1}, \ldots, A_{i_j})$ forms a $\mathcal{B}$-complete system.

For a given graph G a subset $V_0 \subset V(G)$ is called *extendible* to H in G if after turning V_0 into a complete graph, the resulting graph on $V(G)$ contains a copy of H with at least two vertices inside V_0 and at least one vertex outside V_0. Also, we call V_0 *empty* in G if the set of edges it spans is empty.

Let C be an integer and G be a graph. Choose uniformly at random C v.d. ordered k-sets of vertices $A_1, \ldots, A_C$ and add on each of them a copy of H according to the fixed ordering of its vertices. Denote the resulting graph by G'. Let $Q_1 = Q_1(G), Q_2 = Q_2(G), Q_3 = Q_3(G)$ be the following three events:

$$\begin{aligned}
Q_1 &= (G' \not\to H). \\
Q_2 &= (\bigcup_{i=1,\ldots,C} A_i \text{ is empty and not extendible to } H \text{ in } G). \\
Q_3 &= (\{A_1, \ldots, A_C\} \text{ hits } \mathcal{B}(G)).
\end{aligned}$$

Let

$$Q = Q(G) = Q_1 \wedge Q_2 \wedge Q_3. \tag{5}$$

Our proof rests on the following two lemmas which of course immediately show that the Ramsey property has a sharp threshold:

Lemma 3.1 *If the Ramsey property has a coarse threshold, then there exist a graph G_0 and an integer C such that $Pr[Q(G_0)] > 0$, where the probability is over the random choices of $A_1, \ldots, A_C$.*

Lemma 3.2 *For any integer C and graph G one has $Pr[Q(G)] = 0$, i.e. it is impossible for Q_1, Q_2 and Q_3 to occur simultaneously.*

3.1 Proof of Lemma 3.1

Our main technical tool is an ordered version of a theorem of Erdős and Simonovits about supersaturated hypergraphs [4]. Let $\mathcal{B}$ be a family of ordered j-tuples of distinct elements of $\{1, \ldots, n\}$. Recall that an ordered j-tuple $(A_1, \ldots, A_j)$ of pairwise disjoint subsets of $\{1, \ldots, n\}$ is called *$\mathcal{B}$-complete* if for every choice of $v_i \in A_i$, $1 \leq i \leq j$, the resulting ordered j-tuple $(v_1, \ldots, v_j)$ belongs to $\mathcal{B}$. We denote the number of ordered j-tuples in $\{1, \ldots, n\}$ by $(n)_j = n(n-1)\ldots(n-j+1)$.

Lemma 3.3 *For every pair of integers $j, k \geq 2$ and for every constant $\epsilon > 0$ there exist a constant $\epsilon' = f(\epsilon, j, k) > 0$ and an integer $n_0 = g(\epsilon, j, k)$ such that for every $n \geq n_0$ the following holds. Let $\mathcal{B}$ be a family of ordered j-tuples of distinct elements of $\{1, \ldots, n\}$ of cardinality $|\mathcal{B}| \geq \epsilon(n)_j$. If an ordered j-tuple $(A_1, \ldots, A_j)$ of disjoint subsets of size k of $\{1, \ldots, n\}$ is chosen uniformly at random, then $(A_1, \ldots, A_j)$ is $\mathcal{B}$-complete with probability at least ϵ'.*

The proof of the above statement is essentially the same as that of the original theorem, and we omit it.

For given ϵ, j, k we define $\epsilon' = f(\epsilon/2, j, k)$ and require in the sequel that the number of vertices n satisfies $n \geq g(\epsilon/2, j, k)$. Here the functions f and g come from the formulation of Lemma 3.3. We also set C to be the minimal integer divisible by j for which $(1-\epsilon')^{C/j} \leq \epsilon/8$.

As H is strongly strictly balanced (and therefore balanced) and the edge probability p_c satisfies $p_c \sim n^{-1/m_H^1}$, the expected number of copies of H in $G(n, \epsilon p_c)$ is linear in n. By well known results (see, e.g., [2]) $G(n, \epsilon p_c)$ contains almost surely a linear in n number of copies of H. It is also easy to show that in fact $G(n, \epsilon p_c)$ contains almost surely a linear number of *vertex disjoint* copies of H. In particular, for large enough n we have:

$$Pr[G(n, \epsilon p_c) \text{ does not contain } C \text{ v.d. copies of } H] \leq \frac{\epsilon}{8}\ . \qquad (6)$$

Now we use the full strength of our assumption about H being strongly strictly balanced (SSB). Recall that for a given graph $G=(V,E)$ on n vertices, a subset $V_0 \subset V$ is called *extendible* to H in G if after turning V_0 into a complete graph the resulting graph on n vertices contains a copy of H with at least two vertices inside V_0 and at least one vertex outside V_0. Also, V_0 is *empty* if $G[V_0]$ has an empty set of edges. Let $P=P(G)$ be the following property of graphs on n vertices:
We will say a graph G has property P if

$$Pr[Ck \text{ random vertices of } G \text{ are empty and non-extendible to } H] \geq 1-\frac{\epsilon}{8}\ .$$

As H is strongly strictly balanced and $p_c \sim n^{-1/m_H^1}$, a random subset of $G(n,p_c)$ of a fixed size is almost surely non-extendible to H. Indeed, let H_0 be a subgraph of H with $|V(H_0)|=k_0$ vertices, where $2 \leq k_0 < k$. Then, for a fixed subset $F \subset V(G)$ of a constant size, the probability that $G(n,p)$ contains a copy of H, whose intersection with F is exactly H_0, is at most $O(n^{k-k_0}p^{|E(H)|-|E(H_0)|} = O(n^{k-k_0-\frac{(|(E(H)|-|E(H_0)|)k}{|E(H)|-1}})$. Recalling the definition of an SSB graph, we conclude that the exponent of n in the last expression is negative, implying that F is almost surely non-extendible to H. Also, as the number of edges in $G(n,p_c)$ is almost surely sub-quadratic in n, a random subset of a fixed size is almost surely empty. Therefore, for large enough n we get:

$$Pr[(G(n,p_c) \text{ does not have } P] \leq \frac{\epsilon}{2}\ . \tag{7}$$

Lemma 3.4 *Let the conclusions of Corollary 2.3 hold. Let also $p \sim n^{-1/m_H^1}$. Then for infinitely many n there exists a graph G_0 on n vertices, having the following properties:*

1. *$G_0 \not\to H$;*
2. *G_0 has property P;*
3. *$Pr[G_0 \cup G^* \to H] - Pr[G_0 \cup G(n,\epsilon p_c) \to H] \geq \frac{\epsilon}{2}$.*

Here the probability space $G_0 \cup G^*$ is created by placing a random copy of G^* on top of G_0. (Recall Remark 2.2 about the equivalence of adding a fixed or a random copy of G^* to $G(n,p)$.) Similarly, $G_0 \cup G(n,\epsilon p_c)$ is obtained by adding edges of the random graph $G(n,\epsilon p_c)$ to those of G_0.

Proof: We have

$$\begin{aligned}
\epsilon &\leq Pr[G(n,p_c)\cup G^* \to H] - Pr[G(n,p_c)\cup G(n,\epsilon p_c) \to H]\\
&\leq \sum_{G_0 \text{ has } P} Pr[G(n,p_c)=G_0](Pr[G_0\cup G^* \to H] - Pr[G_0 \cup G(n,\epsilon p_c) \to H])\\
&\quad + Pr[G(n,p_c) \text{ does not have } P]
\end{aligned}$$

Therefore by (7)

$$\sum_{G_0 \text{ has } P} Pr[G(n,p_c) = G_0](Pr[G_0 \cup G^* \to H] - Pr[G_0 \cup G(n,\epsilon p_c) \to H]) \geq \epsilon/2 \ .$$

Then there exists a graph G_0 which possesses P and for which

$$Pr[G_0 \cup G^* \to H] - Pr[G_0 \cup G(n,\epsilon p_c) \to H] \geq \epsilon/2.$$

In particular, $Pr[G_0 \cup G(n,\epsilon p_c) \to H] \leq 1 - \epsilon/2$, implying that $G_0 \not\to H$.

□

Recall that the event Q was defined as the intersection of the events Q_1, Q_2, Q_3 defined in (5). The following lemma shows that for a graph G_0 such as described by Lemma 3.4 there is a positive probability of all three occurring simultaneously.

Lemma 3.5 *Let G_0 be the graph guaranteed by Lemma 3.4. Then*

1. $Pr[Q_1(G_0)] > \frac{\epsilon}{4}$.
2. $Pr[Q_2(G_0] \geq 1 - \frac{\epsilon}{8}$.
3. $Pr[Q_3(G_0)] \geq 1 - \frac{\epsilon}{8}$.

In particular $Pr[Q] > 0$.

Proof of lemma 3.5:

1. conditioning on the event that $G(n,\epsilon p_c)$ contains C v.d. copies of H, these copies are uniformly distributed. Therefore, taking into account (6) and Part 3 of Lemma 3.4, we get

$$\begin{aligned} Pr & \ [G_0 + C \text{ random v.d. copies of } H \not\to H] \\ \geq & \ Pr[G_0 \cup G(n,\epsilon p_c) \not\to H | G(n,\epsilon p_c) \text{ contains } C \text{ v.d. copies of } H] \\ \geq & \ Pr[G_0 \cup G(n,\epsilon p_c) \not\to H] - \\ & \ Pr[G(n,\epsilon p_c) \text{ does not contain } C \text{ v.d. copies of } H] \\ \geq & \ \frac{\epsilon}{2} - \frac{\epsilon}{8} > \frac{\epsilon}{4} \end{aligned} \tag{8}$$

(we use the inequality $Pr[A|B] \geq Pr[A \cap B] \geq Pr[A] - Pr[\bar{B}]$).
This proves the first assertion of the lemma.

2. Note that the set of vertices on which C random v.d. copies of H lie form a randomly chosen subset of size Ck. Recalling that G_0 has property P, we derive the second part of the lemma:

$$Pr\left[\begin{array}{c}\text{union of } C \text{ random v.d. copies of } H \text{ is empty} \\ \text{and non-extendible to } H \text{ in } G_0\end{array}\right] \geq 1 - \frac{\epsilon}{8}\ . \tag{9}$$

3. Now is the point in our proof where we use the nice Erdős-Simonovits theorem. This usage may tend to be less noticeable by the reader when surrounded by all the calculations and technicalities, but we wish to stress that the proof is tailored especially to use this essential notion, without which we were not able to reach our goal.

   Recall the definition of $\mathcal{B}$, the family of all bad sets of vertices in $V(G_0)$, those on which placing a copy of G^* results in a Ramsey graph. Returning again to Part 3 of Lemma 3.4, we see that

$$Pr[G_0 \cup G^* \to H] \geq \epsilon/2$$

   therefore, $|\mathcal{B}| \geq (\epsilon/2)(n)_j$.
   When placing C random copies of H on top of G_0, we can place them in groups of size j, group after group, trying to hit $\mathcal{B}$. From Lemma 3.3 the probability of each such group being $\mathcal{B}$-complete, (thus implying that $\{A_1, \ldots, A_C\}$ hits $\mathcal{B}$,) is at least ϵ'. Thus from the definition of C and ϵ'

$$Pr[C \text{ random copies of } H \text{ do not hit } \mathcal{B}] \leq (1-\epsilon')^{\frac{C}{j}} \leq \frac{\epsilon}{8}\ . \tag{10}$$

Equations (8), (9) and (10) are the assertions of Lemma 3.5, thus we have completed the proof of the existence of the graph G_0 of Lemma 3.1 under the assumption of a coarse threshold.

□

3.2 Proof of Lemma 3.2

Our first step in the proof is to observe that as G^* is modest for p_c and H is strongly strictly balanced, it is reasonably easy to color the vertices of G^* without creating a monochromatic copy of H.

Definition 3.6 *A graph G is called Ramsey-2-choosable (or Ramsey-2-choosable with respect to H) if for every assignment of a list of two colors to each vertex of G there exists a coloring of the vertices of G which assigns to each vertex a color from its list in such a way that there is no monochromatic copy of H.*

Lemma 3.7 *Let H be strongly strictly balanced and let $p = cn^{-1/m^1_H}$. Let G^* be modest for this value of p. Then G^* is Ramsey-2-choosable with respect to H.*

Proof: Let $m = m^1_H$. For a vertex $v \in H$ let $d(v)$ denote the degree of v in H. From the fact that H is SSB it follows easily that $d(v) > m$ for any vertex v in H. On the other hand, from the modesty of G^* it follows that $e_{G^*}/v_{G^*} = m$ and that this ratio is no larger for any subgraph of G^*. Hence G^* has a vertex of degree at most $2m$, and the same is true for every subgraph of it (i.e. G^* is $2m$-degenerate.) Hence one can order the vertices of G^*: $u_1, u_2, \ldots, u_j$ so that for each vertex the number of predecessors in this ordering who are its neighbors in G^* is no more than $2m$. Now, given lists of two colors for each vertex in G^* it is easy to supply a proper coloring: color the vertices inductively according to the chosen ordering. Assume that the color list of the current vertex u_i contains colors c_1 and c_2. If coloring u_i in c_1 (c_2, resp.) creates a monochromatic copy of H in G^*, then u_i is connected with at least $m+1$ vertices which have already been colored in c_1 (c_2, resp.) (recall that the minimal degree in H exceeds m). But then u_i is connected to more than $2m$ predecessors – a contradiction. This shows that at least one color in $\{c_1, c_2\}$ is available for coloring u_i.

□

We now are in a position to show that for any C and G_0 the event $Q = Q[G_0]$ is an impossibility. Let C be an integer and G_0 a graph. Assume the probability of Q is positive. This means there exists a placement of C v.d. copies of H on top of G_0, resulting in a graph G'_0 such that Q_1, Q_2, Q_3 hold. Let us fix such a placement and let the vertex sets of the C copies of H be denoted by $A_1, \ldots, A_C$. By the definition of Q_3 these C copies hit $\mathcal{B}$. (Recall that $\mathcal{B}$ is the family of all bad sets of vertices in $V(G_0)$, those on which placing a copy of G^* results in a Ramsey graph.) Hence there exists an ordered j-subset of $\{1, \ldots. C\}$, say $\{1, \ldots, j\}$, so that $(A_1, \ldots, A_j)$ is $\mathcal{B}$-complete.

As $G'_0 \not\to H$, there exists a proper coloring $\phi : V(G'_0) \to \{1, \ldots, r\}$. According to the choice of ϕ, none of the sets $A_1, \ldots, A_j$ is monochromatic under ϕ. For $1 \le i \le j$, let S_i be (some) list of two colors used by ϕ on A_i. By Lemma 3.7 G^* can be colored from the lists $\{S_i\}_{i=1}^j$ without creating a monochromatic copy of H. Let $\psi : \{1, \ldots, j\} \to \{1, \ldots, r\}$ be such a coloring. For $1 \le i \le j$ let v_i be a vertex of A_i, colored by ϕ in color $\psi(i)$ (such a vertex exists by the definition of the lists S_i). Recall that the j-tuple $(A_1, \ldots, A_j)$ is

$\mathcal{B}$-complete. Therefore $(v_1, \ldots, v_j)$ is bad, and when adding to G_0 an ordered copy of G^* on those vertices $(v_1, \ldots, v_j)$, we create a graph G_0'', which is Ramsey with respect to H. However, we will see now that ϕ gives a proper coloring of G_0'', which is a contradiction.

By definition ϕ does not create monochromatic copies of H in G_0. By our choice of $(v_i)_{i=1}^{j}$ it follows that ϕ induces the coloring ψ on the copy of G^* supported by them (i.e., $\phi(v_i) = \psi(i)$). Recalling the definition of ψ and the fact that $\{v_1, \ldots, v_j\}$ spans an empty set in G_0 by Property Q_2, we conclude that $G_0''[\{v_1, \ldots, v_j\}]$ does not contain a monochromatic copy of H. The only remaining chance to get a monochromatic copy of H in G_0'' under ϕ is to take some (at least two) vertices from $\{v_1, \ldots, v_j\}$ and some (at least one) from outside. But the union of $A_1, \ldots, A_j$ is not extendible to H in G_0. Hence this possibility also fails. Thus, ϕ is a coloring of G_0'' without a monochromatic copy of H, contradicting our previous assumptions. The proof of Lemma 3.2 and Theorem 1.3 is complete.

$\square$

4 The Edge Coloring Case

In this section we will prove Theorem 1.4. First note that if T is a star of degree k then, by the pigeon hole principle, for any r a graph is Ramsey if it has a vertex of degree at least $r(k-1)+1$. One can use this to show that this Ramsey property has a coarse threshold. We will refer to the case of P_4 later. Our general strategy will be similar to that of the previous section: we will compare the effect of adding a fixed graph to $G(n,p)$ to that of adding a large number of random edges and show that the conditions of Corollary 2.3 cannot hold.

Proof: For a start let us deal with the case of $r = 2$, two colors which we will call black and white, and T, the tree defining the property not being a path (and as stated in the conditions of the theorem, not a star.)

Let us assume the conditions of Corollary 2.3 hold. Note that by Theorem 1.2, $p = p_c$ is of order $1/n$. Therefore a modest graph G^* must be a disjoint union of unicyclic components. Let $|V(G^*)| = j$, and let the maximal degree in G^* be d. Partition the vertices of $G(n,p)$ into two sets: $V = \{v_1, \ldots, v_j\}$, the vertices on which we will build G^*, and V', the rest. Let $G(V', p)$ be the restriction of $G(n,p)$ to V'.

Remark 4.1 *Throughout this section we will be considering graphs which are the result of a random construction. These graphs will always have $G(V', p)$ as a subgraph, but often the construction will consist of several independent stages, the first of which will be the construction of $G(V', p)$ followed by stages where additional edges or vertices are* *added in either a random or a deterministic manner. For any resulting such graph R we will use the notation $Adv(R)$, (short for the advantage of R) to denote*

$$Adv(R) = Pr[R \rightarrow T | G(V', p) \not\rightarrow T].$$

The probabilities are taken with respect to the process that generates R. All probabilities and expectations calculated in this section are conditional, with the condition $G(V', p) \not\rightarrow T$).

The proof will consist of a series of implications, beginning with the assumption that the Ramsey property has a coarse threshold, and ending with a contradiction. We will now present a series of statements, and show that each of them follows from the previous ones and from the assumption that the threshold is coarse. These statements and implications are understood to hold for infinitely many values of n, and therefore we may assume when needed that n is sufficiently large. The notation used in the statements will be explained later, for now this list may be considered by the reader as an outline of the steps of the proof. Here is the list of statements:
Assume the Ramsey property (with respect to T) has a coarse threshold. Then there exist a constant $\epsilon > 0$, a function $p = p_c(n)$, a graph G^* which is modest for $p_c(n)$ and infinitely many values of n such that:

1. $Adv(G(V', p) \cup G(V', \epsilon p)) \leq 1 - \epsilon$.
2. $Adv(G(n, p) \cup G^*) \geq \epsilon$.
3. $Adv(G(n, p) \cup G^* \text{ with canonical coloring}) \geq \epsilon$.
4. $Adv(G(V', p) + \text{ white stars } + \text{ black stars}) \geq \frac{6}{7}\epsilon$.
5. $Adv(G(V', p) + \text{ one large white star } + \text{ one large black star}) \geq \frac{5}{7}\epsilon$.
6. $Adv(G(V', p) + \text{ one large white star } + \text{ many large trees}) \geq \frac{3}{7}\epsilon$.
7. $Adv(G(V', p) + \text{ white star } \cup G(V', \delta p)) \geq \frac{2}{7}\epsilon$.
8. $Adv(G(V', p) \cup G(V', \delta p) \cup G(V', \delta p)) \geq \frac{1}{7}\epsilon$.
9. $Adv(G(V', p) \cup G(V', \epsilon p)) \geq 1 - \epsilon/2$.

Note that the first and last statements are contradictory as promised.
The "nibbling" in the probabilities on the right hand side of statements 3-8 will be achieved by repeated use of

$$Pr[A|B] \geq Pr[A \cap B] \geq Pr[A] - Pr[\bar{B}]. \tag{11}$$

We now proceed to explain the statements one by one and for each of them prove that they follow from our initial assumption.
Statements (1) and (2) : In statement (2) we refer to a fixed copy of G^* added on the vertices outside V'. Let $p = p_c$. Let us use the notation

$$P = [G(n,p) \cup G^* \to T]$$

and

$$Q = [G(n,p) \cup G(n,\epsilon p) \to T].$$

Corollary 2.3 gives

$$Pr[P] - Pr[Q] \geq \epsilon.$$

But

$$\begin{aligned} Pr[P] &= Pr[G(V',p) \to T] \cdot Pr[G(n,p) \cup G^* \to T | G(V',p) \to T] \\ &+ Pr[G(V',p) \not\to T] \cdot Pr[G(n,p) \cup G^* \to T | G(V',p) \not\to T] \end{aligned}$$

and

$$\begin{aligned} Pr[Q] &= Pr[G(V',p) \to T] \cdot Pr[G(n,p) \cup G(n,\epsilon p) \to T | G(V',p) \to T] \\ &+ Pr[G(V',p) \not\to T] \cdot Pr[G(n,p) \cup G(n,\epsilon p \to T | G(V',p) \not\to T]. \end{aligned}$$

Taking the difference between these two expressions and dividing by $Pr[G(V',p) \not\to T]$ we get

$$Adv(G(n,p) \cup G^*) - Adv(G(n,p) \cup G(n,\epsilon p)) \geq \epsilon.$$

This implies statement (2) immediately. Similarly we get

$$Adv(G(n,p) \cup G(n,\epsilon p)) \leq 1 - \epsilon.$$

Statement (1) now follows simply from the observation that if $G \not\to T$ then $G' \not\to T$ for any $G' \subseteq G$.
Statement (3): First, for every graph with a partial coloring of its edges we will say that the graph is Ramsey if in every coloring of the graph consistent with the partial coloring there is a monochromatic copy of T. We may assume without altering our calculations that $G(n,p)$ has no edges within the vertex set of G^*, as this happens with Probability $1 - o(1)$. We now define a coloring of the edges of G^* and of the edges adjacent to it. We refer to this coloring as the *canonical* coloring. It has the property that it does not create any monochromatic copies of T. Recall that the modesty of G^* implies that it is a disjoint union of unicyclic components. We will assume from here on that G^* is connected, as

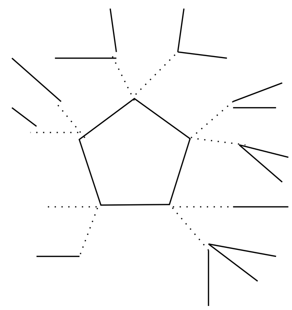

Figure 1: The coloring of G^* and the adjacent edges.

this will not result in loss of generality. The canonical coloring is as follows: color the edges of the cycle in G^* black, all those adjacent to those white, all those adjacent to those black, etc. Except for the black cycle all other monochromatic components formed by this partial coloring are stars. Therefore indeed there are no monochromatic copies of T in the canonical coloring. Statement (3) should now be clear: the event in question is whether every coloring of $G(n,p_c) \cup G^*$ that is consistent with the canonical coloring has a monochromatic copy of T. Since these colorings are a subset of all colorings the probability of this must be at least as large as that of the same statement with no restrictions.
Statement (4): The intuition behind the passage from statement (3) to statement (4) is that once one colors G^* and the adjacent edges by the canonical coloring, this forms one monochromatic cycle that will not be part of any monochromatic copy of T (it is black and all edges adjacent to it are white), and several monochromatic stars that can join other edges to form a monochromatic copy of T.
Recall that j is the number of vertices in G^* and d is its maximal degree. Define j_1 and j_2 according to the canonical coloring: A subset of the vertices of G^* , let us call it W, had all edges from them to V' colored white, whereas the set of other vertices, which we denote by B had all edges from them to V' colored black. Let $|W| = j_1$ and $|B| = j_2$.
A more precise formulation of Statement (4) is as follows: There exist constants M, j_1, j_2 with $j_1 + j_2 = j$ such that the following holds: pick from the vertices of $G(V', p)$ at random j_1 sets $W_1, \dots, W_{j_1}$ of size M and j_2 such sets $B_1, \dots, B_{j_2}$. For every such set glue on to the graph a star of degree $M + d$ by identifying M of the leaves with the vertices in the set. Now color the edges of the stars glued on to the W sets white and those of the B sets black. Call the resulting graph R. Then $Adv(R) \geq \frac{6}{7}\epsilon$.
Now let us prove that this statement follows from Statement (3): Recall that all calculations here will be conditioned on $(G(V', p) \not\to T.)$ For simplicity let us denote this

by

$$N = (G(V',p) \not\to T).$$

Recalling that p_c is of order $1/n$, the expected degree of a vertex in $G(n,p)$ is constant. Therefore there exists a constant M such that

$$Pr[\exists i \leq j \text{ such that degree}(v_i) > M | N] < \frac{\epsilon}{7}.$$

Here degree(v_i) is the degree in $G(n,p)$. Define the events P and Q

$$P = (G(n,p_c) \cup G^* \text{ with canonical coloring} \to T)$$

and

$$Q = (\exists i \leq j \text{ such that degree}(v_i) > M).$$

By the definition of M we have $Pr[Q|N] < \frac{\epsilon}{7}$.
Using (11) we get

$$Pr[P|\bar{Q} \wedge N] \geq Pr[P|N] - \frac{\epsilon}{7}.$$

By monotonicity we have

$$Pr[P|N \wedge (\text{degree}(v_i) = M, i = 1, \ldots, j)] \geq Pr[P|\bar{Q} \wedge N] \geq Pr[P|N] - \frac{\epsilon}{7}.$$

But in this case the monochromatic components of G^* and the adjacent edges that join $G(V',p)$ are white and black stars of degree no more than $d+M$. Again from monotonicity we may assume the degrees are exactly $d+M$, which gives us Statement (4).
Statement (5): A precise formulation of (5) is as follows: Let j_1, j_2, M be the constants from the previous statement. Pick W, a set of Mj_1 random vertices in V', and B, a random set of size Mj_2 in V'. Glue on W a star of degree $Mj_1 + d$ colored white, and on B a star of degree $Mj_2 + d$ colored black. Call the resulting graph R. Then $Adv(R) \geq \frac{5}{7}\epsilon$.

The explanation of why Statement (4) implies Statement (5) is simple: Consider $B_1, B_2, \ldots, B_{j_2}$, the sets of vertices in V' that had stars glued on to them as in Statement (4). As long as the distance in $G(V',p)$ between any two vertices that have a star glued to them is larger than the number of vertices in T any copy of T can intersect at most one such set. In such a case the effect of gluing on the small black stars is exactly the same as that of gluing on one large black star. Thus Statement (5) will follow by using the nibble from (11) if we show that the probability of having any 2 vertices closer than this is less than $\frac{\epsilon}{7}$. This again follows easily from the fact that p_c is of order $1/n$. For any constant k the probability that the distance between two random vertices in $G(V',p_c)$ is less than k is $o(1)$. Therefore the probability that any two of the special vertices chosen have distance smaller than the number of vertices in T is less than $\frac{\epsilon}{7}$ for sufficiently large n.

Statement (6): For any integers r, t let $T_{r,t}$ be the r-ary tree of depth t. The precise formulation of Statement (6) is as follows: There exist constants K, r and t such that the following holds: Add a white star to $G(V', p_c)$ as in the previous statement. Add K copies of $T_{r,t}$ at random to the resulting graph. Denote the resulting graph by R. Then $Adv(R) \geq \frac{3}{7}\epsilon$. To prove this statement we need the following lemma:

Lemma 4.2 *For an integer c let $s = c(d + j_2M)$. For every tree T there exist c and t such that the following holds: Take the tree $T_{s,t}$. Fix an arbitrary partition of the sons of each interior (non-leaf) vertex into c son-sets of size $j_2M + d$. Then in every coloring of the edges of $T_{s,t}$ by black and white there either exists a white copy of T or a vertex such that all of the edges from it to one of its son-sets are colored black.*

Proof: If for a given coloring there is no black set of edges as required then for every interior vertex in the tree and every son-set there is at least one white edge between them. In such a coloring there exists a white $T_{c,t}$. Obviously for large enough values of t and c this tree has T as a subtree.

□

Let us now make some additional definitions:
After constructing $G(V', p)$ and gluing on the white star, define a set of j_2M vertices in V' to be *bad* if gluing a black star of degree $d + j_2M$ on to them results in a Ramsey graph.
Let T' be a copy of $T_{s,t}$ with vertex set within V'. If the distance in $G(V', p)$ between every two vertices of T' is larger than the number of vertices in T call T' remote. If T' is remote *and* there exists a partition of the sons of every interior node in T' into sets all of which are bad we will say T' is bad. This definition is useful due to the following
Observation: ¿From Lemma 4.2 it follows that if after adding the white star to $G(V', p)$ one adds to the graph a bad tree T' the resulting graph has the Ramsey property. In any coloring of the graph one finds within T' either a white copy of T or a black star of degree $d + j_2M$ with a bad set as a subset of its leaves and the rest of the vertices far enough in $G(V', p)$ to have the same effect as gluing a black star on to the bad set. (The difference between gluing a star on or taking the rest of the vertices of the star from V' is that a resulting copy of T may "wrap around itself" if the additional vertices are too close to the bad set.)

To make a transition from Statement (5) to Statement (6) we will define two random variables β and α. They are defined as follows:

$$\beta = Pr[\text{a random set of vertices of size } j_2M \text{ is bad}].$$

$$\alpha = Pr[\text{a random copy of } T_{s,t} \text{ is bad}].$$

Here the probabilities are over the random choice of vertices in V' in the case of β and over the choice of a copy of $T_{s,t}$ for α. The random process that defines the values of β and α however is the choice of edges in $G(V',p)$ and the choice of where to glue on the white star.

The expected value of β is exactly the probability that ($G(V',p)$ + the two random stars) is Ramsey. From Statement (5) it follows

$$Exp[\beta | G(V',p) \not\to T] \geq \frac{5}{7}\epsilon$$

and since $\beta \leq 1$ we also get from a Markov type calculation that

$$Pr[\beta \geq \frac{\epsilon}{7} | G(V',p) \not\to T] \geq \frac{4}{7}\epsilon.$$

Now if n is sufficiently large and β is bounded away from 0, say $\beta > \frac{\epsilon}{7}$, then when sampling subsets of V' to support a copy of $T_{s,t}$ we can disregard the effect of sampling with repetition and bound α from below. Keeping in mind that due to the fact that p is of order $1/n$ a random copy of T' is almost surely remote, an elementary calculation shows that

Claim: There exists a constant c such that for sufficiently large n if $\beta \geq \frac{\epsilon}{7}$ then $\alpha \geq \beta^c$. (Essentially, if there are c subsets of the vertices of $T_{s,t}$ that must be bad then $\alpha \approx \beta^c$.)

From this it follows

$$Pr[\alpha \geq (\frac{\epsilon}{7})^c | G(V',p) \not\to T] \geq Pr[\beta > \frac{\epsilon}{7} | G(V',p) \not\to T] \geq \frac{4}{7}\epsilon.$$

Let $(\frac{\epsilon}{7})^c = \epsilon'$ and K be such that $1-(1-\epsilon')^K > \frac{3}{4}$. Thus, conditioning on $G(V',p) \not\to T$, adding K random copies of $T_{s,t}$ to ($G(V',p)$ + the white star) creates a Ramsey graph with probability at least $\frac{4}{7}\epsilon \cdot \frac{3}{4} = \frac{3}{7}\epsilon$. This completes the proof that Statement (5) implies Statement (6).

Statement (7): The precise statement is that for any $\delta > 0$, for sufficiently large value of n it is true that $Adv(G(V',p) + \text{ white star } \cup G(V',\delta p)) \geq \frac{2}{7}\epsilon$. This follows easily from the previous statement: as δp is of order $1/n$ it is well known (see e.g. [2]) that for any tree T' and any integer K $G(n,\delta p)$ almost surely has more than K vertex disjoint copies of T'. So in particular

$$Pr[G(n,\delta p) \text{ has more than } K \text{ disjoint copies of } T_{s,t}] > 1 - \frac{\epsilon}{7}$$

and using (11) we nibble the right hand side of our estimates down to $\frac{2\epsilon}{7}$.

Statement (8): The passage from Statement (5) to Statement (7) replaced a black star by the addition of $G(n, \delta p)$. This argument can be repeated in more or less the same manner to pass from statement (7) to statement(8).

Statement (9): Follows from Statement (8) if the ratio between ϵ and δ is large enough. In particular, if L is such that

$$1 - (1 - \frac{\epsilon}{7})^L > 1 - \frac{\epsilon}{2} ,$$

then statement (8) implies

$$Adv(G(V', p)) \cup \ 2L \text{ copies of } G(V', \delta p)) > 1 - \frac{\epsilon}{2}.$$

If furthermore we choose δ small enough so

$$1 - (1 - \delta p)^{2L} < \epsilon p$$

we get Statement (9). This completes the proof for the case where $r = 2$ and T is not a star or a path.

If $T = P_4$ and $r = 2$ the threshold is coarse since there exist modest (=unicyclic) graphs G such that $G \to (P_4)^e_2$. Finding an example of such a graph is left as a simple puzzle for the reader.

If $T = P_k$, $k > 4$ the proof is very similar to the previous one. We will point out some necessary modifications, but leave the details to the interested reader.

- The coloring of G^* and the edges adjacent to it in this case is as follows: Assume that G^* has a cycle of odd length (this is the harder case.) Let the vertices of the cycle be $v_1, \dots, v_q$ and the edge from v_i to $v_{(i+1)\text{mod}_q}$ be e_i. Color e_1 black and also $e_2, e_4, \dots, e_{q-1}$. Color all the rest of the edges in the cycle white. Now color all the other edges adjacent to v_1 white. Do the same for $v_2, v_4, \dots, v_{q-1}$. For $v_3, v_5, \dots, v_q$ color the other edges adjacent to them black. Now continue to color the graph inductively by choosing an edge that has been colored and coloring all edges adjacent to it that have not been previously colored by the opposite color. Note that this coloring has a black path of length 3, but no monochromatic path of length larger than 3.

- Define *a broom* of degree d to be a graph consisting of a path of two edges (the "handle") and another d edges adjacent to one end of the path (the "straws"). A set of vertices will now defined to be bad via the gluing to the straws of a copy of

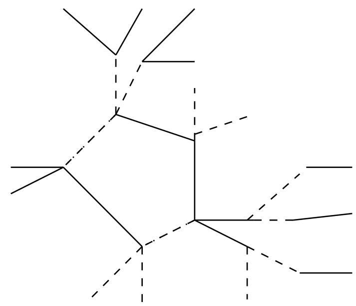

Figure 2: The coloring of G^* and the adjacent edges.

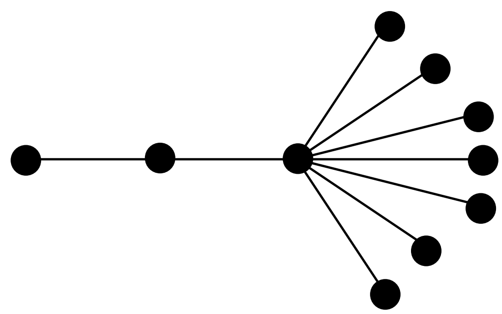

Figure 3: A broom

a black broom rather than a copy of a black star. Observe that if a set of vertices is sufficiently pairwise distant in $G(V',p)$ then it is bad iff the following holds: In every proper coloring of the graph either one of the vertices in the set is the endpoint of a black copy of P_{k-3} or there exist two vertices v and u in the set and an integer x such that u is the endpoint of a black copy of P_x and v is the endpoint of a black copy of P_{k-2-x}. (These are the reasons the black broom may create an obstruction.)

- From this characterization of bad sets we get the following : if in $G(V'p)$ there are two bad sets such that the distance between any two vertices in these sets is sufficiently large (say, $2k$), and one glues on a black star with those sets as leaves then there are no proper colorings left in the resulting graph. Why? Because in any proper coloring of the graph (without the star) one can find two vertices in these sets which are the end points of sufficiently long disjoint black paths: Either a P_x and a P_{k-2-x}, or two copies of P_{k-3}. The black star then connects these paths into a path of length at least k. (We have used the fact that $2(k-3)+2>k$.)

These are the modifications necessary for the case where T is a path.

Now to end the proof we must refer to the case of $r > 2$, more than two colors. Here the proof is essentially the same. Divide the colors $1, \ldots, r$ into two classes, class 1 which consists of color 1 and class 2 which consists of all the rest. Substitute coloring G^* with two colors by a coloring using the two color classes, where the colors of class 2 are chosen arbitrarily. A bad set is now defined to be one where gluing on a star colored by *any* combination of colors of the second class leaves no proper colorings. The rest of the proof remains the same *mutatis mutandis.*

$\square$

5 Conclusion

As is clear many challenging questions are left open. One annoying issue that might be no more than a technicality is getting rid of the SSB restriction in Theorem 1.3. A possibly more difficult problem than this is the case of edge coloring for graphs other than trees, triangles being a natural candidate to start with. The critical p for this case is $p = n^{-1/2}$. We find it hard to conceive that adding a copy of, say, K_5 which is modest in this region, but can be bi-colored with no monochromatic triangle, can have a global effect, and change the probability of a random graph being Ramsey, but we have not been able to rule out this seemingly strange possibility.

Acknowledgments

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