FRACTIONAL PLANKS

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ABSTRACT. In 1950 Bang proposed a conjecture which became known as "the plank conjecture": Suppose that a convex set S contained in the unit cube of $\Re^n$ and touching all its sides is covered by planks. (A *plank* is a set of the form $\{(x_1,\dots,x_n) : x_j \in I\}$ for some $j \in \{1,\dots,n\}$ and a measurable subset I of $[0,1]$. Its *width* is defined as $|I|$.) Then the sum of the widths of the planks is at least 1. We consider a version of the conjecture in which the planks are fractional. Namely, we look at n-tuples $f_1,\dots,f_n$ of nonnegative-valued measurable functions on $[0,1]$ which cover the set S in the sense that $\sum f_j(x_j) \geq 1$ for all $(x_1,\dots,x_n) \in S$. The *width* of a function f_j is defined as $\int_0^1 f_j(x)dx$. In particular, we shall be interested in conditions on a convex subset of the unit cube in $\Re^n$ which ensure that it cannot be covered by fractional planks (functions) whose sum of widths (integrals) is less than 1. We shall prove that this (and, *a-fortiori*, the plank conjecture) is true for sets which touch all edges incident with two antipodal points in the cube. For general convex bodies inscribed in the unit cube in $\Re^n$ we prove that the sum of widths must be at least $\frac{1}{n}$ (the true bound is conjectured to be $\frac{2}{n}$).

1. INTRODUCTION

In 1950 Bang [4, 5] proved the following conjecture of Tarski: if a convex body of width 1 in $\Re^n$ is covered by slabs, then the sum of the widths of the slabs is at least 1 (in other words, the most economical way to cover the body is by just one slab). He then asked the following, more demanding question. Let S be a convex body in $\Re^n$, and let $T_1,\dots,T_m$ be slabs whose union covers S. Is it true that the sum of the *relative* widths of the slabs is at least 1? (The *relative width* of the slab with respect to S is the ratio of the width of the slab to the width of S in the same direction). This conjecture gained the name "the plank conjecture", for obvious reasons. It has a number of equivalent formulations, including a geometric pigeonhole principle suggested by Davenport and generalized by Alexander [2].

For the version of the conjecture which will be used here we shall need a few definitions. The unit cube in $\Re^n$ will be denoted by Q_n (to avoid trivial exceptions, we shall assume throughout that $n \geq 2$). A subset of Q_n is called *packed* in the cube if it touches all facets (sides) of the cube. A *plank* P (of type j) in Q_n is a set of the form $\{\mathbf{x} \in Q_n : x_j \in I\}$ for some measurable subset I of $[0,1]$. (The notation $\mathbf{x}$ will henceforth be reserved for a point $(x_1, x_2, \dots, x_n)$ in Q_n.) We shall write then $P = Pl_j(I)$. The *width* $|P|$ of the plank is $|I|$, the Lebesgue measure of I. Given a subset S of Q_n, a family $\mathcal{P} = (P_1,\dots,P_n)$ of planks (where P_j is of

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type j) is called a *plank cover* of S if its union contains S. The *total width* $|\mathcal{P}|$ of the family is the sum of the widths of the planks in it.

Conjecture 1.1 (The plank conjecture). *A plank cover of a packed convex set in Q_n has total width at least* 1.

This conjecture is known to be true for $n = 2$ [6] and when the set is centrally symmetric [3]. We note that it is customary to reserve the term "plank" for the case when the set I (which we allow to be an arbitrary measurable subset of $[0,1]$) is an interval. It is straightforward to reduce our formulation of the plank conjecture to one which uses only interval-planks, but allows any number of them in each direction.

A plank of type j can be viewed as a $0,1$ function $f_j(x_j)$ on the interval $[0,1]$. Viewing planks this way, it is natural to ask what happens when each plank Pl_j is replaced by a nonnegative real valued measurable function $f_j(x_j)$, instead of a $0,1$ function. The covering condition is then that $\sum f_j(x_j) \geq 1$ for each point $\mathbf{x} \in S$. If this condition holds then we say that the system $f_1, \dots, f_n$ is a *fractional plank cover* of S.

Notation. For a measurable function f from $\Re$ to $\Re$ and a measurable subset T of its domain, we denote by $f(T)$ the integral $\int_T f(x)dx$. If T is an interval $[a,b]$, we abbreviate and write $f[a,b]$ for $f([a,b])$.

The *width* $|f_j|$ of a fractional plank f_j is $f_j[0,1]$. Given a system $f_1, \dots, f_n$ of fractional planks, the *total width* of the system is defined as $\sum |f_j|$. The infimum of the total width over all fractional plank covers of S is called the *fractional plank covering number* of S. By standard arguments (see [9, Theorem 2.21]) this infimum is attained, that is, it is a minimum. This minimum is denoted by $\tau^*(S)$. The source of this notation is in combinatorics. To explain it, we shall need the following terminology (a standard reference for which is, say [7]). A *hypergraph* is a family H of subsets (called *edges*) of some ground set (whose elements are called *vertices*). A *fractional cover* of H is a system of nonnegative real weights on the vertices which sums up to at least 1 on each edge. The minimal sum of weights over all fractional covers is called the *fractional covering number* of H, and is denoted by $\tau^*(H)$. Now, a subset S of Q_n can be viewed as a hypergraph $H = H(S)$. The ground set is the disjoint union of n copies of $[0,1]$, and each point $\mathbf{x} \in S$ corresponds to an edge of $H(S)$, namely the subset $\{x_1, x_2, \dots, x_n\}$ of that union. The fractional plank covering number is thus a continuous version of the fractional covering number of the hypergraph.

In this terminology the plank covering number of S (that is, the infimum of the total width over all plank covers of S), is the analogue of the "covering number" of the hypergraph, which is the minimal number of vertices which meet all edges. Hence it is appropriate to assign to it the usual notation for this parameter, namely $\tau(S)$. (We believe, but cannot prove, that $\tau(S)$ is in fact a minimum, that is, that there exists a plank cover attaining it, for every measurable subset S of Q_n.)

Since the notion of fractional plank covers is more general than that of ordinary plank covers, $\tau^* \leq \tau$, and in fact usually strict inequality obtains. Thus there are packed convex sets in Q_n with $\tau^* < 1$. In fact, τ^* may be as low as $\frac{2}{n}$ for such sets, as the following example shows:

Denote by Δ_n the standard $(n-1)$-dimensional simplex in $\Re^n$, namely the set of all points $\mathbf{x} \in Q_n$ such that $\sum x_j = 1$.

Let $f_j(x) = (\frac{2}{n} - x)^+$ $(j = 1, \dots, n)$, that is: $f_j(x) = \frac{2}{n} - x$ for $0 \leq x \leq \frac{2}{n}$, $f_j(x) = 0$ for $\frac{2}{n} < x \leq 1$. Then, for each point $\mathbf{x} \in \Delta_n$ we have:

$$\sum f_j(x_j) \geq n\tfrac{2}{n} - \sum x_j = 1.$$

The total width of the system is $\sum f_j[0,1] = \frac{2}{n}$, implying that $\tau^*(\Delta_n) \leq \frac{2}{n}$.

Later we shall see that, in fact, $\tau^*(\Delta_n) = \frac{2}{n}$, and that if the plank conjecture is true then $\tau^* \geq \frac{2}{n}$ for all packed convex sets in Q_n.

2. A DUAL CONCEPT

A *measure matching* on a measurable subset S of Q_n is a nonnegative measure defined on the Lebesgue-measurable subsets of Q_n, whose support is contained in S, and whose marginal measure on each of the coordinates has density at most 1. That is, the measure of any plank of width δ is at most δ. (A similar concept, confined to probability measures, was introduced by Gardner [8] as a tool for studying the plank conjecture; he used the term "relative width measure for the coordinate directions".) In the discrete case this corresponds to a system of nonnegative real weights on the points of S, such that for each coordinate j and each $u \in [0,1]$ the sum of the weights of the points $\mathbf{x}$ satisfying $x_j = u$ is at most 1. The name used in combinatorics for such a system is a *fractional matching* of the hypergraph represented by S. The supremum of $\mu(S)$ over all measure matchings μ on S is called the *measure matching number* of S, and is denoted (following the combinatorial convention) by $\nu^*(S)$. By standard measure theoretical arguments (see [9, Theorem 2.19]) it follows that if S is compact then $\nu^*(S)$ is attained.

Given a measure matching μ on S with marginals $\mu_1, \dots, \mu_n$ and a fractional plank cover $f_1, \dots, f_n$, one clearly has:

$$(1) \qquad \mu(S) \leq \int_S \sum_j f_j(x_j) d\mu(\mathbf{x}) = \sum_j \int_0^1 f_j(x_j) d\mu_j(x_j) \leq \sum_j f_j[0,1].$$

This implies that $\nu^*(S) \leq \tau^*(S)$. An analogue of the duality theorem of linear programming [9, Corollary 2.18] yields that $\nu^* = \tau^*$ for all measurable subsets of Q_n.

3. A FRACTIONAL VERSION OF THE PLANK CONJECTURE

A hypergraph is called *n-partite* if it admits a partition of the vertex set into n parts, such that every edge consists of a choice of one vertex from each part. In [10] Lovász proved that for n-partite hypergraphs the inequality $\tau \leq \frac{n}{2}\tau^*$ holds. Since, as noted above, subsets of Q_n may be viewed as n-partite hypergraphs, the same inequality holds for them, too.

Theorem 3.1.

$$\tau(S) \leq \frac{n}{2}\tau^*(S)$$

for any measurable subset S of Q_n.

The proof below is a continuous version of Lovász' proof. At its base lies the following observation, which is a special case of Proposition 2 in [1]:

Lemma 3.2. *There exists a measure matching μ on Δ_n whose support is contained in the intersection of Δ_n with the cube $\{\mathbf{x} : 0 \leq x_j \leq \frac{2}{n}, 1 \leq j \leq n\}$, and whose marginal measure in each direction x_j has density 1 on the interval $0 \leq x_j \leq \frac{2}{n}$.*

Proof. The lemma is trivial for $n = 2$. We shall first prove the case $n = 3$. There are many ways of constructing a measure on Δ_3 which do the job. One of them uses the following:

Theorem 3.3 (Archimedes)**.** *On the unit disk there exists a positive measure with constant marginals in* all *directions.*

In fact, the measure is given by a function: the function

$$\frac{1}{\sqrt{1-||x||^2}}$$

has the property that its integrals on intersections of lines with the unit disk are all equal. Applying the theorem to the disk inscribed in the triangle Δ_3, and normalizing the measure suitably, yields the desired measure.

As the lemma is true for $n = 2, 3$, to prove it in general it suffices to note that if it is true for two values ℓ and m of n, then it holds also for $\ell + m$. Indeed, the Cartesian product $\frac{\ell}{\ell+m}\Delta_\ell \times \frac{m}{\ell+m}\Delta_m$ naturally embeds in $\Delta_{\ell+m}$. On each of the factors of this product we have, by assumption, a suitable measure, and the product of those measures satisfies the requirements on $\Delta_{\ell+m}$. $\square$

Remark. The lemma shows that $\nu^*(\Delta_n) \geq \frac{2}{n}$, and combining this with the reverse inequality proved in the introduction, $\nu^*(\Delta_n) = \tau^*(\Delta_n) = \frac{2}{n}$ for $n \geq 2$.

Proof of Theorem 3.1. Let $f_1, f_2, \dots, f_n$ be a fractional plank cover for the set S, with total width w. We shall prove that there exists a plank cover with total width at most $\frac{n}{2}w$.

For each point $\mathbf{x} \in \Delta_n$ we shall define a plank cover for S, as follows. For each $1 \leq j \leq n$ let $I_j = I_j(\mathbf{x})$ be the set of points $x \in [0, 1]$ for which $f_j(x) \geq x_j$. Let $P_j = Pl_j(I_j)$.

Since $\sum x_j = 1$, and since $\sum f_j(s_j) \geq 1$ for all points $\mathbf{s} = (s_1, s_2, \dots s_n) \in S$, it follows that for each point $\mathbf{s} \in S$ there exists a $j \leq n$ for which $\mathbf{s} \in P_j$, i.e. $\mathcal{P}(\mathbf{x}) = P_1, \dots, P_n$ is a plank cover.

In order to show that there exists a plank cover of width at most $\frac{n}{2}w$ it suffices to show that the normalized μ-average over all $\mathbf{x} \in \Delta_n$ of the widths of $\mathcal{P}(\mathbf{x})$ is at most that number (where μ is a measure on Δ_n satisfying the requirements of Lemma 3.2). For each j we have

$$\int |P_j| d\mu = \int |\{x : f_j(x) \geq x_j\}| d\mu =$$

$$\int_0^{\frac{2}{n}} |\{x : f_j(x) \geq y\}| dy = \int_0^1 \min\left\{\frac{2}{n}, f_j(x)\right\} dx \leq |f_j|.$$

(The second equality follows from the equidistribution of the marginal of μ in the j-th coordinate, between 0 and $\frac{2}{n}$. The third equality is a general property of integrals.)

Thus $\int |\mathcal{P}(\mathbf{x})| d\mu \leq \sum |f_j| = w$, and since μ is of total weight $\frac{2}{n}$, it follows that the normalized μ-average of $|\mathcal{P}|$ is at most $\frac{n}{2}w$, as promised. $\square$

By Theorem 3.1, if the plank conjecture is true then the following conjecture also holds:

Conjecture 3.4. *$\tau^*(S) \geq \frac{2}{n}$ for any convex set S which is packed in Q_n .*

As already noted, the simplex Δ_n is an example where equality holds in Conjecture 3.4. In fact, the simplex is but one member of a family of sets in which equality obtains. We name the members of this family "generalized octahedra", and they are defined as follows.

Let $\mathbf{c} = (c_1, c_2, \dots c_n)$ be a point in Q_n, and let $P(\mathbf{c})$ be the set of projections of $\mathbf{c}$ on the $2n$ facets of Q_n. Any convex set S which satisfies $conv(P(\mathbf{c}) \setminus \{\mathbf{c}\}) \subseteq S \subseteq conv(P(\mathbf{c}))$ is called a *generalized octahedron*, with *center* $\mathbf{c}$. (Here and elsewhere $conv(A)$ denotes the convex hull of the set A. The definition is devised so as to cope with the special case in which the center is a vertex of the cube, in which case it is one of its own projections. This includes the case of the simplex, but also that of the body consisting of the simplex together with all points below it, i.e., the convex hull of the simplex and the vertex of the cube which it encloses.)

Generalized octahedra play a special role with respect to τ and τ^*. First, as already mentioned, they satisfy $\tau^* = \frac{2}{n}$, namely they are extreme cases in Conjecture 3.4. We suspect that these are the only extreme cases for $n \geq 3$.

Generalized octahedra are also the only packed convex sets we know, in which there exists a family of more than one plank covering the set, with total width 1 (i.e., these are the only cases known in which the bound 1 in the plank conjecture is attained in a non-trivial way).

Still another fact about generalized octahedra is that in the case $n = 2$ they are the only packed convex sets in Q_2 in which measure matchings, rather than functions, are really needed. That is, we can prove that these are the only packed convex sets in the square for which no measure matching with mass 1 on the set can be represented as the integral of a function with respect to the 2-dimensional Lebesgue measure.

In the original plank conjecture there is no dimension-independent constant lower bound known. In contrast, Conjecture 3.4 can be proved to within a factor of 2. In fact, we shall prove a little more: not only that τ^* of every convex set packed in Q_n is at least $\frac{1}{n}$, but also ν, the matching number of the set, is at least $\frac{1}{n}$. But first we have to define this notion:

Definition 3.5. *Given a segment T in $\Re^n$, we denote by $\nu(T)$ the minimum among the lengths of its projections on the axes.*

Definition 3.6. *The* matching number *$\nu(S)$ of a subset S of Q_n is the supremum of $\sum_{T \in \mathcal{T}} \nu(T)$, where $\mathcal{T}$ ranges over all finite families of segments contained in S, whose projections on all axes are pairwise disjoint.*

This definition is obtained from the standard definition of the matching number of hypergraphs, by discretisation. Note that the supremum in this definition is not always attained, not even if we allow infinite families of segments. An example in which the supremum is not attained is the set of points in Q_2 satisfying the inequality $y \geq 2|x - \frac{1}{2}|$. The matching number of this set is 1, but it has no matching in the sense of Definition 3.6 with full projections.

It is easily seen that $\nu(S) \leq \nu^*(S)$: put a uniform measure with total mass $\nu(T)$ on each segment $T \in \mathcal{T}$. Hence the following theorem implies that $\tau^*(S) \geq \frac{1}{n}$ for all packed convex subsets S of Q_n:

Theorem 3.7. *$\nu(S) \geq \frac{1}{n}$ for every convex set S which is packed in Q_n.*

Proof. Without loss of generality we may assume that S is closed. The difference body $K = S - S$ of S is then convex, compact, centrally symmetric and packed in the cube $[-1,1]^n$. As noted above, Ball [3] proved the plank conjecture for such bodies. Hence, denoting by D_j the open plank $Pl_j((-\frac{1}{n}, \frac{1}{n}))$ $(1 \leq j \leq n)$, the set K is not contained in the union of the planks D_j (whose sum of widths is 2). (We have used here also the compactness of K.) Thus there exists a point $\mathbf{k} = \mathbf{u} - \mathbf{v} \in K$ not belonging to any D_j, where $\mathbf{u}, \mathbf{v} \in S$. The segment whose endpoints are $\mathbf{u}$ and $\mathbf{v}$ is contained in S, and its projections on all axes are all no shorter than $\frac{1}{n}$. □

Remark 1. The convex hull of the midpoints of all facets of the cube shows that $\frac{1}{n}$ is the best lower bound possible on ν of a *single* segment contained in a packed convex body in Q_n.

Remark 2. A conjecture of Ryser and Lovász (see [7, p. 105]) states that $\tau \leq (n-1)\nu$ for any n-partite hypergraph. Combining this with the plank conjecture would yield that $\nu(S) \geq \frac{1}{n-1}$ for every convex set S which is packed in Q_n. But it is quite possible that this is not the best bound for $n \geq 3$. Indeed, Δ_n, which we conjecture to be extreme for τ^*, has $\nu > \frac{1}{n-1}$. Let us calculate it for $n = 3$:

Proposition 3.8. $\nu(\Delta_3) = \frac{3}{5}$.

Proof. We shall prove here only one direction, namely we shall construct a matching in Δ_3 with $\nu = \frac{3}{5}$. The other direction is rather complicated, and is omitted.

Let $\mathbf{p}_1 = (0, \frac{1}{5}, \frac{4}{5})$, $\mathbf{q}_1 = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$, and let $\mathbf{p}_2, \mathbf{p}_3$ be the two cyclic permutations of $\mathbf{p}_1$, and $\mathbf{q}_2, \mathbf{q}_3$ the two cyclic permutations of $\mathbf{q}_1$. Finally, let $I_j (j = 1,2,3)$ be the segments joining $\mathbf{p}_j$ with $\mathbf{q}_j$. Then it is easy to check that the projections of the three segments on each axis have disjoint interiors, while $\nu(I_j) = \frac{1}{5}$, yielding $\nu(\Delta_3) \geq \frac{3}{5}$. □

4. Hefty sets

One aim of this paper is to study conditions which imply that a subset of Q_n has $\tau^* = 1$.

Definition 4.1. *A measurable subset S of Q_n is called* hefty *if $\tau^*(S) = 1$.*

Proposition 4.2. *The following four conditions on a measurable subset S of Q_n are equivalent:*

(1) S is hefty.

(2) There exists a probability measure on S with uniform marginals (i.e., the marginal on each axis is the ordinary Lebesgue measure on $[0,1]$).

(3) Given any system $\beta_1, \beta_2, \ldots \beta_n$ of measurable functions on $[0,1]$, if $\sum \beta_j(x_j) \geq 1$ for each point $(x_1, x_2, \ldots x_n) \in S$, then $\sum \beta_j[0,1] \geq 1$.

(4) Given any system $\alpha_1, \alpha_2, \ldots \alpha_n$ of measurable functions on $[0,1]$, if $\sum \alpha_j(x_j) \geq 0$ for each point $(x_1, x_2, \ldots x_n) \in S$, then $\sum \alpha_j[0,1] \geq 0$.

Proof. The equivalence between (1) and (2) follows from Section 2. That (3) implies (1) is clear, since the conclusion in both is the same, while the condition in (1) is stronger - there is the additional condition that the functions are nonnegative-valued.

The equivalence of (3) and (4) is also easy: assuming that (3) holds, given a system of functions α_j as in (4) define $\beta_1 = \alpha_1 + 1, \beta_j = \alpha_j$ for $j > 1$. Applying (3) to the system β_j yields then (4). The reverse implication is similar.

It remains to show that (2) implies (3). Assume that there exists a probability measure μ on S with uniform marginals. Let $\beta_1, \dots \beta_n$ be functions as in (3). Write the analogue of (1) for the functions β_j:

$$1 = \mu(S) \leq \int_S \sum_j \beta_j(x_j) d\mu(\mathbf{x}) = \sum_j \int_0^1 \beta_j(x_j) d\mu_j(x_j) = \sum_j \beta_j[0,1].$$

Note that while in (1) the nonnegativity of the functions f_j was needed to obtain the right-hand inequality, here it is not necessary that the functions β_j be nonnegative, since the marginals μ_j are known to be equal to the Lebesgue measure. □

5. The special role of the center

Notation. For a real number t we shall denote by $\overrightarrow{\mathbf{t}}$ the vector all of whose entries are t (the dimension of the vector is to be understood from the context).

It turns out that the center of the cube, the point $\overrightarrow{\mathbf{\tfrac{1}{2}}}$, plays a special role with regard to heftiness:

Proposition 5.1. *A closed convex hefty set contains* $\overrightarrow{\mathbf{\tfrac{1}{2}}}$.

Proof. If S is closed and convex and $\overrightarrow{\mathbf{\tfrac{1}{2}}} \notin S$, then there exist real numbers $r_1, \dots, r_n$ and a positive ϵ such that $\sum r_j x_j \geq \sum \frac{1}{2} r_j + \epsilon$ for all points $\mathbf{x} \in S$. Applying condition (4) of Proposition 4.2 to the functions $\alpha_j(x_j) = r_j(x_j - \frac{1}{2}) - \frac{\epsilon}{n}$ yields a contradiction to the heftiness of S. □

The converse of the proposition is false. Even the ball B_n inscribed in Q_n is not hefty, for $n \geq 4$. (B_3 is hefty, a fact which was known already to Archimedes - the measure matching showing this is similar to that appearing in the proof of Theorem 3.3.) The following proposition settles in the negative a problem of Gardner [8]:

Proposition 5.2. *For $n \geq 4$ the ball B_n is not hefty.*

Proof. A point $\mathbf{x} \in B_n$ satisfies $\sum_i (x_i - \frac{1}{2})^2 \leq \frac{1}{4}$, which implies, by the Cauchy-Schwartz inequality,

$$\sum_i |x_i - \tfrac{1}{2}| \leq \frac{\sqrt{n}}{2}.$$

Hence, defining for every $1 \leq i \leq n$

$$f_i(x) = \sqrt{\frac{1}{4n}} - |x - \tfrac{1}{2}|,$$

we have

$$\sum f_i(x_i) \geq \frac{\sqrt{n}}{2} - \frac{\sqrt{n}}{2} = 0. \tag{2}$$

But, as is easily seen, $\sum f_i[0,1] = \frac{\sqrt{n}}{2} - \frac{n}{4}$, the last expression being negative for $n > 4$ and 0 for $n = 4$. Thus, for $n > 4$ the proposition follows from Proposition 4.2. In the case $n = 4$, note that equality is attained in (2) only at a finite number of points. Hence by changing one of the functions $f_i(x)$ slightly in a small neighborhood of some value of x_i such that there is no point with equality having that value of x_i, we can still maintain property (2), while having negative sum of integrals. □

The following proposition presents a case in which the converse of Proposition 5.1 is true:

Proposition 5.3. *If a set of vertices of Q_n has $\overrightarrow{\frac{1}{2}}$ in its convex hull, then the convex hull is hefty.*

Proof. Let $V = \{\mathbf{v}_k : k \in K\}$ be the set of vertices in question, with $\mathbf{v}_k = (v_1^k, \ldots, v_n^k)$. Write $\overrightarrow{\frac{1}{2}} = \sum_{k \in K} \alpha_k \mathbf{v}_k$, where $\alpha_k \geq 0$ and $\sum \alpha_k = 1$. For each $k \in K$ let T_k be the segment connecting $\mathbf{v}_k$ with $\overrightarrow{\frac{1}{2}}$, and distribute uniformly on T_k a measure with total mass α_k. Let μ be the measure on $conv(V)$ which is concentrated on the union of the segments T_k and is the sum of all the above measures. We shall show that μ is a measure matching: this will complete the proof, since obviously the mass of μ is 1.

Let $1 \leq j \leq n$. Since $\sum\{\alpha_k : v_j^k = 1\} = \sum\{\alpha_k : v_j^k = 0\} = \frac{1}{2}$, the mass of μ on $Pl_j[0, \frac{1}{2}]$ is $\frac{1}{2}$. Since μ is evenly distributed on each segment T_k, it follows that this mass of $\frac{1}{2}$ is evenly distributed on $[0, \frac{1}{2}]$, which means that $\mu(Pl_j(I)) = |I|$ for every interval $I \subset [0, \frac{1}{2}]$. The same is true for all subintervals of $[\frac{1}{2}, 1]$, which implies the desired conclusion. □

Another result in the converse direction to that of Proposition 5.1 was proved in [1]. A subset of Q_n will be called *hexagonal* if it is packed, and is the intersection of a hyperplane with Q_n. (The source of the name is that in the case $n = 3$ such a set has a (possibly degenerate) hexagonal shape.)

Theorem 5.4. *A hexagonal set containing $\overrightarrow{\frac{1}{2}}$ is hefty.*

Since, in fact, Proposition 2 in [1] is formulated a bit differently, we shall give here an outline of the proof. The proof will use the following easy observation (which is needed if one wants to translate Proposition 2 of [1] into the terms of Theorem 5.4).

Lemma 5.5. *(i) If a hyperplane $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ meets all facets of Q_n then*

$$|a_i| \leq \sum_{j \neq i} |a_j| \tag{3}$$

for all $1 \leq i \leq n$.

(ii) If a hyperplane $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ passes through $\overrightarrow{\frac{1}{2}}$ and satisfies (3), then it meets all facets of the cube. □

Outline of proof of Theorem 5.4. For $n = 3$ it is possible to provide concretely a measure matching with total mass 1 on the hexagon, concentrated on certain of its diagonals.

The case $n > 3$ is done by induction on n. Let X be a hexagonal set, namely X is the set of points $\mathbf{x} \in Q_n$ satisfying the equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$. Without loss of generality, assume that $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. Now, "join" the first two variables, i.e., look at the hyperplane H in $\Re^{n-1}$ defined by $(a_1 + a_2)y + \sum_{i>2} a_ix_i = c$. Since $n > 3$, H satisfies (3). Hence, by the induction hypothesis, the intersection of H with Q_{n-1} is hefty, and the probability measure on this intersection readily yields a probability measure on the original hexagonal set. □

6. Other conditions implying heftiness

As already mentioned, the plank conjecture is true for $n = 2$. Since the combinatorial interpretation of this case is that of bipartite graphs, and since for such graphs $\nu = \tau^* = \tau$, this implies that all convex sets packed in Q_2 are hefty. (Gardner [8] reached the same conclusion by constructing measure matchings for such sets.) As noted above, for $n > 2$ this is no longer true. It is tempting to ascribe the difference between the two cases to the fact that in the case $n = 2$ "facets" and "edges" coincide, and thus being packed means not only touching the facets of the cube, but also the edges. Indeed, it is not hard to prove the following:

Proposition 6.1. *A convex subset of Q_n touching all of its edges is hefty.*

In fact, a much weaker condition (though equivalent in the case $n = 2$) suffices:

Definition 6.2. *A subset S of Q_n is called* strongly packed *if there exist two antipodal vertices of Q_n such that S touches all $2n$ edges incident with them.*

The main theorem of this paper is:

Theorem 6.3. *A strongly packed convex subset of Q_n is hefty.*

In order to understand the intuition behind this theorem, note, first, that a subset of Q_n containing a main diagonal (a segment connecting two antipodal vertices) is hefty (this will be shown below). The idea behind the theorem is that the property of having a hefty convex hull is preserved upon replacing each of these two vertices by n points on the edges incident with it. That is, the n "splinters" of the vertex do the work that the single vertex had done. In fact, we believe that this is true in general, namely:

Conjecture 6.4. *If a vertex $\mathbf{v}$ of the cube belongs to a set S having the property that its convex hull is hefty, then replacing $\mathbf{v}$ by n points on the edges incident with it preserves this property of S.*

It will also be useful to have a name for the operation in this conjecture:

Definition 6.5. *A replacement as in the conjecture is called* splitting *of* $\mathbf{v}$.

For the property of having $\overrightarrow{\frac{1}{2}}$ in the convex hull, the analogue of Conjecture 6.4 is indeed true:

Proposition 6.6. *If* $\overrightarrow{\frac{1}{2}} \in conv(S)$ *and* S' *is obtained by splitting a vertex in* S, *then* $\overrightarrow{\frac{1}{2}} \in conv(S')$.

Proof. We may assume that the split vertex is $\overrightarrow{\mathbf{0}}$. Then there exists a point $\mathbf{u} \in conv(S \setminus \{\overrightarrow{\mathbf{0}}\})$ such that $\overrightarrow{\frac{1}{2}}$ is on the segment connecting $\overrightarrow{\mathbf{0}}$ and $\mathbf{u}$. Let $\mathbf{z}$ be the point on the line connecting $\overrightarrow{\mathbf{0}}$ and $\overrightarrow{\frac{1}{2}}$ which lies on the hyperplane spanned by the n splinters of $\overrightarrow{\mathbf{0}}$. Clearly, the order of the points on the line $(t, t, \dots, t)$ is $\overrightarrow{\mathbf{0}}, \mathbf{z}, \overrightarrow{\frac{1}{2}}, \mathbf{u}$, and thus $\overrightarrow{\frac{1}{2}}$ is on the segment connecting $\mathbf{z}$ and $\mathbf{u}$, and thus is in the convex hull of S'. □

The remainder of the paper will be mainly devoted to the proof of Theorem 6.3. The main tool used in the proof will be a certain property of families of sub-boxes of Q_n, which we shall study in the next section.

7. τ^*-DETERMINING AND VOLUME-DETERMINING FAMILIES OF BOXES

A main tool in our investigations will be that of τ^*-*determining* families of boxes. These are families of sub-boxes of Q_n having the property that if a given set S is hefty relative to every box in the family, then S is hefty in Q_n. We shall find a surprising equivalent condition: that the boxes cannot all be enlarged simultaneously by infinitesimal changes in their boundaries.

Here are precise definitions of these notions. By a *box* in $\Re^n$ we always mean a box whose sides are parallel to the axes. For such a box B we shall denote by $I_j(B)$ $(1 \leq j \leq n)$ the interval which is the projection of B on the x_j-axis, and by $\ell_j(B)$ the length of $I_j(B)$. Unless otherwise stated, we shall assume that boxes are not degenerate, namely $\ell_j(B) > 0$ for all j. By $vol(B)$ we denote the volume of B.

Given a box B and a system $\mathcal{F} = (f_1, f_2, \dots, f_n)$ of measurable real valued functions, where the domain of f_j contains $I_j(B)$, we shall write $t(B, \mathcal{F}) = \sum_j \frac{f_j(I_j(B))}{\ell_j(B)}$. This is a "normalized" version of the linear functional $\sum f_j[0,1]$ which is used in the definition of τ^* (and indeed, coincides with it when $B = Q_n$). If the functions in $\mathcal{F}$ are nonnegative valued, and if $\sum_j f_j(x_j) \geq 1$ for each point $\mathbf{x} \in S \cap B$, then we say that $\mathcal{F}$ is a *fractional plank cover* of S relative to B.

The minimum of $t(B, \mathcal{F})$ over all fractional plank covers of S relative to B will be denoted by $\tau^*_B(S)$. An equivalent definition is that $\tau^*_B(S)$ is τ^* of the subset of Q_n obtained by stretching $S \cap B$ by a factor of $\frac{1}{\ell_j(B)}$ in the direction of each coordinate j.

Definition 7.1. *A subset* S *of* Q_n *is called* B-hefty *if* $\tau^*_B(S) = 1$.

Definition 7.2. *A family* $\mathcal{B}$ *of sub-boxes of* Q_n *is* τ^* -determining *if for each family* $\mathcal{F} = (f_1, \dots, f_n)$ *of nonnegative-valued measurable functions the inequalities* $t(B, \mathcal{F}) \geq 1,\ B \in \mathcal{B}$, *imply the inequality* $t(Q_n, \mathcal{F}) \geq 1$.

Remark. Similarly to Proposition 4.2, it is easy to show that equivalently one can remove the nonnegativity condition, while requiring that $t(B, \mathcal{F}) \geq 0,\ B \in \mathcal{B}$, imply $t(Q_n, \mathcal{F}) \geq 0$.

The definition implies:

Lemma 7.3. *If a family* $\mathcal{B}$ *of sub-boxes of* Q_n *is* τ^**-determining then any set which is* B*-hefty for all boxes* $B \in \mathcal{B}$ *is hefty.* □

Let $\mathcal{B}$ be a finite family of boxes in $\Re^n$. We write $M(\mathcal{B})$ for the smallest box containing all boxes in $\mathcal{B}$. We shall denote by $\mathcal{P}(\mathcal{B})$ the set of hyperplanes perpendicular to the axes and supporting the boxes in $\mathcal{B}$. For each j, these hyperplanes partition $I_j(M(\mathcal{B}))$ into $m_j(\mathcal{B})$ intervals $I_j^k,\ 1 \le k \le m_j(\mathcal{B})$, each being of length (say) ℓ_j^k. Let $K_j(B)$ denote the set of those k for which the box B contains points with $x_j \in I_j^k$ (i.e., $I_j(B)$ is the union of the intervals $I_j^k,\ k \in K_j(B)$).

Assigning a variable w_j^k to each interval I_j^k, we can define a linear form $t_{\mathcal{B}}(B)(\mathbf{w})$ by

$$t_{\mathcal{B}}(B)(\mathbf{w}) = \sum_j \frac{\sum_{k\in K_j(B)} w_j^k}{\sum_{k\in K_j(B)} \ell_j^k}.$$

Using this terminology, we obtain the following characterization of a τ^*-determining family.

Lemma 7.4. *Let $\mathcal{B}$ be a family of sub-boxes of Q_n. Let $\hat{\mathcal{B}} = \mathcal{B} \cup \{Q_n\}$. Then the following are equivalent:*
(i) $\mathcal{B}$ is τ^*-determining.
(ii) $t_{\hat{\mathcal{B}}}(Q_n)$ is a convex combination of $t_{\hat{\mathcal{B}}}(B),\ B \in \mathcal{B}$.
(iii) $t_{\hat{\mathcal{B}}}(Q_n)$ is a nonnegative linear combination of $t_{\hat{\mathcal{B}}}(B),\ B \in \mathcal{B}$. $\square$

Two families of boxes $\mathcal{C}$ and $\mathcal{D}$ will be called *similar* if $m_j(\mathcal{C}) = m_j(\mathcal{D})$ for each $1 \le j \le n$, and if there is a bijection $\phi : \mathcal{C} \to \mathcal{D}$ such that $K_j(\phi(C)) = K_j(C),\ j = 1, \ldots, n$, for all $C \in \mathcal{C}$. An equivalence class of the relation of similarity will be called a *configuration of boxes.*

A configuration of boxes Γ will be called *volume-determining* if for every two families of boxes $\mathcal{C}, \mathcal{D} \in \Gamma$, if $vol(C) \le vol(\phi(C))$ for all $C \in \mathcal{C}$ (where ϕ is as above) then $vol(M(\mathcal{C})) \le vol(M(\mathcal{D}))$.

A configuration of boxes Γ will be called τ^**-determining* if every $\mathcal{C} \in \Gamma$ satisfying $M(\mathcal{C}) = Q_n$ is τ^*-determining. A somewhat surprising fact is that these two conditions are equivalent:

Theorem 7.5. *A configuration of boxes is τ^*-determining if and only if it is volume-determining.*

Proof. Let Γ be a volume-determining configuration of boxes in $\Re^n$, choose $\mathcal{B} \in \Gamma$ and let $C = M(\mathcal{B})$. Consider the expressions $vol(B), B \in \mathcal{B}$, as functions of the variables ℓ_j^k (the lengths of the intervals in the partitions induced by $\mathcal{P}(\mathcal{B})$). By the assumption that Γ is volume-determining, for each vector $\mathbf{v}$ in $\Re^{\sum_j m_j(\mathcal{B})}$, if all functions $vol(B),\ B \in \mathcal{B}$ increase in the direction of $\mathbf{v}$, then so does the function $vol(C) = \prod_j \sum_k \ell_j^k$. It follows that for all $\mathbf{v}$, if $\nabla vol(B) \cdot \mathbf{v} > 0$ for each $B \in \mathcal{B}$ then $\nabla vol(C) \cdot \mathbf{v} \ge 0$. Replacing $\mathbf{v}$ by $\mathbf{v} + \overrightarrow{\epsilon}$ for $\epsilon > 0$ and letting ϵ tend to 0, it follows that already $\nabla vol(B) \cdot \mathbf{v} \ge 0$ for each $B \in \mathcal{B}$ implies $\nabla vol(C) \cdot \mathbf{v} \ge 0$. By the theory of linear inequalities this means that $\nabla vol(C)$ is a nonnegative combination of the vectors $\nabla vol(B), B \in \mathcal{B}$. When $C = Q_n$, this is easily seen to be equivalent to condition (iii) in Lemma 7.4.

For the proof of the converse, assume that the configuration Γ is τ^*-determining, and let $\mathcal{C}, \mathcal{D}$ be two families in Γ with a correspondence $\phi : \mathcal{C} \to \mathcal{D}$ between them. We have to show that if

$$vol(C) \leq vol(\phi(C)) \tag{4}$$

for every $C \in \mathcal{C}$, then $vol(M(\mathcal{C})) \leq vol(M(\mathcal{D}))$. Without loss of generality, we assume that $M(\mathcal{C}) = Q_n$.

For each $C \in \mathcal{C}$ let

$$\mathbf{p}(C) = \left(\frac{\sum_{k \in K_j(C)} {\ell'}_j^k}{\sum_{k \in K_j(C)} \ell_j^k} \right)_{1 \leq j \leq n}$$

where $\ell_j^k = \ell_j^k(\mathcal{C})$ and ${\ell'}_j^k = \ell_j^k(\mathcal{D})$.

The inequality (4) can be written as:

$$\prod_{j=1}^{n} \frac{\sum_{k \in K_j(C)} {\ell'}_j^k}{\sum_{k \in K_j(C)} \ell_j^k} \geq 1.$$

But this means that the points $\mathbf{p}(C), C \in \mathcal{C}$, belong to the convex subset AH (the notation standing for "above hyperbola") of the positive orthant of $\Re^n$, consisting of those points whose product of coordinates is at least 1.

Apply now condition (ii) of Lemma 7.4 to the family $\mathcal{C}$, substituting ${\ell'}_j^k$ for w_j^k. The conclusion obtained is that the point $(\sum_k {\ell'}_j^k)_{1 \leq j \leq n}$ is a convex combination of the points $\mathbf{p}(C)$, and since AH is convex, this point belongs to AH. But this means precisely that $vol(M(\mathcal{D})) \geq 1 = vol(M(\mathcal{C}))$, as required. □

The proof yields a little more: it suffices to consider volume changes corresponding to infinitesimal changes in the partitions. This leads to:

Corollary 7.6. *A family $\mathcal{B}$ of sub-boxes of Q_n satisfying $M(\mathcal{B}) = Q_n$ is τ^*-determining if and only if one cannot move the hyperplanes in $\mathcal{P}(\mathcal{B})$, while fixing those supporting Q_n itself, in such a way that all boxes in $\mathcal{B}$ strictly grow in volume.* □

Corollary 7.7. *If $\mathcal{B}$ is a family of sub-boxes of Q_n satisfying $M(\mathcal{B}) = Q_n$ whose configuration is τ^*-determining, then $\bigcup \mathcal{B} = Q_n$ (we say then that $\mathcal{B}$ is* covering*).*

Proof. Assume that all boxes in $\mathcal{B}$ miss some cell C in the partition of Q_n formed by the hyperplanes in $\mathcal{P}(\mathcal{B})$. Then one can move the boundaries of C so as to make C fill almost all of Q_n, while all boxes in $\mathcal{B}$ get smaller. This contradicts Corollary 7.6, applied to the new family of boxes. □

Note that a family of sub-boxes which is τ^*-determining does not have to be covering: take the two boxes $x, y \leq \frac{1}{2}$ and $x, y \geq \frac{1}{2}$ in Q_2.

Here is an example, in Q_2, of a configuration which is covering but not τ^*-determining: choose two points (a, b) and (c, d) in the unit square, where $a < c$ and $b < d$. Let $\mathcal{B}$ consist of the four boxes $0 \leq x \leq c, 0 \leq y \leq d; a \leq x \leq 1, b \leq y \leq 1; 0 \leq x \leq a, d \leq y \leq 1$ and $c \leq x \leq 1, 0 \leq y \leq b$.

However, we conjecture that for all n, if $m_j(\mathcal{B}) = 2$ for each j, then the condition is sufficient.

Another simple corollary of Theorem 7.5, which can also be proved directly, is:

Corollary 7.8. *If the family of boxes $\mathcal{B}$ partitions Q_n, then it is τ^*-determining.* □

We shall mainly use one particular τ^*-determining configuration of boxes:

Lemma 7.9. *Let $a_1, a_2, \ldots, a_n$ be numbers in the open interval $(0,1)$, and let $B_0 = \prod_{j=1}^n [0, a_j]$, $B_k = Pl_k([a_k, 1])(1 \le k \le n)$. Then the family $B_i, 0 \le i \le n$, is τ^* - determining.*

Proof. Enlarging the volumes of the boxes B_i, $1 \le i \le n$, means making the a_i's smaller, which means making B_0 smaller. By Corollary 7.6 the result follows.

It is also not hard to find the coefficients of the expression of $t_{\hat{\mathcal{B}}}(Q_n)$ as a combination of the $t_{\hat{\mathcal{B}}}(B_i)$'s:

Note that

$$t_{\hat{\mathcal{B}}}(B_0)(\mathbf{w}) = \sum_j \frac{w_j^1}{a_j},$$

$$t_{\hat{\mathcal{B}}}(B_i)(\mathbf{w}) = \frac{w_i^2}{1-a_i} + \sum_{k \ne i}(w_k^1 + w_k^2) \ (1 \le i \le n).$$

Hence we have:

$$(\prod_j a_j) t_{\hat{\mathcal{B}}}(B_0)(\mathbf{w}) + \sum_{i=1}^n (1-a_i)(\prod_{k\ne i} a_k) t_{\hat{\mathcal{B}}}(B_i)(\mathbf{w}) =$$

$$(\sum_j \frac{1}{a_j} - (n-1))(\prod_j a_j) \sum_j (w_j^1 + w_j^2) = \alpha t_{\hat{\mathcal{B}}}(Q_n)(\mathbf{w})$$

where α is some positive number. □

8. Proof of Theorem 6.3

8.1. **Splitting one vertex.** We start with a basic observation, already used in the special case $B = Q_n$:

Proposition 8.1. *A set containing a main diagonal of B is B-hefty.*

Proof. Let S be a set containing a main diagonal D of B, and let $\mathcal{F}$ be a fractional plank cover of S relative to B.

Let $x_j(t) = a_j + t(b_j - a_j)$ $(0 \le t \le 1)$ range over $I_j(B)$ in such a way that $(x_1(t), \ldots, x_n(t))$ ranges over D. Since $\sum f_j(x_j) \ge 1$ for every $\mathbf{x} \in S \cap B$, we have $\sum \int_0^1 f_j(x_j(t))dt \ge 1$. But this means that $t(B, \mathcal{F}) \ge 1$. □

We first prove the theorem in the case that just one of the two antipodal vertices is split. In this case the theorem is:

Theorem 8.2. *Let $a_1, \ldots, a_n$ be real numbers between 0 and 1, and let $\mathbf{p}_j = a_j \mathbf{e}_j$, where $\mathbf{e}_j$ is the j-th unit vector, $(0, \ldots, 1, \ldots, 0)$. Then the set*

$$S = conv\{\mathbf{p}_1, \ldots, \mathbf{p}_n, \overrightarrow{1}\}$$

is hefty.

Proof. Let $B_0, B_1, \dots, B_n$ be the sub-boxes of Q_n as defined in Lemma 7.9, and let $\mathcal{F} = (f_1, \dots, f_n)$ be a system of nonnegative-valued functions covering S. We have to prove that $\sum f_j[0,1] \geq 1$. For each $0 \leq s \leq 1$ let $\phi(s) = \sum_j f_j(s)$.

For each pair $i \neq j$ of coordinates let D_{ij} be the degenerate box having the segment between the points $\mathbf{p}_i$ and $\mathbf{p}_j$ as a main diagonal.

Note that each $B_i, 1 \leq i \leq n$, has a main diagonal contained in S, and thus S is B_i-hefty for $1 \leq i \leq n$. What is missing for an application of Lemma 7.9 is the B_0-heftiness of S. This, as it turns out, is true for $n = 3$ (the proof of this fact requires the theorem itself!) but fails in general for $n > 3$. However, the diagonals of D_{ij} are contained in S, and $t(B_0, \mathcal{F})$ can be nicely expressed by $t(D_{ij}, \mathcal{F}) = \sum_{k \neq i,j} f_k(0) + \frac{f_i[0,a_i]}{a_i} + \frac{f_j[0,a_j]}{a_j}$. We have:

$$t(B_0, \mathcal{F}) = \frac{1}{n-1} \sum_{i<j} t(D_{ij}, \mathcal{F}) - \frac{n-2}{2}\phi(0) \geq \frac{n}{2} - \frac{n-2}{2}\phi(0).$$

This implies that if $\phi(0) \leq 1$ then we are done. On the other hand, if $\phi(s) > 1$ for all $0 \leq s \leq 1$, then $\sum_j f_j[0,1] = \phi[0,1] > 1$. This leads us to consider $s_0 = \inf\{s : \phi(s) \leq 1\}$. If the point $\overrightarrow{\mathbf{s_0}} = (s_0, s_0, \dots, s_0)$ lies on or above the hyperplane H determined by $\mathbf{p}_1, \dots, \mathbf{p}_n$ then the segment $[\overrightarrow{\mathbf{s_0}}, \overrightarrow{\mathbf{1}}]$ is contained in S, and hence $\sum_j f_j[0,1] = \phi[0, s_0] + \phi[s_0, 1] \geq s_0 + 1 - s_0 = 1$. Thus we may assume that $\overrightarrow{\mathbf{s_0}}$ lies below H.

Let $r \geq s_0$ be such that $\phi(r) \leq 1$ and $\overrightarrow{\mathbf{r}} = (r, r, \dots, r)$ still lies below H. Applying the above argument to the cube with the two antipodal vertices $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{1}}$ we deduce that $\phi[r,1] \geq 1 - r$. By the choice of s_0 and the nonnegativity of $f_1, \dots, f_n$ we obtain that $\phi[0,1] = \phi[0,s_0] + \phi[s_0, r] + \phi[r,1] \geq s_0 + 1 - r$, and letting $r \to s_0$ yields $\phi[0,1] \geq 1$, as required. □

In order to prove Theorem 6.3, we shall need a slight extension of Theorem 8.2, where one of the splinters is allowed to lie outside the cube.

Proposition 8.3. *Suppose $n \geq 3$. Let $a_1, \dots, a_n$ be nonnegative real numbers, at most one of which exceeds 1, and let $\mathbf{p}_j = a_j \mathbf{e}_j$. Then the set*

$$S = conv\{\mathbf{p}_1, \dots, \mathbf{p}_n, \overrightarrow{\mathbf{1}}\} \cap Q_n$$

is hefty.

Proof. We may assume that $a_j \leq 1$ for all $j \neq 1$. Let k be the least nonnegative integer such that $a_1 \leq \left(\frac{n-1}{n-2}\right)^k$. We proceed by induction on k. The case $k = 0$ was handled in Theorem 8.2, so we assume that $k \geq 1$, i.e., $a_1 > 1$.

Consider the box $B_0 = \prod_{j=1}^n [0, b_j]$, where $b_1 = 1$ and for $2 \leq j \leq n$ we have $b_j = q a_j$, with $q = \frac{2 - \frac{1}{a_1}}{n-1}$. It is straightforward to check that the hyperplane determined by $\mathbf{p}_1, \dots, \mathbf{p}_n$ passes through the center of B_0 and meets all its facets. It follows by Theorem 5.4 that S is B_0-hefty.

The box B_0, together with the boxes $B_j = Pl_j([b_j, 1])$ for $2 \leq j \leq n$, forms a τ^*-determining family of sub-boxes of Q_n. (Note that because we set $b_1 = 1$, the box B_1 has vanished, but the argument showing that the family is τ^*-determining remains valid.) Thus, it suffices to show that S is B_j-hefty for $2 \leq j \leq n$. We show this for $j = n$, for example.

The set S contains the set $S' = conv\{\mathbf{p'}_1, \dots, \mathbf{p'}_n, \vec{\mathbf{1}}\} \cap B_n$, where

$$\mathbf{p'}_i = (1-q)\mathbf{p}_i + q\mathbf{p}_n$$

for $1 \le i \le n-1$ and

$$\mathbf{p'}_n = \mathbf{p}_n.$$

Now, S' relates to B_n in the same way as S relates to Q_n in the statement of the proposition (up to a normalization in the n-th direction). Moreover, denoting by a'_1 the first coordinate of $\mathbf{p'}_1$, we have

$$a'_1 = (1-q)a_1 < \frac{n-2}{n-1}a_1 \le \left(\frac{n-1}{n-2}\right)^{k-1}.$$

Hence, by the induction hypothesis, the set S' (and therefore S) is B_n-hefty. □

8.2. **Proof of the general case of Theorem 6.3.** We may assume that $n \ge 3$, and that the set S in question is of the form $S = conv\{\mathbf{p}_1, \dots, \mathbf{p}_n, \ \mathbf{q}_1, \dots, \mathbf{q}_n\}$, where $\mathbf{p}_j = a_j\mathbf{e}_j$ and $\mathbf{q}_j = \vec{\mathbf{1}} - b_j\mathbf{e}_j$ for some numbers a_j, b_j between 0 and 1. Let $\mathcal{F} = (f_1, \dots, f_n)$ be a system of nonnegative-valued functions covering S. We have to prove that $\sum f_j[0,1] \ge 1$, or equivalently, using the notation $\phi(s) = \sum_j f_j(s)$, that $\phi[0,1] \ge 1$. This is trivially true if $\{s : \phi(s) \le 1\} = \emptyset$, so we assume that this set is non-empty, and consider:

$$\begin{aligned} s_0 &= \inf\{s : \phi(s) \le 1\}, \\ s_1 &= \inf\{s : \phi(1-s) \le 1\}, \\ s^* &= \min\{s_0, s_1\}. \end{aligned}$$

Clearly, $\phi[0,s^*] + \phi[1-s^*,1] \ge 2s^*$, and it remains to prove that $\phi[s^*, 1-s^*] \ge 1 - 2s^*$.

Let us consider the two points $\overrightarrow{\mathbf{s}^*}$ and $\overrightarrow{\mathbf{1}-\mathbf{s}^*}$ on the diagonal of Q_n, and the cube B^* having these two points as antipodal vertices. Let us denote by H_0 (respectively H_1) the hyperplane determined by $\mathbf{p}_1, \dots, \mathbf{p}_n$ (respectively $\mathbf{q}_1, \dots, \mathbf{q}_n$), and by I the segment of the diagonal between H_0 and H_1. We distinguish between several cases concerning the position of $\overrightarrow{\mathbf{s}^*}$ and $\overrightarrow{\mathbf{1}-\mathbf{s}^*}$ relative to I.

Case 1. Both points lie in I.

In this case the diagonal $\left[\overrightarrow{\mathbf{s}^*}, \overrightarrow{\mathbf{1}-\mathbf{s}^*}\right]$ of B^* is contained in S and hence $\phi[s^*, 1-s^*] \ge 1-2s^*$.

Case 2. Exactly one of the points lies in I.

Let us assume without loss of generality that $\overrightarrow{\mathbf{1}-\mathbf{s}^*}$ lies in I (and hence in S) and that $\overrightarrow{\mathbf{s}^*}$ lies below H_0. For $1 \le i \le n$, let $\mathbf{p'}_i$ be the point on the intersection of H_0 with the line parallel to the x_i-axis going through $\overrightarrow{\mathbf{s}^*}$. We check that $\mathbf{p'}_i$ is in B^*, that is, its i-th coordinate a'_i does not exceed $1-s^*$. Indeed, we have $\frac{1}{a_i}a'_i + \sum_{j\ne i}\frac{1}{a_j}s^* = 1$, and hence $a'_i = a_i\left(1 - s^*\sum_{j\ne i}\frac{1}{a_j}\right) \le 1-s^*$. Thus, we may apply Theorem 8.2 to the cube B^* and obtain that $\phi[s^*, 1-s^*] \ge 1-2s^*$.

Case 3. None of the points lies in I.

Let us assume without loss of generality that $s^* = s_0$. Let $r \ge s^*$ be such that $\phi(r) \le 1$ and $\vec{\mathbf{r}}$, $\overrightarrow{\mathbf{1}-\mathbf{r}}$ still lie outside I. For $1 \le i \le n$, let $\mathbf{p'}_i$ (respectively $\mathbf{q'}_i$)

be the point on the intersection of H_0 (respectively H_1) with the line parallel to the x_i-axis going through $\overrightarrow{\mathbf{r}}$ (respectively $\overrightarrow{\mathbf{1}-\mathbf{r}}$). As was shown above, the points $\mathbf{p'}_i$ all lie in the cube $B = [r, 1-r]^n$, and the same holds for the points $\mathbf{q'}_i$.

We consider a τ^*-determining family of sub-boxes of B as in Lemma 7.9. It consists of $B_0 = \prod_j [r, a'_j]$, where a'_j is the j-th coordinate of $\mathbf{p'}_j$, and of $B_k = [a'_k, 1-r] \times \prod_{j \neq k} [r, 1-r]$, $1 \leq k \leq n$. As it was shown in the proof of Theorem 8.2, the fact that $\phi(r) \leq 1$ implies that $t(B_0, \mathcal{F}) \geq 1$. To each of the B_k's we may apply Proposition 8.3. Indeed, the set S contains the splinters $\mathbf{q'}_1, \dots, \mathbf{q'}_n$ of the vertex $\overrightarrow{\mathbf{1}-\mathbf{r}}$ of B_k, as well as the antipodal vertex of B_k, namely $\mathbf{p'}_k$. Among the splinters $\mathbf{q'}_1, \dots, \mathbf{q'}_n$, only $\mathbf{q'}_k$ may lie outside B_k. Thus, we conclude from Proposition 8.3 that S is B_k-hefty for $1 \leq k \leq n$. It follows that $t(B, \mathcal{F}) \geq 1$, that is, $\phi[r, 1-r] \geq 1 - 2r$. Using the nonnegativity of $f_1, \dots, f_n$ and letting $r \to s^*$, we obtain that $\phi[s^*, 1-s^*] \geq 1 - 2s^*$. □

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