Bounding Ramsey numbers through large deviation inequalities

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Abstract

We develop a new approach for proving lower bounds for various Ramsey numbers, based on using large deviation inequalities. This approach enables us to obtain the bounds for the off-diagonal Ramsey numbers $R(K^r, K^k), r < k$, that match the best known bounds, obtained through the local lemma. We discuss also the bounds for a related Ramsey-type problem and show, for example, that there exists a K^4-free graph G on n vertices in which every $cn^{3/5}\log^{1/2} n$ vertices span a copy of K^3.

1 Introduction

Let G and H be two fixed graphs. The *Ramsey number* $R(G, H)$ is the smallest number n_0 such that in any red-blue coloring of the edges of a complete graph on n_0 vertices K^{n_0} there either is a red copy of G (i.e., a

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copy of G all of whose edges are colored red) or there is a blue copy of H. Here we deal with asymptotic bounds for $R(G,H)$.

The probabilistic method (see [2] as a general reference) turns out to be the most powerful tool for finding lower bounds for various Ramsey numbers. This method shows that in some appropriately chosen probability space there exists with positive probability a graph on n vertices that does not contain a copy of G and its complement does not contain a copy of H, yielding $R(G,H) > n$.

In many cases the best known results have been obtained by using the *Lovász local lemma* ([5]). For example, the local lemma provides the best known bounds for the so called diagonal Ramsey numbers $R(K^k, K^k)$ and off-diagonal Ramsey numbers $R(K^r, K^k)$, $r < k$.

In this paper we develop a different approach for finding lower bounds for Ramsey numbers, based on using *large deviation inequalities*. Our technique suffices to match the best known lower bounds for various Ramsey numbers including $R(K^r, K^k)$ and to improve the best known estimates for a Ramsey-type function, introduced by Erdős and Rogers in [6] and generalizing the classical Ramsey numbers $R(K^r, K^k)$.

2 Large deviation inequalities

The simplest example of large deviation inequalities is the bounds on the tail of binomial distribution. If X is the sum of n mutually independent random variables each of which takes the value 1 with probability p and the value 0 with probability $1-p$, then for every constant $0 < \epsilon < 1$ one has:

$$Pr[X < (1-\epsilon)np] \ < \ e^{-\epsilon^2 pn/2}\ , \qquad (1)$$

$$Pr[X > (1+\epsilon)np] \ < \ e^{-\epsilon^2(1-\epsilon)np/2}\ . \qquad (2)$$

(see, e.g., Theorems A.13 and A.11 of [2]).

In words, the binomially distributed random variable is highly concentrated around its mean and its tails are exponentially small.

We will also use the following bound on the lower tail of a Poisson random variable P_λ, defined by $Pr[P_\lambda = i] = e^{-\lambda}\lambda^i/i!,\ i = 0, 1, \ldots$:

$$Pr[P_\lambda < (1-\epsilon)\lambda] \leq e^{-\epsilon^2\lambda/2} \tag{3}$$

(see, e.g., Theorem A.15 of [2]).

When X is the sum of many 'mostly independent' indicator random variables, there is some hope that X is also highly concentrated around its mean as a binomial random variable. Recent results ([9], [7], [8], [12]) give sufficient conditions for this phenomenon to hold. Below we describe a general framework for applying these results.

Suppose Q is a finite set (in our instances Q is the edge set of a complete graph on n vertices). Let $\{J_i : i \in Q\}$ be a set of independent random indicator variables, $Pr[J_i = 1] = p_i$ for every $i \in Q$ ($J_i = 1$ if the corresponding edge belongs to $E(G)$, $G \sim G(n,p)$). Let $\{Q(\alpha)\}_{\alpha\in I}$ be a family of subsets of Q, where I is a finite index set. Define $X_\alpha = \prod_{i\in Q(\alpha)} J_i$ (then $X_\alpha = 1$ if and only if all edges of $Q(\alpha)$ belong to $E(G)$). Now define

$$X = \sum_{\alpha\in I} X_\alpha$$

(in our instances X counts the number of subgraphs of G having specified properties).

For our purposes it will be more convenient to use the estimates on the tails of another random variable X_0 which is tightly connected with X and is defined as

$$X_0 = \max\{\ m : \exists \text{ distinct } \alpha_1, \ldots, \alpha_m \in I \text{ with } X_{\alpha_i} = 1 \text{ and } Q(\alpha_i) \cap Q(\alpha_j) = \emptyset,\ i \neq j\} \tag{4}$$

(X_0 is equal to the size of the maximum family of pairwise disjoint subsets $Q(\alpha_j)$, $1 \leq j \leq m$, such that the indicator random variables J_i for all members i of $Q(\alpha_i)$ attain the value 1). Obviously, $X_0 \leq X$.

Let $\mu = EX$ be the expectation of X, then the following holds (see [7]):
Claim 1

$$Pr[X_0 \geq k] \leq \frac{\mu^k}{k!}$$

for every natural k.

Proof.

$$\begin{aligned} Pr[X_0 \geq k] &\leq \sum^{1} Pr[(X_{\alpha_1} = 1) \wedge \ldots \wedge (X_{\alpha_k} = 1)] \\ &= \frac{1}{k!} \sum^{2} Pr[(X_{\alpha_1} = 1) \wedge \ldots \wedge (X_{\alpha_k} = 1)] \\ &= \frac{1}{k!} \sum^{2} Pr[X_{\alpha_1} = 1] \ldots Pr[X_{\alpha_k} = 1] \\ &\leq \frac{1}{k!} \sum^{3} Pr[X_{\alpha_1} = 1] \ldots Pr[X_{\alpha_k} = 1] = \frac{\mu^k}{k!}\ , \end{aligned}$$

where $\sum^1$ is over sets of k mutually independent events $X_{\alpha_i} = 1$, while $\sum^2$ is over ordered k-tuples of mutually independent events and $\sum^3$ is over all ordered k-tuples of events.

(It is worth mentioning that Janson ([8]) obtained a similar result which yields exponential estimates of $Pr[X_0 \geq (1+\epsilon)\mu]$ for *every* $\epsilon > 0$. However, his proof is a bit more complicated than the one cited above).

In particular, we deduce from the above Claim that

$$Pr[X_0 \geq 5\mu] < \left(\frac{e}{5}\right)^{5\mu}\ . \tag{5}$$

The inequalities (1) and (5) are sufficient to obtain the lower bound for the classical Ramsey numbers $R(K^r, K^k)$. In Section 4 we will describe also an exponential bound on the lower tail of X_0, which will be used to obtain new estimates on a Ramsey-type function $f_{r,s}(n)$, defined explicitly in Section 5.

3 Lower bounds for the Ramsey numbers $R(H, K^k)$

Let H be a graph with $v \geq 3$ vertices and $f \geq 1$ edges. The *density* $\rho(H)$ of H is defined as $\rho(H) = (f-1)/(v-2)$. (Note that our definition of density differs slightly from the usual one, used, for example, in [2]). We call H *balanced* if for every subgraph $H' \subseteq H$ one has $\rho(H') \leq \rho(H)$. Define also $\rho_0(H) = \max_{H' \subseteq H} \rho(H')$. Thus, if H is balanced then $\rho_0(H) = \rho(H)$.

Theorem 1 *There exists a constant $c > 0$ depending only on H such that*

$$R(H, K^k) \geq c(k/\log k)^{\rho_0(H)} .$$

Proof. Suppose first that H is balanced , then $\rho_0(H) = \rho(H)$. Consider a random graph $G(n,p)$ - a graph on n vertices in which all edges are chosen independently and with probability p, set $p = c_1 n^{-1/\rho(H)}$ for some constant c_1 to be chosen later. Let also $k = c_2 n^{1/\rho(H)} \ln n$ where c_2 is a constant. We will show that if c_2 is large enough then there exists an H-free graph on n vertices in which every subset of k vertices spans an edge.

For every subset $S \subseteq V$ of size $|S| = k$ denote by X_S the number of edges spanned by S, by Y_S the number of copies of H sharing at least one edge with S, and by Z_S the maximal number of pairwise edge disjoint copies of H, each sharing at least one edge with S. Obviously, $Z_S \leq Y_S$. Let A_S be the event $X_S > fZ_S$.

Claim 2 *If A_S holds for every $S \subseteq V$ of size $|S| = k$, then G contains an H-free subgraph G' on n vertices in which every k vertices span at least one edge.*

Proof. Let $\mathcal{H}$ be any maximal (under inclusion) family of pairwise edge disjoint copies of H in G. Deleting all edges of all graphs from $\mathcal{H}$ we obtain an H-free graph G' on n vertices. For a subset $S \subset V$ of size $|S| = k$ denote by $\mathcal{H}_S$, $|S| = k$, the subfamily of $\mathcal{H}$, consisting of all graphs from $\mathcal{H}$, sharing at least one edge with S. Clearly $|\mathcal{H}_S| \leq Z_s$. While deleting the edges of graphs from $\mathcal{H}$, we delete at most $f|\mathcal{H}_S| \leq fZ_s$ edges from S, therefore after the deletion every set S of k vertices has at least one edge. $\Box$

Returning to the proof of the theorem, we now prove that the constants c_1 and c_2 can be chosen so that $Pr[\bigwedge_{|S|=k} A_s] > 0$. To this end, we show that both X_S and Z_S are highly concentrated around their means, and hence if, say, $EX_S > 10fEZ_S$, then the probability of $\overline{A_S}$ is exponentially small. This implies that the probability that there exists some S such that $\overline{A_S}$ holds is less than 1.

Obviously, the random variable X_S is distributed binomially with parameters $\binom{k}{2}$ and p, and hence, by (1) for every $0 < \epsilon < 1$ there holds

$$Pr[X_S \leq (1-\epsilon)\binom{k}{2}p] < e^{-\epsilon^2\binom{k}{2}p/2} . \tag{6}$$

We now turn to bound the upper tail of Z_S. Clearly, $EZ_S \leq EY_S$. In order to compute EY_S we represent Y_S as a sum of indicator random variables $Y_{T,S}$ over all subsets T of size v such that $|T \cap S| \geq 2$. Let $Y_{T,S} = 1$ if $G[T]$ contains a copy of H. Then

$$p^f \leq EY_{T,S} \leq v!p^f ,$$

so

$$\binom{k}{2}\binom{n-k}{v-2}p^f \leq EY_S = \sum_{|T\cap S|\geq 2} EY_{T,S} \leq \binom{k}{2}\binom{n-2}{v-2}v!p^f .$$

Recall that $k = o(n)$, therefore

$$c_3\binom{k}{2}n^{v-2}p^f \leq EY_S \leq c_4\binom{k}{2}n^{v-2}p^f ,$$

where $c_4 > c_3 > 0$ are some constants, depending only on v and f.

Denoting by I the set of all possible copies of H sharing at least one edge with S, we can easily see that Y_S, EY_S, Z_S instead of X, μ, X_0, respectively, fit in the framework from the introduction. Then (5) gives

$$Pr[Z_S \geq 5EY_S] \leq e^{-5(\ln 5-1)EY_S} \ . \tag{7}$$

Comparing EX_S and EY_S observe that

$$\frac{\binom{k}{2}p}{c_4\binom{k}{2}n^{v-2}p^f} \leq \frac{EX_S}{EY_S} \leq \frac{\binom{k}{2}p}{c_3\binom{k}{2}n^{v-2}p^f} \ .$$

Returning to the definition of p, we choose the constant $c_1 = c_1(H)$ so that the expression $c_4 n^{v-2}p^{f-1}$ will be equal, say, to $1/(10f)$, then

$$10f \leq \frac{EX_S}{EY_S} \leq \frac{c_4}{c_3}10f \ . \tag{8}$$

Now we estimate $Pr[\overline{A_S}]$. By (6) with $\epsilon = 1/2$, (7) and (8)

$$\begin{aligned} Pr[\overline{A_S}] &= Pr[X_S \leq fZ_S] \leq Pr[X_S \leq EX_S/2] + Pr[fZ_S \geq EX_S/2] \\ &\leq Pr[X_S \leq EX_S/2] + Pr[Z_S \geq 5EY_S] \\ &\leq e^{-\binom{k}{2}p/8} + e^{-\frac{c_3\binom{k}{2}p}{10c_4 f}[5\ln 5 - 5]} \leq 2e^{-c_5 k^2 p} \end{aligned}$$

for some positive constant c_5. Therefore

$$Pr[\exists S : \overline{A_S}] \leq \binom{n}{k} Pr[\overline{A_S}] \leq \binom{n}{k} 2e^{-c_5 k^2 p} \ .$$

Using the inequality $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$, we write

$$\binom{n}{k} 2e^{-c_5 k^2 p} < 2\left(\frac{en}{k} e^{-c_5 kp}\right)^k \ .$$

Choose now the constant c_2 in the definition of k to be sufficiently large, then the above estimates imply the desired inequality $Pr[\bigwedge_{|S|=k} A_S] > 0$.

In case H is not balanced we take a subgraph $H' \subset H$ such that $\rho_0(H) = \rho(H')$. Obviously, an H'-free graph is also H-free, so $R(H, K^k) \geq R(H', K^k) \geq c(k/\log k)^{\rho_0(H)}$. $\Box$

Remarks: 1) For $H = K^r$ the above bound

$$R(K^r, K^k) \geq c_r \left(\frac{k}{\log k}\right)^{\frac{r+1}{2}}$$

matches the best known bounds for the off-diagonal Ramsey numbers $R(K^r, K^k)$, obtained using the local lemma (see [11]);

2) In fact, we have shown not only existence of an H-free graph G on n vertices such that every subset of $k = cn^{1/\rho_0(H)} \log n$ vertices spans an edge, but even more: there exists a sufficiently large constant c' such that *almost every* graph G from $G(n,p)$ for $p = c''n^{-1/\rho_0(H)}$ contains an H-free subgraph G' on n vertices such that every subset of $k' = c'n^{1/\rho_0(H)} \log n$ vertices spans an edge, where the subgraph G' is obtained by a simple deletion procedure. Moreover, the above proof implies for every H a randomized polynomial time algorithm for constructing such a subgraph.

3) There is a simple meaning in the choice of the value of p: when $p = c_1 n^{-1/\rho(H)}$, the expectation of the number of copies of H through an edge e tends to a *constant*, hence when this constant is small we may hope that in every subset S there will be an edge that does not belong to any copy of H.

4 More large deviation inequalities

In this short section we present an exponential bound on the lower tail of the random variable X_0, defined in (4).

Let

$$\Delta = \sum_{\alpha \sim \beta} Pr[(X_\alpha = 1) \wedge (X_\beta = 1)] \ .$$

where the sum is over ordered pairs and $\alpha \sim \beta$ if $\alpha \neq \beta$ and $Q(\alpha) \cap Q(\beta) \neq \emptyset$. Let also

$$\nu = \max_{\alpha \in I} \sum_{\beta \sim \alpha} Pr[X_\beta = 1] \ .$$

Assume $Pr[X_\alpha = 1] \leq \eta$ for all $\alpha \in I$. Then Lemma 4.2 of Chapter 8 of [2] states in our notation that the probability $p(s)$ of the existence of a subset $J \subseteq I$ of size $|J| = s$ such that for every $\alpha_1 \neq \alpha_2 \in J$ there holds $Q(\alpha_1) \cap Q(\alpha_2) = \emptyset$ and also for every $\alpha \in I \setminus J$ there exists an $\alpha' \in J$ for which $Q(\alpha) \cap Q(\alpha') \neq \emptyset$ (such a subset J is called a maximal disjoint family, or maxdisfam, in [2]) can be bounded by

$$p(s) \leq \frac{\mu^s}{s!} \exp(-\mu) \exp(s\nu) \exp\left\{\frac{\Delta}{2(1-\eta)}\right\} .$$

Therefore for every constant $0 < \epsilon < 1$ one has the following estimate on the lower tail of X_0.

$$\begin{aligned} Pr[X_0 \leq (1-\epsilon)\mu] &\leq \sum_{s \leq (1-\epsilon)\mu} p(s) \leq \sum_{s \leq (1-\epsilon)\mu} \frac{\mu^s}{s!} \exp(-\mu) \exp(s\nu) \exp\left\{\frac{\Delta}{2(1-\eta)}\right\} \\ &\leq \exp\left\{\frac{\Delta}{2(1-\eta)}\right\} \sum_{s \leq (1-\epsilon)\mu)} \frac{\mu^s}{s!} \exp(-\mu) \exp((1-\epsilon)\mu\nu) \\ &= \exp\left\{\frac{\Delta}{2(1-\eta)} + (1-\epsilon)\mu\nu\right\} Pr[P_\mu \leq (1-\epsilon)\mu], \qquad (9) \end{aligned}$$

where P_μ is a Poisson random variable with mean μ.

5 K^s-free graphs without large K^r-free subgraphs - a new bound

Consider now the following Ramsey-type problem. Let $2 \leq r < s \leq n$ be natural numbers. Let G be a K^s-free graph on n vertices. What is the largest size of a K^r-free subgraph of G? Formally, let

$$f_{r,s}(n) = \min \max\{|V_0|, V_0 \subseteq V(G) : K^r \not\subseteq G[V_0]\} ,$$

where the minimum is taken over all K^s-free graphs G of order n. It is easy to see that for $r = 2$ we obtain the classical problem of determining the Ramsey numbers $R(K^s, K^k)$. Indeed, if $f_{2,s}(n) \geq k$, then every graph G on n vertices contains either a copy of K^s or a subset of k vertices that spans no edge, showing that $R(K^s, K^k) \leq n$. The special case $s = n - r + 2$ was considered by Erdős and Gallai([4]) in 1961, they established $f_{r,n-r+2}(n) = 2r - 2$ for all n satisfying $n > 2r - 2$. (In fact, Erdős and Gallai proved in their paper that if G is a graph on n vertices such that for some integer p the covering number of every induced subgraph of G on $2p + 2$ vertices does not exceed p, then the covering number of G also does not exceed p, they also showed the tightness of this result. The related result about $f_{r,n-r+2}(n)$, cited above, can be easily derived from their result just by reformulating it in terms of the complement of G.) Erdős and Rogers([6]) in 1962 showed that for a fixed s, $r = s - 1$, and n tending to infinity there exist K^s-free graphs of order n such that every induced subgraph of order $n^{1-\epsilon(s)}$ contains a K^{s-1}, where $\epsilon(s) \sim c/s^4 \log s$ for large values of s. The problem was considered again only about thirty years later by Bollobás and Hind([3]). Improving their results it is shown in [10] that

$$f_{r,s}(n) \geq c' n^{1/(s-r+1)} (\log \log n)^{1-1/(s-r+1)}$$

and

$$f_{r,s}(n) \leq c'' n^{(s-2)r/(s(s-1)-r)} (\log n)^{\left(\binom{s}{2}-\binom{r}{2}\right)/\left(\binom{s}{2}(r-1)-\binom{r}{2}\right)}, \qquad (10)$$

where c' and c'' are constants depending only on the values of r and s and in the particular case $r = 3, s = 4$

$$f_{3,4}(n) \leq c n^{2/3} (\log n)^{1/3}, \qquad (11)$$

where c is an absolute constant. The bounds (10) and (11) were obtained by combining the local lemma with the Janson inequality([9]). Some constructive bounds for $f_{r,s}(n)$ are presented in [1]. Here we improve the bound (11) using the technique based on large deviation inequalities. Our treatment is rather similar to that of previous section and can be used to improve (10) in a similar manner.

Theorem 2 *There exists an absolute constant c_0 such that*

$$f_{3,4}(n) \le c_0 n^{3/5} (\log n)^{1/2} \ .$$

Proof. Denote $k = c_0 n^{3/5}(\log n)^{1/2}$. In order to show $f_{3,4}(n) < k$ we will prove that in the probability space $G(n,p)$, with $p = c_1 n^{-2/5}$ and c_1 small enough, there exists with positive probability a graph G that *can be converted* by edge deletion to a K^4-free graph G' on n vertices in which every k vertices of it span a K^3.

Let us say that a graph H is of *type* $K^3 + K^4$ if it can be represented as a union of a copy of K^4 and a copy of K^3, sharing at least one edge. It is easy to see that there are exactly two $K^3 + K^4$-type graphs, one of them being K^4 itself. For every subset $S \subseteq V$ of size $|S| = k$ denote by X_S the maximal number of pairwise edge disjoint copies of K^3, spanned by S, by Y_S the number of $K^3 + K^4$-type graphs H such that $K^3 \subseteq G[S \cap V(H)]$ and by Z_S the maximal number of pairwise edge disjoint $K^3 + K^4$-type graphs, having a copy of K^3 in S. Obviously, $Z_S \le Y_S$. Let A_S be the event $X_S > 16 Z_S$.

Claim 3 *If A_S holds for every $S \subseteq V$ of size k then G contains a K^4-free subgraph G' on n vertices in which every k vertices span a K^3.*

Proof. Let $\mathcal{H}$ be a maximal (under inclusion) family of pairwise edge disjoint copies of K^4. Deleting all edges of all graphs from $\mathcal{H}$ we obtain a K^4-free graph G' on n vertices. Now we claim that every k vertices of G' span a copy of K^3. Suppose this is false, and let S be a set of k vertices having no triangles after deleting the edges of all graphs from $\mathcal{H}$. Fix a family $\mathcal{T}_S$ of pairwise edge disjoint triangles in S of size $|\mathcal{T}_S| = X_S$ and define an auxiliary graph G_S with vertex set $\mathcal{T}_S$ and with an edge joining two triangles t_1, t_2 from $\mathcal{T}_S$ if there exists a copy of K^4 from $\mathcal{H}$, having at least one edge in common with each of the triangles t_1, t_2. Since every triangle from $\mathcal{T}_S$ has a common edge with at most three graphs from $\mathcal{H}$ and every graph from $\mathcal{H}$

has a common edge with at most five other triangles from $\mathcal{T}_S$, the maximum degree in G_S does not exceed $3 \cdot 5 = 15$, therefore G_S has an independent set of vertices (triangles) of size at least $|V(G_S)|/16 = X_S/16$. Now take each triangle from such an independent set of size $X_S/16$ and join to it a copy of K^4 from $\mathcal{H}$ sharing an edge with it (such a copy exists since according to our assumption all triangles from $\mathcal{T}_S$ are destroyed when deleting edges of K^4-s from $\mathcal{H}$). By the definition of G_S we obtain a family of pairwise edge-disjoint $K^3 + K^4$-type subgraphs, each having a triangle in S, of size at least $X_S/16 > Z_S$, which is a contradiction. $\square$

Next we find k and p such that $Pr[\bigwedge_{|S|=k} A_S] > 0$. We will again show that both X_S and Z_S have exponentially small tails, so if, say, $EX_S > 160 EZ_S$ then $Pr[\overline{A_S}]$ is exponentially small.

Let us first estimate the lower tail of X_S. We are going to use inequality (9). To do so, define

$$\begin{aligned}
\mu &= \binom{k}{3} p^3 , \\
\Delta &= \sum_{\substack{T,T' \subseteq S \\ |T|=|T'|=3 \\ |T \cap T'|=2}} Pr[(G[T] \cong K^3) \wedge (G[T'] \cong K^3)] \\
&= \binom{k}{3} p^3 3(k-3)p^2 = O(k^4 p^5) , \\
\nu &= \sum_{T' : |T \cap T'|=2} Pr[G[T'] \cong K^3] = 3(k-3)p^3 = O(kp^3) ,
\end{aligned}$$

(note that the above sum is the same for every $T \subseteq S, |T| = 3$), and

$$\eta = Pr[G[T] \cong K^3] = p^3 .$$

By (9) and (3), for every constant $0 < \epsilon < 1$ one has:

$$\begin{aligned}
Pr[X_S \leq (1-\epsilon)\mu] &\leq \exp\left\{(1-\epsilon)\mu\nu + \frac{\Delta}{2(1-\eta)}\right\} Pr[P_\mu \leq (1-\epsilon)\mu] \\
&\leq \exp\left\{(1-\epsilon)\mu\nu + \frac{\Delta}{2(1-\eta)} - \epsilon^2 \mu/2\right\}
\end{aligned}$$

Both $\mu\nu$ and Δ are $o(\mu)$ and so we have for n sufficiently large

$$Pr[X_S \leq (1-\epsilon)\mu] \leq e^{-\epsilon^2\mu/3} \ . \tag{12}$$

Now we obtain a bound on the upper tail of Z_S. Obviously, $EZ_S \leq EY_S$. A simple argument gives

$$c_2\binom{k}{3}n^2p^8 \leq EY_S \leq c_3\binom{k}{3}n^2p^8 \ .$$

Thus, (5) implies

$$Pr[Z_S \geq 5EY_S] \leq e^{-5(\ln 5-1)EY_S} \ . \tag{13}$$

Comparing μ and EY_S we have

$$\frac{\binom{k}{3}p^3}{c_3\binom{k}{3}n^2p^8} \leq \frac{\mu}{EY_S} \leq \frac{\binom{k}{3}p^3}{c_2\binom{k}{3}n^2p^8} \ .$$

Let us choose c_1 in the definition of p so that $c_3n^2p^5 = 1/(10\cdot 16) = 1/160$, then

$$160 \leq \frac{\mu}{EY_S} \leq 160\frac{c_3}{c_2} \ . \tag{14}$$

Now we are ready to estimate $Pr[\overline{A_S}]$. From (12) with $\epsilon = 1/2$, (13) and (14) it follows that

$$\begin{aligned} Pr[\overline{A_S}] &= Pr[X_S \leq 16Z_S] \leq Pr[X_S \leq \mu/2] + Pr[16Z_S \geq \mu/2] \\ &\leq Pr[X_S \leq \mu/2] + Pr[Z_S \geq 5EY_S] \\ &\leq e^{-\binom{k}{3}p^3/12} + e^{-\frac{c_2\binom{k}{3}p^3}{160c_3}[5\ln 5-5]} \leq 2e^{-c_4k^3p^3} \end{aligned}$$

for some absolute positive constant c_4. Therefore

$$Pr[\exists S : \overline{A_S}] \leq \binom{n}{k}Pr[\overline{A_S}] \leq \binom{n}{k}2e^{-c_4k^3p^3} \leq 2\left(\frac{en}{k}e^{-c_4k^2p^3}\right)^k < 1$$

for sufficiently large value of the constant c_0. $\square$

Remarks: 1)Again, we have shown in fact that there exists a constant c such that *almost every* graph G from $G(n,p)$ with $p = c_1 n^{-2/5}$ contains a K^4-free subgraph G' on n vertices, obtained by simple edge deletion procedure, in which every $k = cn^{3/5}(\ln n)^{1/2}$ vertices span a K^3. Our proof yields a randomized polynomial time algorithm for constructing such a subgraph;

2) When $p = c_1 n^{-2/5}$, the expectation of the number of copies of K^4 through an edge e tends to a *constant*;

3) The above proof can easily be extended for the case of general r, s. We omit the detailed, somewhat tedious computation, which gives the following

Theorem 3 *For every pair $2 \leq r < s$ there exist constants $c_0, c_1 > 0$ depending only on the values of r and s such that almost every graph G from $G(n,p)$ for $p = c_1 n^{-2/(s+1)}$ can be converted by edge deletion to a K^s-free graph G' on n vertices in which every $k = c_0 n^{r/(s+1)}(\log n)^{1/(r-1)}$ vertices span a copy of K^r. Therefore*

$$f_{r,s}(n) \leq c_0 n^{\frac{r}{s+1}}(\log n)^{\frac{1}{r-1}} \ ;$$

4) It would be interesting to improve the bounds for $f_{r,s}$ and in particular to decide if for every $0 < \alpha < 1$ the value of $f_{s-1,s}$ is larger than $n^{1-\alpha}$ for sufficiently large s.

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