Random regular graphs of high degree

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Abstract

Random d-regular graphs have been well studied when d is fixed and the number of vertices goes to infinity. We obtain results on many of the properties of a random d-regular graph when $d = d(n)$ grows more quickly than $\sqrt{n}$. These properties include connectivity, hamiltonicity, independent set size, chromatic number, choice number, and the size of the second eigenvalue, amongst others.

1 Introduction

The concept of random graphs is one of the central notions in modern Discrete Mathematics. Random graphs have been studied intensively during the last 40 years, with thousands of papers and two excellent monographs by Bollobás [8] and by Janson, Łuczak and Ruciński [19] devoted to the subject and its diverse applications.

Strictly speaking, the term "random graph" comprises several models of random graphs which are quite different in many aspects. A common feature of practically all of these models is that the ground set of the probability space is composed of all graphs on n labeled vertices. Usually *asymptotic* properties of random graphs are studied, that is, the number of vertices n is assumed tending to infinity. Putting different probabilities on n-vertex graphs results in different probability spaces.

The most commonly used model of random graphs, sometimes synonymous with the term "random graphs", is the so-called *binomial random graph* $G(n,p)$. This is a probability

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space of all labeled graphs on n vertices, where the probability assigned to a graph $G = (V, E)$ is $p(G) = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$. It is easy to see that $G(n,p)$ is in fact a product probability space, in which every edge is chosen independently and with probability $p = p(n)$. The relative simplicity of the above definition indicates that this model is quite accessible, and indeed many properties of the random graph $G(n,p)$ are known and well understood. Below we state just a few of them, directly relevant to the contents of this paper. We will use the notation $G(n,p)$ to denote both the probability space defined above, and a random graph on n vertices chosen from the probability space.

As mentioned above, asymptotic properties of $G(n,p)$ are usually of interest. We thus will assume that the number of vertices n tends to infinity. A sequence of graph properties $\{A_n\}$ holds *asymptotically almost surely* in $G(n,p)$, or a.a.s. for brevity, if A_n is a set of graphs on n vertices and $\lim_{n\to\infty} \mathbf{P}_{G(n,p)}[A_n] = 1$.

Here is a list of some of the asymptotic properties of $G(n,p)$ relevant to the topic of the present paper. Let c be any constant satisfying $0 < c < 1$.

- If $p(n) \le c$ and $np \to \infty$ then a.a.s. in $G(n,p)$ the independence number is:

$$\alpha(G) = (1+o(1))\frac{2\log(np)}{\log\frac{1}{1-p}}$$

  (see Bollobás and Erdős [10], Matula [28] and Frieze [15]);

- If $p(n) \le c$ and $np \to \infty$ then a.a.s. in $G(n,p)$ the chromatic number of G is:

$$\chi(G) = (1+o(1))\frac{n\log\frac{1}{1-p}}{2\log(np)}$$

  (see Bollobás [9] and Łuczak [23]);

- The *choice number* (also known as the list-chromatic number) of a graph G is the minimum k such that G is k-choosable, which means that if each vertex v of G is given a list L_v of k permitted colors, there is (for every choice of such lists) a proper vertex coloring of G such that for each vertex v, the color of v is a member of L_v. If $p(n) \le c$ and $np \to \infty$ then a.a.s. in $G(n,p)$ the choice number $\chi_l(G)$ of G satisfies:

$$\chi(G) \le \chi_l(G) \le (2+o(1))\chi(G)$$

  (see Krivelevich and Vu [22]). If, moreover, $p(n) \ge n^{-1/4+\epsilon}$ for some $\epsilon > 0$, then a.a.s. in $G(n,p)$ the choice number of G and its chromatic number have the same asymptotic value (see Alon [1] and Krivelevich [21]);

- If $p(n) = \frac{\log n + \omega(n)}{n}$, where $\omega(n)$ is any function tending to infinity arbitrarily slowly with n, then a.a.s. $G(n,p)$ is connected (see Erdős and Rényi [14]). Also, for arbitrary

$p(n)$, a.a.s. in $G(n,p)$ the vertex connectivity of G is equal to its minimal degree (see Bollobás and Thomason [11]);

- If $p(n) = \frac{\log n + \log\log n + \omega(n)}{n}$ for any function $\omega(n)$ tending to infinity arbitrarily slowly with n, then a.a.s. $G(n,p)$ is hamiltonian (see Komlós and Szemerédi [20]);

- Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of G and let $p(n) = c$. Then a.a.s. in $G(n,p)$

$$\lambda_1 = (1+o(1))np \qquad \text{and} \qquad \max_{2\leq i\leq n} |\lambda_i| = 2\sqrt{pqn} + O(n^{1/3}\log n)$$

(see Füredi and Komlós [17]).

In this paper we study a different model of random graphs — random regular graphs. For a positive integer-valued function $d = d(n)$ we define the model $G_{n,d}$ of random regular graphs consisting of all regular graphs on n vertices of degree d with the uniform probability distribution. We say that an event holds a.a.s. in $G_{n,d}$ if its probability tends to 1 as $n \to \infty$, with n restricted to those integers for which dn is even. As with $G(n,p)$, we will use the notation $G_{n,d}$ to denote both the probability space and a random graph in it.

Properties of random regular graphs were first studied in the late 70's, almost all the results being obtained using the configuration or pairing model of random regular graphs introduced in its simplest form by Bollobás [6]. This model is only amenable for d relatively small compared to n, and at the present time nothing has really been published for d bigger than $\sqrt{n}$. However, Boldi and Vigna [5] and Cooper et al. [13] have obtained such results simultaneously with the present paper.

For comparison with the results given above about $G(n,p)$, we note here that for fixed $d \geq 3$, $G_{n,d}$ is a.a.s. d-connected, as shown by Bollobás [7] and Wormald [33], and hamiltonian as shown by Robinson and Wormald [29, 30]. For properties such as independent set size and chromatic number, various bounds are known. Wormald [34] gave a description of what is known about these and other properties. Note that $G_{n,d}$ is *not* a monotone model, so monotone properties which are a.a.s. true in $G_{n,d}$ are not necessarily so in $G_{n,d+1}$. However, this statement is true for every fixed d except $d = 1$, by contiguity (see [19] and [34]).

Our aim here is to provide a range of results for random regular graphs of high degree, by which we mean d rather larger than $\sqrt{n}$. Some of our arguments could easily be extended below this lower bound, whilst others did not reach that low. We use this as a general cutoff mainly because results already in the literature for fixed d have a reasonable chance to be extended up to $\sqrt{n}$. This is because the switching method has already provided quite accurate results for such d (as by McKay and Wormald [24] for example), as well as permitting enumeration of graphs with given degrees up to this point as by McKay and Wormald [26]. We focus on some of the main properties, such as those given above for

$G(n,p)$. Since the standard model is difficult to analyze for such d we use other tools, of which there are two main ones. The first is the method of switchings, which we extend in the present paper to give information on random regular graphs with degrees $o(n)$. This is less not powerful enough to give an asymptotic formula for the number of d-regular graphs but is sufficient for our purposes. The second main tool is the asymptotic formula for the number of near-regular graphs of degrees approximately cn given by McKay and Wormald [25]. Additionally, we are able to transfer properties of $G(n,p)$ to $G_{n,d}$ in some cases using bounds on the numbers of regular graphs obtained by Shamir and Upfal [31]. Our results overlap partly with those in [13], which obtains similar results on connectivity and hamiltonicity but for a different range of d ($3 \leq d \leq cn$ for some small constant c). One of the main results in [5] is similar to some of ours, being that in $G_{n,d}$ for $n/\log n < d = o(n)$, a.a.s. every three vertices have at least one neighbour in common (c.f. Theorem 2.1). The method used there is basically the use of the asymptotic formula mentioned above.

We close this section with some conventions and notation. The vertex set of graphs with n vertices is assumed to be $\{1, 2, \ldots, n\} = [n]$. A function which tends to infinity arbitrarily slowly with n is denoted $\omega(n)$. If log has no suffix, it denotes the natural logarithm. The neighborhood $N(u)$ of a vertex u is the set of vertices adjacent to it. The *codegree* of two vertices u and v is $\text{codeg}(u,v) = |N(u) \cap N(v)|$.

2 Results and consequences

The proofs of the main results in this section are given in the appropriate later sections.

Theorem 2.1 *(i) Suppose that $d^2/n > \omega(n)\log n$ and $d < n - cn/\log n$ for some constant $c > 2/3$. Then in $G_{n,d}$,*

$$\mathbf{P}(\max_{u,v} |\text{codeg}(u,v) - d^2/n| < Cd^3/n^2 + 6d\sqrt{\log n}/\sqrt{n}) \to 1$$

where C is some absolute constant. If d is bounded below by $cn/\log n$, then C can be defined to be 0.

(ii) For any $\delta > 0$, there exists $\epsilon > 0$ such that if $3 \leq d = O(n^{1-\delta})$ we have that a.a.s. for all u and v in $[n]$, $\text{codeg}(u,v) < \max\{d^{1-\epsilon}, 3\}$.

For the case $d \ll n^{1/3}$, the asymptotic behaviour of the independence number of $G_{n,d}$ was computed by Frieze and Łuczak [16].

Theorem 2.2 *Let $\sqrt{n}\log n < d = d(n) < n - cn/\log n$ for some constant $c > 2/3$. Then for some function $g(n) \to 0$, the independence number of $G_{n,d}$ is a.a.s. at most $(2 + g(n))\log_b d$ where $b = n/(n-d)$.*

Theorem 2.3 *Let $\epsilon > 0$. If $n^{\frac{6}{7}+\epsilon} \le d \le 0.9n$, then there exists a function $h(n) \to 0$ so that a.a.s. in $G_{n,d}$ every subset $V_0 \subset V(G)$ of size $|V_0| \ge n/\log^4 n$ contains an independent set of size at least $(2 - h(n)) \log_b d$, where $b = n/(n-d)$.*

The above two theorems clearly imply that in the range $n^{6/7+\epsilon} \le d \le 0.9n$ the asymptotic value of the independence number of $G_{n,d}$ is $2\log_b d$ with $b = n/(n-d)$. An easy analysis shows that this value coincides with the asymptotic value of the independence number of the binomial random graph $G(n,p)$ with edge probability $p = d/n$.

In [34] it was conjectured that $G_{n,d}$ is a.a.s. d-connected provided $3 \le d \le n-4$. The following result overlaps with one of the main results of [13] to prove this conjecture.

Theorem 2.4 *Let $G \in G_{n,d}$ and let $\sqrt{n}\log n < d \le n-4$. Then a.a.s. G is d-connected and hamiltonian.*

The statement of the theorem trivially holds (using say Dirac's theorem for hamiltonicity) also for $d > n-4$ except for d-connectivity, when $d = n-3$. In that case the complement of $G_{n,d}$ is a random 2-regular graph which with nonzero probability has a cycle of length 4. Deleting the other $n-4$ vertices from G gives a disconnected graph.

For large d we can show, using the greedy algorithm combined with the bound on the independence number, that $\chi(G_{n,d}) = (1 \pm o(1))\chi(G(n, d/n))$. The same can be done for the choice number.

Theorem 2.5 *For every constant $\epsilon > 0$, if $n^{6/7+\epsilon} \le d \le 0.9n$, then a.a.s. in $G_{n,d}$*

$$\chi(G) = (1 + o(1))n/2\log_b d,$$

where $b = n/(n-d)$.

It is important to observe that by the results in [9] and [23], the chromatic number of $G(n,p)$ also has asymptotic value $n/2\log_b d$ with $d = np$ and $b = n/(n-d)$. (For the case $d = o(1)$ we get a more familiar expression $\chi(G) = (1 + o(1))d/2\log d$). Therefore the chromatic numbers in the random regular graph model $G_{n,d}$ and the binomial random graph model $G(n,p)$ of the corresponding density $p = d/n$ coincide asymptotically.

A similar estimate for $\chi(G_{n,d})$ was given in [16] for small d, $d \le n^{1/3-\delta}$, where δ is an arbitrary positive constant less than $1/3$. It was shown that for d in this range again the chromatic numbers in $G_{n,d}$ and in $G(n, d/n)$ are asymptotically equal.

It was proved in [21] that for $n^{3/4+\epsilon} \le p(n)n \le 0.9n$, the choice number of the random graph $G(n,p)$ has the same asymptotic value as its chromatic number. A crucial technical instrument of the argument in [21] was an exponential estimate on the probability of non-existence of an independent set of an almost optimal size in $G(n,p)$, parallel to our Theorem

2.3. Using the technique developed in [21], we can show a similar result for the choice number of $G_{n,d}$.

Theorem 2.6 *Let d satisfy $n^{6/7+\epsilon} \leq d \leq 0.9n$ for a positive constant ϵ. Then a.a.s. in $G_{n,d}$*

$$\chi_l(G) = (1+o(1))\chi(G) = (1+o(1))\frac{n}{2\log_b d},$$

where $b = n/(n-d)$.

We omit the details of the proof.

For the remaining cases, we do not know the asymptotic behaviour of $\chi(G_{n,d})$ and $\chi_l(G_{n,d})$. However, we can show that a.a.s. they both have the same order of magnitude as $\chi(G(n,d/n))$. It is, of course, plausible to think that all three are a.a.s. asymptotically equal.

Theorem 2.7 *Let $0 < \alpha < 1/2$ be a positive constant; then for any $n^{\alpha} < d < n^{1-\alpha}$, a.a.s.*

$$\Omega(d/\log d) = \chi(G_{n,d}) \leq \chi_l(G_{n,d}) = O(d/\log d).$$

Let $\lambda_1(G_{n,d}) \geq \lambda_2(G_{n,d}) \geq \cdots \geq \lambda_n(G_{n,d})$ be the eigenvalues of the adjacency matrix of $G_{n,d}$. Since a graph in $G_{n,d}$ is d-regular and a.a.s. connected, it is clear by the Perron-Frobenius theorem that $\lambda_1(G_{n,d}) = d$ and it has multiplicity 1, a.a.s. The really exciting parameter is, in fact, $\rho(G_{n,d}) = \max(|\lambda_2(G_{n,d})|, |\lambda_n(G_{n,d})|)$. The following theorem says that a.a.s., $\rho(G_{n,d})$ is significantly less than d.

First set $x = K\sqrt{n\log n}/d$ if $d > cn/\log n$ or $\omega(n)\sqrt{n\log n} \leq d < n^{3/4}\log^{1/4} n$ and $x = Kd/n$ if $cn/\log n \geq d \geq n^{3/4}\log^{1/4} n$, where K is a sufficiently large constant so that almost surely the number of common neighbors of any two vertices is at most $(1+x)d^2/n$ (see Theorem 2.1). It is clear that $x = o(1)$.

Theorem 2.8 *For all d satisfying the assumption of Theorem 2.1 and x defined as above, a.a.s.*

$$\rho(G_{n,d}) = O\left(d\left(\frac{n^{1/4}}{d^{1/2}} + x^{1/4}\right)\right) = o(d).$$

For a d-regular graph G, the quantity $\rho(G)/d$ is called the mixing rate of G. Theorem 2.8 implies that in most regular d-regular graphs, a random walk, starting from a fixed vertex (say 1), mixes very fast. Here we say that a random walk mixes in T steps, if after T steps, the variation distance between the obtained distribution and the stationary distribution is at most $1/2$.

Corollary 2.9 *For all $d > \omega(n)\sqrt{n \log n}$, set $s = (\frac{n^{1/4}}{d^{1/2}} + x^{1/4})^{-1}$. Then a.a.s. in $G_{n,d}$, a random walk starting from a fixed vertex mixes in $O(\log_s n)$ steps.*

The isoperimetric number of a graph G, $I(G)$, is the minimum value of the ratio $|\partial X|/|X|$ over all subsets $X \subset V$ with at most $n/2$ vertices, where ∂X is the set of edges with exactly one point in X.

Corollary 2.10 *For all $d > \omega(n)\sqrt{n \log n}$, a.a.s. $I(G_{n,d}) \geq (\frac{1}{2} - o(1))d$.*

Proof. Given a d-regular graph G, the conductance $\phi(G)$ is defined as the minimum of $\frac{n|\partial X|}{d|X||V \setminus X|}$ over all subsets $X \subset V$ with at most $n/2$ vertices. It is well known (see [18]) that

$$\phi(G) \geq 1 - \rho(G)/d.$$

By Theorem 2.8, it follows that $\phi(G_{n,d}) \geq 1 - o(1)$ a.a.s. On the other hand, $I(G) \geq d\phi(G)/2$ by definition and this completes the proof. It is also clear that the constant $\frac{1}{2}$ is best possible. ∎

For a special value of d, Theorem 2.8 also has a consequence for the number of induced copies of a fixed graph. Assume that G is $n/2$-regular and $\rho(G) = o(n)$. For such G, it has been proved that for any fixed graph H on r vertices, the number of induced copies of H in G is $(1 + o(1))n^r 2^{-\binom{r}{2}}$. (See, e.g., Alon and Spencer [4, Chapter 9].)

Corollary 2.11 *For any fixed graph H on r vertices, the number of induced copies of H in $G_{n,n/2}$ is a.a.s. $(1 + o(1))n^r 2^{-\binom{r}{2}}$.*

3 Bounds on codegrees and independent sets

When d is not $o(n)$, we will need the following corollary of the asymptotic formula for the number of graphs with given degrees which was proved in [25]. Let $N(d_1, \ldots, d_n)$ denote the number of labeled simple graphs of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Then [25, Theorem 3] implies the following.

Proposition 3.1 *Let $d_j = d_j(n)$, $1 \leq j \leq n$ be integers such that $\min\{\lambda, 1 - \lambda\} > c/\log n$ for some $c > \frac{2}{3}$, where $\sum_{j=1}^{n} d_j = \lambda n(n-1)$ is an even integer, and $|\lambda n - d_j| = O(n^{1/2+\epsilon})$ uniformly over j for sufficiently small fixed $\epsilon > 0$. Then*

$$N(d_1, \ldots, d_n) = f(d_1, \ldots, d_n)(\lambda^\lambda (1 - \lambda)^{1-\lambda})^{\binom{n}{2}} \prod_{j=1}^{n} \binom{n-1}{d_j}$$

where

(i) $f(d_1, \ldots, d_n) = O(1)$, *and*

(ii) if $\max\{|\lambda n - d_j| = o(\sqrt{n})$ *then* $f(d_1, \ldots, d_n) \sim \sqrt{2}e^{1/4}$ *(uniformly over the choice of such* d_j*).* ∎

The restriction on λ is perhaps artificial since this result is known to hold also for $\lambda = o(n^{-1/2})$, and it is conjectured in [25] to hold for all λ for which $n^2 \min\{\lambda, 1-\lambda\} \to \infty$.

Proof of Theorem 2.1. We begin with (i). For each of the following two cases (which have a non-trivial overlap), let $u, v \in [n]$ and let $\mathcal{C}_k$ denote the set of d-regular graphs on n vertices with $\mathrm{codeg}(u, v) = k$. Let $G \in \mathcal{C}_k$.
Case 1: $d = o(n)$. For this case we use switchings (see [34]) of the type introduced by McKay and Wormald [27].

An operation called a *forward switching* consists of choosing a vertex $w \in N(u) \cap N(v)$, and edges xy and $x'y'$ of G, deleting the edges uw, vw, xy and $x'y'$ and inserting new edges xu, yw, $y'w$ and $x'v$ so as to obtain another d-regular graph H on the same vertex set. Choices of w, x, y, x' and y' which would cause multiple edges, where for instance xu is already an edge, or unwanted duplication of vertices such as x and y or y', are forbidden. Cases causing a multiple edge number at most $O(kd^3n)$, and those with a duplicated vertex are $O(kd^2n)$. So the number of forward switchings possible is

$$kd^2n^2 - O(kd^3n).$$

Note that the constant implicit in $O()$ here is independent of k. A *reverse switching* is applied to $H \in \mathcal{C}_{k-1}$ by choosing $x \in N(u) \setminus N(v)$, $x' \in N(v) \setminus N(u)$ and a path ywy'. Delete the edges xu, yw, $y'w$ and $x'v$ and insert uw, vw, xy and $x'y'$. Again choices causing any of these vertices to be repeated, or multiple edges to occur, are forbidden, and are $O(d-k)^2d^3$ in number. So the number of reverse switchings is

$$(d-k)^2nd^2 - O(d-k)^2d^3.$$

It follows that

$$\frac{|\mathcal{C}_k|}{|\mathcal{C}_{k-1}|} = \frac{(d-k)^2}{kn}(1 + O(d/n)). \tag{1}$$

So define k_0 to be the (real) solution of $(d-k)^2 = kn$. Then

$$k_0 = \frac{d^2}{n}\left(1 + O\Big(\frac{d}{n}\Big)\right). \tag{2}$$

We can now choose a C sufficiently large that for $k > k_0 + Cd^3/n^2 + xd/\sqrt{n}$, (1) implies

$$\frac{|\mathcal{C}_k|}{|\mathcal{C}_{k-1}|} < \frac{(d-k_0)^2}{k_0 n} \times \frac{1}{1 + x\sqrt{n}/d} < e^{-x\sqrt{n}/2d}$$

for $x = o(d/\sqrt{n})$. It follows that for $k_1 = k_0 + Cd^3/n^2 + Bd\sqrt{\log n}/\sqrt{n}$ and $k_2 = k_1 + Bd\sqrt{\log n}/\sqrt{n}$,

$$\frac{|\mathcal{C}_{k_2}|}{|\mathcal{C}_{k_1}|} < n^{-B^2/2}.$$

By (1), $|C_k| < |C_{k_2}|$ for $k > k_2$. Thus $\mathbf{P}(\mathrm{codeg}(u,v) > k_2) = O(n^{1-B^2/2})$. The lower tail is bounded in exactly the same way, again using (1). Choosing $B = 3$ and noting (2) gives

$$\mathbf{P}(|\mathrm{codeg}(u,v) - k_0| > Cd^3/n^2 + 6d\sqrt{\log n}/\sqrt{n}) = o(n^{-2}). \tag{3}$$

Case 2: $\min\{d, n-d\} > cn/\log n$ for $c > 2/3$. Assume firstly that u and v are nonadjacent. Delete u and v from G to obtain a graph with k vertices of degree $d-2$, $2d-2k$ of degree $d-1$ and the rest of degree d. Such graphs are in one-one correspondence with elements of $\mathcal{C}_k$ provided that the vertices of degree $d-1$ are divided into two classes: $d-k$ which were the neighbours of u and the rest which were the neighbours of v. The number of graphs with such a degree sequence is given asymptotically by Proposition 3.1 (ii). Multiplying by the number of orderings of the degrees of the vertices and the selection of the two classes as above gives

$$|\mathcal{C}_k| \sim \frac{(n-2)!\sqrt{2}e^{1/4}(\lambda^\lambda(1-\lambda)^{1-\lambda})^{\binom{n-2}{2}}}{k!(d-k)!^2(n-2-2d+k)!}\binom{n-3}{d}^{n-2-2d+k}\binom{n-3}{d-1}^{2d-2k}\binom{n-3}{d-2}^{k}$$

where $\lambda = \frac{d(n-4)}{(n-2)(n-3)}$. The asymptotics is uniform over all k. So letting $t_{n,d,k}$ denote the product of the factorials and binomials in the above formula, we only have to bound $t_{n,d,k}$ for k away from d^2/n. Note that

$$\frac{t_{n,d,k}}{t_{n,d,k-1}} = \frac{(d-k)^2}{k(n-2d+k)}\left(1 + O(\tfrac{1}{k}) + O(\tfrac{1}{d-k})\right)$$

and so this ratio is 1 if $k = d^2/n + O(1)$. The rest of the argument is as in Case 1 to obtain (3) with $C = 0$.

If u and v are adjacent, almost the same argument gives the same result.

Finally, in both Case 1 and Case 2, summing (3) over all pairs (u, v) gives an upper bound on the probability required for part (i) of the theorem.

To verify part (ii), note that $\epsilon > 0$ and $\epsilon' > 0$ can be chosen sufficiently small so that for $k \geq k_0 = \lfloor d^{1-2\epsilon} \rfloor$ the ratio (1) is $O(n^{-\epsilon'})$, and hence for $\epsilon'(d^\epsilon - 1) > 4$,

$$\begin{aligned}
\mathbf{P}(\mathrm{codeg}(u,v) \geq d^{1-\epsilon}) &\leq \sum_{k \geq d^{1-\epsilon}} \frac{|\mathcal{C}_k|}{|\mathcal{C}_{k_0}|} \\
&= O\left(n\left(n^{-\epsilon'}\right)^{d^\epsilon - 1}\right) \\
&= O(n^{-3}).
\end{aligned}$$

On the other hand, for bounded d, (1) gives $\mathbf{P}(\mathrm{codeg}(u,v) \geq 3) = O(n^{-3})$. Summing the bounds over all u and v gives (ii). ∎

Proof of Theorem 2.2.

Case 1: $d = o(n)$. We use switchings as in Case 1 of the proof of Theorem 2.1. Let $A \subseteq [n]$, let $a = |A|$, and let $\mathcal{C}_k$ denote the event that exactly k edges have both ends in A. For $G \in \mathcal{C}_k$ with $k > 0$, choose an edge uv with u, $v \in A$ and choose two other edges $u'v'$ and $u''v''$ of G. Delete these three edges and add the edges vu', $v'u''$ and $v''u$. This procedure is called a forward switching if it produces a graph H in $\mathcal{C}_{k-1}$. The reverse switching is applied to such an H by choosing two edges uv'' and vu' in H where u, $v \in A$, and an edge $v'u''$, and deleting these three edges and adding edges uv, $u'v'$ and $u''v''$ (provided that a member of $\mathcal{C}_k$ results). The number of ways of applying a forward switching is

$$kn^2d^2(1 + O(d/n)).$$

For a reverse switching, it is

$$\left(\binom{a}{2} - (k-1)\right) nd^3(1 + O(d/n)). \tag{4}$$

(Note that if the switching is chosen to be the simpler version in which only one random edge is chosen, the reverse switching count is highly dependent on the graph H, in particular, on how many paths of length 3 have both ends in A.) Hence

$$\frac{|\mathcal{C}_k|}{|\mathcal{C}_{k-1}|} \sim \frac{\binom{a}{2} d}{kn}$$

provided $k = o(a^2)$. Now fix $k = \lfloor \binom{a}{2} d/n \rfloor$ and assume $k \to \infty$. Then

$$\frac{|\mathcal{C}_k|}{|\mathcal{C}_0|} = e^{o(k)} \left(\binom{a}{2} d/n\right)^k / k! = e^{k+o(k)},$$

and hence $\mathbf{P}(\mathcal{C}_0) = e^{-k+o(k)}$. The probability there is at least one independent set of vertices of G of cardinality a is at most $\binom{n}{a}\mathbf{P}(\mathcal{C}_0) < (\frac{en}{a})^a e^{-k+o(k)}$. Summing this over $a > (2 + o(1))n \log d/d$ for a suitable function $o(1)$, and using (4) to show that values of k larger than $o(a^2)$ really can be ignored, gives a result which is $o(1)$. So the theorem follows in this case since $\log b = -\log(1 - d/n) \sim d/n$.

Case 2: $\min\{d, n-d\} > cn/\log n$ for $c > 2/3$. We first give a lemma which assumes only that Proposition 3.1 holds for the values of d involved. In $G_{n,d}$, define X_a to be the number of independent sets of cardinality a.

Lemma 3.2 *Assume that $\lambda(n) = d/(n-1)$ satisfies the hypotheses of Proposition 3.1, and suppose that $a = O(n^{1/2+\epsilon})$ for ϵ as in that proposition. Then*

$$\mathbf{E}X_a \leq \exp\left(\tfrac{1}{2}a(a-1)\log(1-\lambda) + a\log n + o(a \log n)\right)$$

where $\lambda = d/(n-1)$.

The lemma is proved next. Combined with Markov's inequality, the lemma implies the theorem in this second case, since here $\log(1-\lambda) \sim -\log b$ and $\log n \sim \log d$. ∎

Proof of Lemma 3.2. If the vertices $n-a+1, \ldots, n$ form an independent set in $G \in G_{n,d}$, then deleting all these vertices gives a graph with degree sequence $d-h_1, \ldots, d-h_{n-a}$ where h_i is the degree of i in the graph H consisting of just those edges incident with deleted vertices. Hence, using Proposition 3.1(i) to bound the number of possibilities for $G - E(H)$, and (ii) to estimate $|G_{n,d}|$,

$$\mathbf{E}X_a = O(1)\binom{n}{a}\sum_H \frac{g(\tilde{\lambda})^{(n-a)(n-a-1)/2}}{g(\lambda)^{n(n-1)/2}}\binom{n-1}{d}^{-a}\prod_{i=1}^{n-a}\frac{\binom{n-a-1}{d-h_i}}{\binom{n-1}{d}} \tag{5}$$

where $\tilde{\lambda} = \frac{d(n-2a)}{(n-a)(n-a-1)}$ and $g(x) = x^x(1-x)^{(1-x)}$. Here $\binom{n}{a}$ is the number of choices of an independent set.

We can bring order into chaos by eliminating the summation over H. Note that

$$\begin{aligned}\frac{\binom{n-a-1}{d-h_i}}{\binom{n-1}{d}} &= \frac{[d]_{h_i}[n-d-1]_{a-h_i}}{[n-1]_a}\\ &\le \frac{d^{h_i}(n-d-1)^{a-h_i}}{(n-1)^a}\times\frac{n-1}{n-a}\end{aligned}$$

where $[x]_k$ denotes $x(x-1)\cdots(x-k+1)$. An easy way to see why the inequality holds is by considering the equivalent inequality

$$\prod_{i=1}^{h_i-1}\left(1-\frac{i}{d}\right)\prod_{i=1}^{a-h_i-1}\left(1-\frac{i}{n-d-1}\right) \le \prod_{i=1}^{a-2}\left(1-\frac{i}{n-1}\right).$$

Rewrite the $a-2$ factors in the expression on the left hand side as $s_1 \ge \cdots \ge s_{a-2}$. The number of these which are strictly greater than $1-\frac{i}{n-1}$ is less than or equal to $\lfloor\frac{id}{n-1}\rfloor$ from the first product and less than or equal to $\lfloor\frac{i(n-1-d)}{n-1}\rfloor$ from the second product. Hence $s_i \le 1-\frac{i}{n-1}$, and the inequality follows. We now have

$$\begin{aligned}\prod_{i=1}^{n-a}\frac{\binom{n-a-1}{d-h_i}}{\binom{n-1}{d}} &\le \frac{d^{ad}(n-d-1)^{a(n-a-d)}}{(n-1)^{a(n-a)}}\times\left(\frac{n-1}{n-a}\right)^{n-a}\\ &= \lambda^{\lambda a(n-1)}(1-\lambda)^{(1-\lambda)a(n-1)-a(a-1)}\times\left(\frac{n-1}{n-a}\right)^{n-a}.\end{aligned} \tag{6}$$

Also note that $\tilde{\lambda}-\lambda = -\lambda a(a-1)/((n-a)(n-a-1))$, therefore using the Taylor expansion of the function $\log g(\lambda)$ we obtain

$$\log g(\tilde{\lambda}) - \log g(\lambda) = -\frac{a(a-1)\lambda}{(n-a)(n-a-1)}\log\left(\frac{\lambda}{1-\lambda}\right) + O(a^4(1-\lambda)^{-1}n^{-4}).$$

Hence

$$\begin{aligned}\frac{g(\tilde{\lambda})^{(n-a)(n-a-1)/2}}{g(\lambda)^{n(n-1)/2}} &= g(\lambda)^{-an+a(a-1)/2}\exp\left(-\tfrac{a(a-1)\lambda}{2}\log\tfrac{\lambda}{1-\lambda}+O(a^4\log n/n^2)\right)\\ &= \lambda^{-\lambda an}(1-\lambda)^{(1-\lambda)(-an+a(a-1)/2)}\exp(o(a\log n)),\end{aligned}\tag{7}$$

where we used $1/(1-\lambda)=O(\log n)$, $\log(\lambda/(1-\lambda))=o(\log n)$ and $a=o(n^{2/3})$.

It is convenient to multiply the factor $\binom{n-1}{d}^{-a}$ in (5) by the number of terms in the summation, which is at most $\binom{n-a}{d}^a$ since each vertex of the independent set chooses d neighbors from the other $n-a$ neighbors. This product is

$$\left(\frac{[n-d]_{a-1}}{[n-1]_{a-1}}\right)^a<(1-\lambda)^{a(a-1)}.\tag{8}$$

Thus (5) is bounded above by the product of $\binom{n}{a}$, (6), (7) and (8). Using $\binom{n}{a}<n^a$ and $\log\frac{n-1}{n-a}=a/n+o(a/n)$ gives the bound stated in the lemma (again using $-\log\lambda=o(\log n)$ and $-\log(1-\lambda)=o(\log n)$). ∎

Proof of Theorem 2.3. Let $p=d/n$. We will compare the probability that the random graph $G(n,p)$ contains a subset V_0 of size $|V_0|=n/\log^4 n$ without an independent set of the desired size with the probability that $G(n,p)$ is d-regular. The former probability will be estimated from above using arguments which are by now quite standard, to be presented here in a somewhat abridged form.

Let $k_0=k_0(n,p)$ be defined by

$$k_0=\max\left\{k:\binom{n/\log^4 n}{k}(1-p)^{\binom{k}{2}}\geq n^3\right\}.$$

One can show that k_0 satisfies $k_0\sim 2\log_b d$ with $b=n/(n-d)$.

Denote $m=n/\log^4 n$ and consider a random graph $G(m,p)$. Let X be the random variable counting the number of independent sets of size k_0 in $G(m,p)$. Denoting the expectation of X by μ and recalling the definition of k_0, we get:

$$\mu=\binom{m}{k_0}(1-p)^{\binom{k_0}{2}}\geq n^3.$$

Let

$$\Delta=2\sum_{\substack{|S|,|S'|=k_0\\ 2\leq|S\cap S'|\leq k_0-1}}\mathbf{P}[S,S'\text{ form an independent set in }G(m,p)].$$

Then

$$\begin{aligned}\Delta &= \binom{m}{k_0}(1-p)^{\binom{k_0}{2}}\sum_{i=2}^{k_0-1}\binom{k_0}{i}\binom{m-k_0}{k_0-i}(1-p)^{\binom{k_0}{2}-\binom{i}{2}}\\ &= \mu^2\sum_{i=2}^{k_0-1}g(i),\quad\text{where}\quad g(i)=\frac{\binom{k_0}{i}\binom{m-k_0}{k_0-i}(1-p)^{-\binom{i}{2}}}{\binom{m}{k_0}}.\end{aligned}$$

One can check that $g(2) = \Theta(k_0^4/m^2)$ is the dominating term in the above sum, while the summands decrease quickly as i goes away from the ends of the interval $[2, k_0 - 1]$. This implies that $\Delta = \Theta(\mu^2 k_0^4/m^2)$. Also, as $\mu \geq n^3$ we get $\Delta \geq \mu$. Then by the generalized Janson inequality (see, e.g., [4, Chapter 7])

$$\mathbf{P}[X = 0] \leq e^{-\frac{\mu^2(1+o(1))}{2\Delta}} = e^{-\Omega\left(m^2/k_0^4\right)} .$$

Recalling the definition of k_0, we see that the exponent above is of order $d^4/n^2 \text{polylog}\, n$.

We next need a lower bound on the probability that a random graph in $G(n,p)$ is regular. We use the result of Shamir and Upfal [31, equation (35)] with $\phi(n) = d$, $\theta = \frac{1}{2} + \delta$ for some $\delta > 0$, choosing $w(n) - \phi(n) = \lceil w(n)^{1-\delta} \rceil$, to deduce that the number of d-regular graphs on n vertices is at least

$$\binom{\binom{n}{2}}{nd/2} \exp(-O(nd^{1/2+2\delta})).$$

(Here there is a condition on $d(n)$; growing faster than $\log^2 n$ is sufficient.) It follows that for any fixed $\delta > 0$

$$\mathbf{P}(G(n, d/n) \text{ is } d\text{-regular}) \geq \exp(-nd^{1/2+\delta}). \qquad (9)$$

Comparing the last two exponents and using the assumption $d \geq n^{6/7+\epsilon}$, we obtain that the probability that $G(n, d/n)$ is d-regular is much higher asymptotically than the probability that $G(n, d/n)$ contains a large subset without an independent set of size k_0. Therefore, a.a.s. for $G_{n,d}$ where d lies in the range given in the theorem statement, every subset V_0 of size $|V_0| \geq n/\log^4 n$ spans an independent set of size k_0. The theorem is proven. ∎

4 Connectivity and hamiltonicity

In this section we prove that dense d-regular graphs with some pseudo-random properties are d-connected. As an immediate corollary we obtain that for $\sqrt{n}\log n < d \leq n - 4$ a random d-regular graph on n vertices is a.a.s. d-connected and hamiltonian.

First we need the following lemma, which is very similar in spirit to one obtained by Alon et al. [2].

Lemma 4.1 *Let $G = (V, E)$ be a d-regular graph on n vertices such that $\sqrt{n}\log n < d$ and for all $u \neq v \in V$, $\text{codeg}(u, v) = (1 + o(1))d^2/n$. Then the number of edges between any two $B_1, B_2 \subseteq V$, satisfying $|B_1| \geq \Omega(n)$ and $|B_2| \geq \sqrt{n}$, is at least $(1 + o(1))\frac{d}{n}|B_1||B_2|$.*

Proof. Let A be the adjacency matrix of G, let J be the all 1 matrix whose rows and columns are indexed by V and put $H = A - (d/n)J = (h_{uv})_{u,v \in V}$. An easy computation shows that the inner product of any two columns of H is relatively small. Indeed, if $N(v)$ and $N(v')$ denote the sets of all neighbors of v and v', respectively, and $v \neq v'$ then,

$$\sum_{u \in V} h_{uv} h_{uv'} = |N(v) \cap N(v')| - \frac{d}{n}(|N(v)| + |N(v')|) + n(d/n)^2 = o(d^2/n).$$

Also note that for any vertex v we have

$$\sum_{u \in V} h_{uv}^2 = d(1-d/n)^2 + (n-d)(d/n)^2 = d - d^2/n < d.$$

Therefore

$$\begin{aligned}
\sum_{u \in B_1} \Big(\sum_{v \in B_2} h_{uv} \Big)^2 &\leq \sum_{u \in V} \Big(\sum_{v \in B_2} h_{uv} \Big)^2 \\
&= \sum_{u \in V} \Big(\sum_{v \in B_2} h_{uv}^2 + \sum_{v,v' \in B_2, v \neq v'} h_{uv} h_{uv'} \Big) \\
&= \sum_{v \in B_2} \sum_{u \in V} h_{uv}^2 + \sum_{v,v' \in B_2, v \neq v'} \sum_{u \in V} h_{uv} h_{uv'} \\
&\leq |B_2| d + |B_2|^2 o(d^2/n).
\end{aligned}$$

Let $e(B_1, B_2)$ denote the total number of edges of G between B_1 and B_2. By the Cauchy-Schwartz inequality and the last estimate

$$\begin{aligned}
\Big(e(B_1, B_2) - \frac{d}{n} |B_1||B_2| \Big)^2 &= \Big(\sum_{u \in B_1} \sum_{v \in B_2} h_{uv} \Big)^2 \\
&\leq |B_1| \sum_{u \in B_1} \Big(\sum_{v \in B_2} h_{uv} \Big)^2 \\
&\leq |B_1||B_2| d + |B_1||B_2|^2 o(d^2/n).
\end{aligned}$$

Hence

$$\begin{aligned}
e(B_1, B_2) &\geq \frac{d}{n}|B_1||B_2| - \sqrt{|B_1||B_2|d} - |B_1|^{1/2} |B_2| o(d/\sqrt{n}) \\
&> (1+o(1)) \frac{d}{n} |B_1||B_2|,
\end{aligned}$$

where the last inequality follows from the facts that $d > \sqrt{n} \log n$, $|B_1| \geq \Omega(n)$, $|B_2| \geq \sqrt{n}$ and n is sufficiently large. This completes the proof. ∎

Using this lemma we obtain the following result about connectivity of pseudo-random graphs.

Proposition 4.2 *Let $G = (V, E)$ be a d-regular graph on n vertices such that $\sqrt{n} \log n < d \leq 3n/4$ and the number of common neighbors for every two distinct vertices in G is $(1+o(1))d^2/n$. Then the graph G is d-connected.*

Proof. Suppose that there is a subset $S \subset V$ of size at most $d-1$ such that the induced graph $G[V-S]$ is disconnected. Denote by B_2 the set of vertices of the smallest connected component of $G[V-S]$ and set $B_1 = V - (S \cup B_2)$. Then $|B_1| \geq \Omega(n)$, B_2 contains at least two vertices and there is no edge between B_1 and B_2. Therefore by Lemma 4.1 we obtain $|B_2| < \sqrt{n}$. Let u and v be any two distinct vertices in B_2. Clearly all the neighbors of these two vertices belong to the set $B_2 \cup S$ which has size at most $d + \sqrt{n}$. Since the degrees of v and u are equal to d, this implies that the number of common neighbors of these two vertices is at least $2d - (d + \sqrt{n}) = d - \sqrt{n}$. This is asymptotically much bigger than $(1 + o(1))d^2/n$, a contradiction. ∎

Remark. Note that the same proof is valid for any upper bound on the degree d of the form $(1-\delta)n$ for any fixed $\delta > 0$.

To complete the proof of Theorem 2.4 we need the following well known result of Chvátal and Erdős [12].

Proposition 4.3 *Let G be a k-connected simple graph such that G contains no independent set of size $k+1$. Then G has a hamiltonian cycle.* ∎

Proof of Theorem 2.4. Since by Theorem 2.1 for $\sqrt{n} \log n < d \leq 3n/4$, $G_{n,d}$ almost surely satisfies the conditions of Proposition 4.2, we obtain that a.a.s. it is d-connected. This together with result of Chvátal and Erdős implies that for the same values of d, the random d-regular graph almost surely contains a hamiltonian cycle. Here we used the fact that by Theorem 2.2, a.a.s. the size of the maximal independent set in $G_{n,d}$ is at most $O(n \log d/d) = O(\sqrt{n}) < d$.

Next we consider the case when $3n/4 \leq d \leq n-4$. Since it is well known that any graph on n vertices with minimum degree at least $n/2$ contains a hamiltonian cycle, we obtain that $G = G_{n,d}$ is hamiltonian. Let G' be the complement of G. Note that by definition $G' = G_{n,d'}$ is a random d'-regular graph with $d' = n - 1 - d \geq 3$. Suppose that there is a subset $S \subset V(G)$ of size at most $d-1$ such that the induced graph $G[V-S]$ is disconnected. Denote by B_2 the set of vertices of the smallest connected component of $G[V-S]$ and set $B_1 = V - (S \cup B_2)$. Then obviously $|B_1| + |B_2| = n - (d-1) = d' + 2$, B_2 contains at least two vertices and there is no edge between B_1 and B_2. This implies that G' contains a complete bipartite graph with bipartition (B_1, B_2). We claim that this is a.a.s. impossible. Since $|B_1| \geq (|B_1| + |B_2|)/2 = (d'+2)/2$ we conclude that any two vertices in B_2 have at least $(d'+2)/2$ common neighbors. Therefore if $(d')^{\epsilon} > 2$ then the number of common neighbors is at least $(d'+2)/2 > \max\{(d')^{1-\epsilon}, 3\}$. This contradicts the assertion of Theorem 2.1. For $3 \leq d' \leq 2^{1/\epsilon}$ note that the number of edges in the subgraph of G' induced by $B_1 \cup B_2$ is at least $2d' > d' + 2 = |B_1| + |B_2|$. This again is a.a.s. not possible since it is known (see, e.g., [34]) that for a fixed d', any subgraph on $d'+2$ vertices of the random d'-regular graph almost surely contains at most $d'+2$ edges. This completes the proof of the theorem. ∎

5 Coloring

Proof of Theorem 2.5. The lower bound on $\chi(G)$ follows immediately from the upper bound on the independence number, given by Theorem 2.2, and the inequality $\chi(G) \geq |V(G)|/\alpha(G)$.

To prove the upper bound, we will apply the approach of Bollobás [9] by first covering most of the vertices of the graph by independent sets of asymptotically optimal size, and then coloring the remaining subgraph using degeneracy arguments. We need the following proposition.

Proposition 5.1 *Let $n^{6/7+\epsilon} \leq d \leq 0.9n$ for a positive constant ϵ. Then a.a.s. in $G_{n,d}$ every $s \leq n/\log^4 n$ vertices of G span less than $sd/\log^2 d$ edges.*

Proof. Set $p = d/n$. The probability that the random graph $G(n,p)$ contains a subset of size $s \leq n/\log^4 n$ spanning at least $sd/\log^2 d$ edges, can be bounded from above by

$$\sum_{i=2d/\log^2 d}^{n/\log^4 n} \binom{n}{i} \binom{\binom{i}{2}}{\frac{id}{\log^2 d}} p^{\frac{id}{\log^2 d}} \leq \sum_{i=2d/\log^2 d}^{n/\log^4 n} \left[\frac{en}{i} \left(\frac{ei \log^2 d}{2d} \right)^{\frac{d}{\log^2 d}} p^{\frac{d}{\log^2 d}} \right]^i$$
$$\leq \sum_{i=2d/\log^2 d}^{n/\log^4 n} \left[n \left(\frac{O(1) ip \log^2 d}{d} \right)^{\frac{d}{\log^2 d}} \right]^i \leq e^{-\frac{d^2}{\log^4 d}}$$

for n sufficiently large.

Comparing the above probability with the probability that $G(n,p)$ is d-regular, which is bounded below in (9), and recalling our assumption $d \geq n^{6/7+\epsilon}$, we conclude that a random regular graph $G_{n,d}$ has a.a.s. the property stated in the proposition. ∎

Recall that a graph G is called *s-degenerate* if every subgraph of it contains a vertex of degree at most s. It is easy to see that every s-degenerate graph is $(s+1)$-colorable. Going back to the above proposition we see that a.a.s. in $G_{n,d}$ every subset $V_0 \subset V$ of size $|V_0| \leq n/\log^4 n$ spans a $(2d/\log^2 d - 1)$-degenerate and thus $2d/\log^2 d$-colorable subgraph.

Now we can present an argument for the upper bound on $\chi(G_{n,d})$. As long as G still has at least $n/\log^4 n$ uncolored vertices, we find an independent set of size $(2-o(1))\log_b d$, which exists a.a.s. by Theorem 2.3. We color it by a fresh color and discard. If less than $n/\log^4 n$ vertices remain uncolored, then we color the spanning subgraph of uncolored vertices by using at most $2d/\log^2 d$ colors, based on Proposition 5.1. Altogether we use at most $(1+o(1))n/2\log_b d + 2d/\log^2 d = (1+o(1))n/2\log_b d$ colors. ∎

Proof of Theorem 2.7. The upper bound $\chi_l(G_{n,d}) = O(d/\log d)$ is a simple corollary of Theorem 2.1 and the following theorem, proved by Vu [32, Theorem 4.1]. A similar result was proved for $d = \Omega(\text{polylog}\, n)$, by Alon et al. [3].

Theorem 5.2 *Let G be a d-regular graph on n vertices. Assume that the codegrees of G are at most $d^{1-\epsilon}$, for some positive constant ϵ. Then*

$$\chi_l(G) = O(d/\log d).$$

The lower bound $\Omega(d/\log d) = \chi(G_{n,d})$, for the case $d > n^{1/2}\log n$, follows from the upper bound on the independence number (Theorem 2.2). If $n^\alpha \leq d \leq n^{1/2}\log n$, the same upper bound $O(n\log d/d)$ (with a more generous multiplicative constant) still holds (a.a.s.) for the independence number. The proof is similar to that of Theorem 2.2 and is omitted. ∎

6 The second eigenvalue

Proof of Theorem 2.8. Let $A(G_{n,d})$ be the adjacency matrix. Observe that

$$\sum_{i=1}^{n} \lambda_i^4(G_{n,d}) = \mathbf{tr}(A^4(G_{n,d})).$$

It follows that $\mathbf{tr}(A^4(G_{n,d})) - d^4 \geq \rho^4(G_{n,d})$. On the other hand, $\mathbf{tr}(A^4(G_{n,d}))$ is the number of closed walks of length 4 in $G_{n,d}$ and can be expressed as follows

$$\mathbf{tr}(A^4(G_{n,d})) = nd^2 + nd(d-1) + 8C_4(G_{n,d}),$$

where $C_4(G_{n,d})$ is the number of cycles of length 4 in $G_{n,d}$. By the definition of x,

$$C_4(G_{n,d}) \leq \frac{1}{2}\binom{n}{2}\binom{\frac{d^2}{n}(1+x)}{2},$$

and a routine calculation yields that

$$\rho^4(G_{n,d}) = O(nd^2 + d^4x).$$

The theorem follows. ∎

7 Concluding remarks

One of the main obstacles to deriving results on $G_{n,d}$ for large d is the lack of an accessible model of dense random regular graphs. Such a model would be desired to simplify the computations such as those in the proof of Theorems 2.1 and 2.2. Even the asymptotic probability that $G(n,p)$ is regular for $n^{-1/2} < p = o(1/\log n)$ is not known (but would follow immediately from the asymptotic enumeration conjecture in [25]).

On coloring problems, the main unknowns remaining from the results in this paper are the asymptotic values of $\chi(G)$ and $\chi_l(G)$ for d below $n^{6/7+\epsilon}$ and the range of asymptotic equality of these two parameters. The range of concentration of $\chi(G)$, $\alpha(G)$ and $\chi_l(G)$ is yet to be established.

For the eigenvalues we still have no great knowledge of their asymptotic distribution (though some results for small d occur in the references to [34]). In particular, a result on ρ which sharpens Theorem 2.8, like that in [17] mentioned in Section 1, would have useful applications.

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References

[1] N. Alon, Restricted colorings of graphs, in *Surveys in Combinatorics 1993*, London Math. Soc. Lecture Notes Series 187 (K. Walker, ed.), Cambridge Univ. Press 1993, 1–33.

[2] N. Alon, M. Krivelevich and B. Sudakov, List coloring of random and pseudo-random graphs, *Combinatorica* **19** (1999), 453–472.

[3] N. Alon, M. Krivelevich and B. Sudakov, Coloring graphs with sparse neighborhoods, *J. Combinatorial Theory Ser. B* **77** (1999), 73–82,

[4] N. Alon and J. Spencer, **The probabilistic method**, Wiley, New York 1992.

[5] P. Boldi and S. Vigna, More lower bounds for weak sense of direction: the case of regular graphs, manuscript.

[6] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European Journal on Combinatorics* **1** (1980), 311–316.

[7] B. Bollobás, Random graphs, in *Combinatorics* (H.N.V. Temperley, ed.), London Mathematical Society Lecture Note Series **52**, Cambridge University Press, Cambridge, 1981, pp. 80–102.

[8] B. Bollobás, **Random graphs**, Academic Press, New York, 1985.

[9] B. Bollobás, The chromatic number of random graphs, *Combinatorica* **8** (1988), 49–55.

[10] B. Bollobás and P. Erdős, Cliques in random graphs, *Math. Proc. Camb. Phil. Soc.* **80** (1976), 419–427.

[11] B. Bollobás and A. Thomason, Random graphs of small order, in *Random Graphs'83*, Proceedings, Poznań, 1983, eds. M. Karoński and A. Ruciński, North-Holland, Amsterdam-New York, 1985, pp. 47–97.

[12] V. Chvátal and P. Erdős, A note on Hamiltonian circuits, *Discrete Math.* **2** (1972), 111–113.

[13] C. Cooper, A. Frieze and B. Reed, Random regular graphs of non-constant degree (manuscript).

[14] P. Erdős and A. Rényi, On random graphs I, *Publ. Math. Debrecen* **6** (1959), 290–297.

[15] A. Frieze, On the independence number of random graphs, *Discrete Math.* **81** (1990), 171–175.

[16] A. Frieze and T. Łuczak, On the independence and chromatic numbers of random regular graphs, *J. Combinatorial Theory (Series B)*, **54** (1992), 123–132.

[17] Z. Füredi and J. Komlós, The eigenvalues of random symmetric matrices, *Combinatorica* **1** (1981), 233–241.

[18] Appendix of Chapter 31, **Handbook of Combinatorics**, edited by R. Graham, M. Grötschel and L. Lovász, North-Holland 1995.

[19] S. Janson, T. Łuczak and A. Ruciński, **Random graphs**, Wiley, New York, 2000.

[20] J. Komlós and E. Szemerédi, Limit distributions for the existence of Hamilton cycles in a random graph, *Discrete Math.* **43** (1983), 55–63.

[21] M. Krivelevich, The choice number of dense random graphs, *Combinatorics, Probability and Computing* **9** (2000), 19–26.

[22] M. Krivelevich and V. H. Vu, Choosability in random hypergraphs, submitted.

[23] T. Łuczak, The chromatic number of random graphs, *Combinatorica* **11** (1991), 45–54.

[24] B. D. McKay and N. C. Wormald, Automorphisms of random graphs with specified degrees, *Combinatorica*, **4** (1984), 325–338.

[25] B. D. McKay and N. C. Wormald, Asymptotic enumeration by degree sequence of graphs of high degree, *Europ. J. Combinatorics* **11** (1990), 565–580.

[26] B.D. McKay and N.C. Wormald, Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$, *Combinatorica* **11** (1991) 369–382.

[27] B. D. McKay and N. C. Wormald, Uniform generation of random regular graphs of moderate degree, *J. Algorithms* **11** (1991), 52–67.

[28] D. Matula, The largest clique size in a random graph, Tech. Rep. Dept. Comp. Sci., Southern Methodist University, Dallas, Texas, 1976.

[29] R. W. Robinson and N. C. Wormald, Almost all cubic graphs are hamiltonian, *Random Structures and Algorithms* **3** (1992), 117–125.

[30] R. W. Robinson and N. C. Wormald, Almost all regular graphs are hamiltonian, *Random Structures and Algorithms* **5** (1994), 363–374.

[31] E. Shamir and E. Upfal, Large regular factors in random graphs, *Convexity and graph theory* (Jerusalem 1981), North Holland Math. Stud. 87, North Holland, Amsterdam-New York 1984, pp. 271–282.

[32] V.H. Vu, On some degree conditions which guarantee the upper bound of chromatic (choice) number of random graphs, *Journal of Graph Theory* **31** 1999, 201–226.

[33] N.C. Wormald, The asymptotic connectivity of labelled regular graphs, *J. Combinatorial Theory (Series B)* **31** (1981), 156–167.

[34] N.C. Wormald, Models of random regular graphs, In *Surveys in Combinatorics, 1999*, J.D. Lamb and D.A. Preece, eds, pp. 239–298.