ON A THEOREM OF LOVÁSZ ON COVERS IN r-PARTITE HYPERGRAPHS

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Abstract. A theorem of Lovász asserts that $\tau(H)/\tau^*(H) \le r/2$ for every r-partite hypergraph H (where τ and τ^* denote the covering number and fractional covering number respectively). Here it is shown that the same upper bound is valid for a more general class of hypergraphs: those which admit a partition $(V_1, \dots, V_k)$ of the vertex set and a partition $p_1 + \dots + p_k$ of r such that $|e \cap V_i| \le p_i \le r/2$ for every edge e and every $1 \le i \le k$. Moreover, strict inequality holds when $r > 2$, and in this form the bound is tight. The investigation of the ratio τ/τ^* is extended to some other classes of hypergraphs, defined by conditions of similar flavour. Upper bounds on this ratio are obtained for k-colourable, strongly k-colourable and (what we call) k-partitionable hypergraphs.

1. Introduction

A *hypergraph* H is an ordered pair $H = (V, E)$, where $V = V(H)$ is a finite set (the set of *vertices*) and $E = E(H)$ is a non-empty collection of non-empty subsets of V called *edges*. The set V is called the *vertex set* of H, the set E is the *edge set* of H. The *rank* of H is $r(H) = \max\{|e| : e \in E(H)\}$. If all edges of H are of size r, then H is *r-uniform*, or simply an *r-graph*. $\binom{V}{r}$ denotes the hypergraph with vertex set V and edge set E, consisting of all subsets of V of size r.

The set $\{1, \dots, n\}$ is denoted by $[n]$.

A set $T \subseteq V$ is called a *cover* (or a transversal) of the hypergraph $H = (V, E)$ if $T \cap e \ne \emptyset$ for every $e \in E(H)$. The minimum cardinality of a cover of H is called the *covering number* of H and denoted by $\tau(H)$. E.g., $\tau\binom{[n]}{r} = n - r + 1$ for all positive integers $n \ge r$.

A set $M \subseteq E$ is called a *matching* in the hypergraph $H = (V, E)$ if all edges of M are pairwise disjoint. The maximum cardinality of a matching in H is called the *matching number* of H and denoted by $\nu(H)$.

Many problems of combinatorics can be formulated as the determination of the covering number of a hypergraph. The exact calculation of the covering number of an arbitrary hypergraph is known to be NP-hard. Hence the question of 'good' approximation of the covering number is of great importance. One of the simplest ways to estimate the covering number is by using the linear programming bound.

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A *fractional cover* of the hypergraph $H = (V, E)$ is a function $g : V \to \mathbb{R}^+$ such that $\sum_{v \in e} g(v) \geq 1$ for every $e \in E(H)$. The *value* of the fractional cover g is $|g| = \sum_{v \in V} g(v)$. The minimum of $|g|$ over all fractional covers of H is the *fractional covering number* of H, denoted by $\tau^*(H)$.

Similarly, a *fractional matching* in $H = (V, E)$ is a function $f : E \to \mathbb{R}^+$ such that $\sum_{e \ni v} f(e) \leq 1$ for every $v \in V(H)$. The *value* of the fractional matching f is $|f| = \sum_{e \in E} f(e)$. The maximum of $|f|$ over all fractional matchings of H is the *fractional matching number* of H, denoted by $\nu^*(H)$.

For every hypergraph H one has : $\tau(H) \geq \tau^*(H)$, $\nu^*(H) \geq \nu(H)$. It is easy to see that the above two problems are in fact a pair of dual linear programming problems. The Duality Theorem of Linear Programming asserts that:

- (i) for every fractional cover g and every fractional matching f one has: $|g| \geq |f|$;
- (ii) $\tau^* = \nu^*$;
- (iii) if g is an optimal fractional cover (i.e., $|g| = \tau^*$) and f is an optimal fractional matching (i.e., $|f| = \nu^*$), then:

$$\begin{aligned} &f(e) > 0 \text{ implies } \sum_{v \in e} g(v) = 1, \\ &g(v) > 0 \text{ implies } \sum_{e \ni v} f(e) = 1 \ . \end{aligned} \tag{1}$$

(These are the so called *complementary slackness conditions*.)

Example 1. $H = \binom{[n]}{r}$.

Define a fractional cover $g : V \to \mathbb{R}^+$ by $g(v) = 1/r$ for every $v \in V$ and a fractional matching $f : E \to \mathbb{R}^+$ by $f(e) = 1 \Big/ \binom{n-1}{r-1}$ for every $e \in E$. Then $|g| = |f| = n/r$, so g and f are an optimal fractional cover and fractional matching, respectively, and $\tau^* = \nu^* = n/r$.

As mentioned above, the fractional covering number may be used as an estimate for the covering number. It is natural to ask how good this estimate is, or, in other words, how large the ratio τ/τ^* can be for certain types of hypergraphs. A very useful upper bound on the ratio τ/τ^* was obtained independently by Lovász([3]), Sapozhenko([6]) and Stein([7]); this bound asserts that $\tau/\tau^* \leq 1 + \log D$, where $D = \max_{v \in V} \left|\{e : v \in e\}\right|$ - the maximum degree in the hypergraph H.

In this paper we focus on bounds based on the rank $r = r(H)$. Since the union of a maximum matching forms a cover, we have $\tau \leq r\nu$, so $\tau \leq r\tau^*$. Even more is true([5]): $r\tau^* \geq \tau + r - 1$, so $\tau/\tau^* < r$ for every $r > 1$. The hypergraph $H_n = \binom{[n]}{r}$, $n \to \infty$, with $\tau(H_n) = n - r + 1$, $\tau^*(H_n) = n/r$ shows that the bound $\tau/\tau^* < r$ is tight for general hypergraphs of rank r.

However, for certain types of hypergraphs this trivial bound can be improved. For example, the famous theorem of König asserts that $\tau = \tau^*$ for any bipartite graph ($r = 2$) $G = (A \cup B, E)$. This result motivates looking at the ratio τ/τ^* in hypergraphs that admit a vertex partition of some kind.

A natural generalization of bipartite graphs is r-partite hypergraphs. An r-graph $H = (V, E), r \geq 2$, is called *r-partite* if there exists a partition of the vertex set V into subsets $V_1, \ldots, V_r$ such that for every edge $e \in E$ one has: $|e \cap V_i| = 1$, $1 \leq i \leq r$. In 1975 Lovász proved([4]):

Theorem. *$\tau(H)/\tau^*(H) \le r/2$ for every r-partite hypergraph H.*

This generalizes the König Theorem. Since the proof of the theorem is based on an idea which turns out to be very fruitful, it is worthwhile to give an outline of the proof here.

Proof([4], see also [1]). Let $H = (V, E)$ be an r-partite hypergraph with a vertex partition $(V_1, \dots, V_r)$. Since $\tau^*(H)$ is defined to be the value of an optimal solution of an LP problem with integral coefficients there exists a minimal fractional cover $g : V \to \mathbb{R}^+$ such that $g(v)$ is rational for every $v \in V$. Therefore we can choose an integer d (as large as we want) so that $g(v)d$ is integral for every $v \in V$. Define a new function $t : V \to \{0, 1, \dots, d\}$ by putting $t(v) := g(v)d$.

It is easy to see that for all integers $r \ge 2$, $m \ge 0$ there exists an $r \times (m+1)$ matrix $A = (a_{ij})_{i=1,\dots,r,\, j=0,\dots,m}$ with the following properties:

(i) every row of A is a permutation of $\{0, 1, \dots, m\}$;
(ii) the sum of every column is at most $\lceil \frac{rm}{2} \rceil$.

Let $m = \lfloor \frac{2(d-1)}{r} \rfloor$. For every $0 \le j \le m$ define a set T_j as follows:

$$T_j = \bigcup_{i=1}^{r} \{v \in V_i : t(v) > a_{ij}\}.$$

Then every T_j is a cover. Indeed, suppose on the contrary that there exists an edge $e = \{v_1, \dots, v_r\} \in E(H)$ such that $e \cap T_j = \emptyset$. It means that $t(v_i) \le a_{ij}$ for every $1 \le i \le r$. But then

$$\sum_{i=1}^{r} g(v_i) = \sum_{i=1}^{r} \frac{t(v_i)}{d} \le \frac{\sum_{i=1}^{r} a_{ij}}{d} \le \frac{\lceil \frac{rm}{2} \rceil}{d} = \frac{\left\lceil \frac{r}{2} \lfloor \frac{2(d-1)}{r} \rfloor \right\rceil}{d} \le \frac{d-1}{d} < 1$$

- a contradiction, since g is a fractional cover.

Since each row of the matrix A is a permutation, for every $v \in V$ we have $|\{0 \le j \le m : v \in T_j\}| \le t(v)$. Then $\sum_{j=0}^{m} |T_j| \le \sum_{v \in V} t(v) = d\tau^*$. Since every T_j is a cover, we obtain:

$$\tau \le \frac{\sum_{j=0}^{m} |T_j|}{m+1} \le \frac{d\tau^*}{\frac{2(d-1)}{r}}.$$

If $d \to \infty$, we obtain:

$$\tau \le \frac{r}{2}\tau^*. \qquad \square$$

Of course, for $r = 2$ the bound of the previous theorem is tight, but for arbitrary r the tightness question was open. We will answer it in the affirmative by building an appropriate family of examples in Section 2 (see Example 2).

Let us make one step further and generalize the concept of an r-partite hypergraph in the following manner. An r-graph $H = (V, E)$ is called a *$(p_1, \dots, p_k)$-graph* for fixed positive integers $p_1, \dots, p_k$ with $\sum_{i=1}^{k} p_i = r$ if there exists a vertex partition $(V_1, \dots, V_k)$ such that for every $e \in E$ one has $|e \cap V_i| = p_i$, $1 \le i \le k$. If every $p_i = 1$ we are back to the definition of an r-partite hypergraph. We will use an even more general concept: a hypergraph $H = (V, E)$ is *$(p_1, \dots, p_k)$-bounded* for fixed positive integers $p_1, \dots, p_k$ if there exists a vertex partition $(V_1, \dots, V_k)$ such that for every $e \in E$ one has $|e \cap V_i| \le p_i$, $1 \le i \le k$. The question is again what can be said about the upper bound on the ratio τ/τ^* for these types of hypergraphs. We managed to obtain a complete answer which is contained in the following theorems.

Theorem 1. *Let $p_1, \dots, p_k$ be positive integers with $\sum_{i=1}^{k} p_i = r > 2$ and suppose that $p_i \leq r/2$ for every $1 \leq i \leq k$. If H is $(p_1, \dots, p_k)$-bounded then:*

$$\frac{\tau(H)}{\tau^*(H)} < \frac{r}{2} \quad .$$

This bound is tight (even for $(p_1, \dots, p_k)$-graphs).

Theorem 2. *Let $p_1, \dots, p_k$ be positive integers with $\sum_{i=1}^{k} p_i = r > 2$ and suppose that $p_{i_0} \geq r/2$ for some $1 \leq i_0 \leq k$. If H is $(p_1, \dots, p_k)$-bounded then:*

$$\frac{\tau(H)}{\tau^*(H)} < p_{i_0} \quad .$$

This bound is tight (even for $(p_1, \dots, p_k)$-graphs).

The above cited theorem of Lovász is a special case of our Theorem 1 (all $p_i = 1$). Moreover, Theorem 1 shows that the weak inequality sign in Lovász' theorem can be replaced by strong inequality. We prove these theorems in Section 2.

Let us turn now to hypergraphs that admit a vertex partition induced by hypergraph colouring.

A *k-colouring* of the hypergraph $H = (V, E)$ is a partition $(C_1, \dots, C_k)$ of the set of vertices V into k classes (colours) such that every edge (of size at least two) meets at least two classes of the partition. H is called *k-colourable* if it admits a k-colouring.

Clearly, a k-colourable hypergraph of rank at most r is $(p_1, \dots, p_k)$-bounded, with $p_1 = \dots = p_k = r - 1$. Thus, Theorem 1 provides an upper bound on τ/τ^* for such hypergraphs. But here the rank r is much smaller than $\sum_{i=1}^{k} p_i$, so we can do much better. The upper bound on the ratio τ/τ^* for k-colourable hypergraphs of rank at most r is given by the following two theorems.

Theorem 3. *Let $2 \leq k < r$ be integers. If H is a k-colourable hypergraph of rank at most r, then:*

$$\frac{\tau(H)}{\tau^*(H)} < r - 1 \quad .$$

This bound is tight.

Theorem 4. *Let $2 \leq r \leq k$ be integers. If H is a k-colourable hypergraph of rank at most r, then:*

$$\frac{\tau(H)}{\tau^*(H)} \leq \frac{k-1}{k} r \quad .$$

This bound is tight.

These results are described in Section 3. In a subsequent paper [2] the above two theorems are applied to the design of approximation algorithms for the set covering problem.

A *strong k-colouring* of the hypergraph $H = (V, E)$ is a partition $(C_1, \dots, C_k)$ of the set of vertices V into k classes (colours) such that no colour appears more than once in the same edge. H is *strongly k-colourable* if it admits a strong k-colouring. Note that if $k = r$ then the definition of a strongly k-colourable r-graph coincides with that of an r-partite hypergraph.

A strongly k-colourable hypergraph is $(1,\dots,1)$-bounded (with k 1's). Hence, by Theorem 1, the ratio τ/τ^* cannot exceed $k/2$ in such a hypergraph. Thus, for the ratio τ/τ^* in a strongly k-colourable hypergraph of rank at most r we have two upper bounds: $k/2$ and r. (We may improve on the latter by observing that strong k-colourability implies k-colourability and invoking Theorem 4.) But we can do better than the minimum of the above two bounds.

For the case $k \geq (r-1)r$ we succeeded to find the exact upper bound for the ratio τ/τ^*:

Theorem 5. *Let $k, r \geq 2$ be integers and suppose $k \geq (r-1)r$. If H is a strongly k-colourable hypergraph of rank at most r, then:*

$$\frac{\tau(H)}{\tau^*(H)} \leq \frac{k-r+1}{k} r \quad .$$

This bound is tight.

In the case $r < k < (r-1)r$ the situation is not so clear. The best result we succeeded to obtain is formulated in the following theorem.

Theorem 6. *Let $k, r \geq 2$ be integers and suppose $r \leq k \leq (r-1)r$. If H is a strongly k-colourable hypergraph of rank at most r, then:*

$$\frac{\tau(H)}{\tau^*(H)} \leq \frac{kr}{k+r} + \min\left(\frac{k-r}{2k}\{u\}, \frac{r}{k}(1-\{u\})\right),$$

where $u = k^2/(k+r)$ and $\{u\} = u - \lfloor u \rfloor$.

This result motivates

Conjecture 1. *Let $k, r \geq 3$ be integers and suppose $r \leq k < (r-1)r$. If H is a strongly k-colourable hypergraph of rank at most r, then :*

$$\frac{\tau(H)}{\tau^*(H)} < \frac{kr}{k+r} \quad .$$

We also conjecture that this bound is tight.

The bound of the above conjecture coincides with the bound of Lovász' theorem for $k = r$ and with the bound of Theorem 5 for $k = (r-1)r$. It should be mentioned that the difference between the bounds of Theorem 6 and Conjecture 1 does not exceed $3 - 2\sqrt{2}$. We consider this problem in Section 4.

We introduce now another generalization of r-partite hypergraphs. A hypergraph $H = (V, E)$ is called *k-partitionable* if there exists a vertex partition $(T_1, \dots, T_k)$ of the vertex set V into k covers T_i, $1 \leq i \leq k$. Again, in the case $k = r$ an r-partitionable r-graph is just an r-partite hypergraph.

The situation here is similar to that of the previous problem. For the case $r \geq (k-1)k$ we know a complete answer:

Theorem 7. *Let $k \geq 2, r \geq 3$ be integers and suppose $r \geq (k-1)k$. If H is a k-partitionable hypergraph of rank at most r, then:*

$$\frac{\tau(H)}{\tau^*(H)} < r - k + 1 \quad .$$

This bound is tight.

For the case $k \leq r \leq (k-1)k$ our result is:

Theorem 8. *Let $k, r \geq 2$ be integers and suppose $k \leq r \leq (k-1)k$. If H is a k-partitionable hypergraph of rank at most r, then:*

$$\frac{\tau(H)}{\tau^*(H)} \leq \frac{r^2}{k+r} + \min\left(\frac{r-k}{2k}\{u\}, 1-\{u\}\right)$$

(with $\{u\}$ defined as in Theorem 6).

But we believe that the following is true:

Conjecture 2. *Let $k, r \geq 3$ be integers and suppose $k \leq r \leq (k-1)k$. If H is a k-partitionable hypergraph of rank at most r, then:*

$$\frac{\tau(H)}{\tau^*(H)} < \frac{r^2}{k+r} \quad .$$

We also conjecture that this bound is tight.

Again, at the endpoints of the interval $k \leq r \leq (k-1)k$ this conjecture coincides with the previously cited results: the case $k = r$ is again Lovász' theorem while in the case $r = (k-1)k$ we obtain the bound from Theorem 7. The difference between the bounds of Theorem 8 and Conjecture 2 is not more than 1. We discuss this problem in Section 5.

Before starting our proofs let us write a few words about the ideas we are going to use. Our main instrument is in a sense a generalization of the core idea of Lovász' proof. It is described in the following lemma.

Lemma 1. *Let $H = (V, E)$ be a hypergraph with a vertex partition $(V_1, \ldots, V_k)$. Let $g : V \to \mathbb{R}^+$ be an optimal fractional cover of H with value $|g| = \tau^*(H)$. Suppose $\delta > 0$, and the set $B \subseteq [0, \delta]^k$ is such that for every $\bar{x} = (x_1, \ldots, x_k) \in B$ the set*

$$T(\bar{x}) = \bigcup_{i=1}^{k} \{v \in V_i : g(v) \geq x_i\}$$

is a cover of H. If there exists a probability measure μ defined on B ($\mu(B) = 1$) such that all marginal distributions μ_i, $1 \leq i \leq k$, are uniform on the interval $[0, \delta]$ (that is, if $\bar{x} \in B$ is randomly chosen according to the measure μ, then $Pr(a \leq x_i \leq b) = (b-a)/\delta$ for every $0 \leq a \leq b \leq \delta$), then:

$$\frac{\tau(H)}{\tau^*(H)} \leq \frac{1}{\delta} \quad .$$

Proof. Let $\bar{x} \in B$ be randomly chosen from B according to the measure μ. Define a random variable $Y = |T(\bar{x})|$ where $T(\bar{x})$ is as defined above. Let us estimate the expectation of Y. Due to linearity of expectation $E(Y) = \sum_{v \in V} E(Y_v)$, where Y_v is the indicator random variable for $v \in V$ being selected to T. Since μ has uniform marginal distributions on the interval $[0, \delta]$, for every $1 \leq i \leq k$ and for every $v \in V_i$ we have:

$$E(Y_v) = Pr(v \in T) = Pr(g(v) \geq x_i) = \min\left(1, g(v)/\delta\right) \leq g(v)/\delta \quad .$$

Then

$$E(Y) = \sum_{v \in V} E(Y_v) \leq \sum_{v \in V} g(v)/\delta = \tau^*/\delta \quad .$$

Therefore there exists $\bar{x} \in B$ such that $|T(\bar{x})| \leq \tau^*/\delta$, and since $T(\bar{x})$ is a cover for every $\bar{x} \in B$ it follows that $\tau \leq \tau^*/\delta$. $\square$

Of course, a probability measure μ having uniform marginal distributions μ_i is a continuous analog of the matrix A from Lovász' proof. For each particular problem considered in this paper, our strategy is to find an appropriate set B and a measure μ and to use Lemma 1. In some cases, when we will prove a strict inequality for the ratio τ/τ^*, we will use not only the existence of such a measure, but also its structure.

Although the above probabilistic method could be used to prove all our results, in some cases we chose to present a proof by a more constructive method. That other method proceeds by induction on the number of vertices and uses the above mentioned complementary slackness conditions (1).

2. The ratio τ/τ^* in $(p_1, \ldots, p_k)$-bounded hypergraphs

Recall first that a hypergraph $H = (V, E)$ is $(p_1, \ldots, p_k)$-bounded, for a fixed k-tuple of positive integers $(p_1, \ldots, p_k)$, if there exists a vertex partition $(V_1, \ldots, V_k)$ such that for every edge $e \in E$ one has $|e \cap V_i| \leq p_i$, $1 \leq i \leq k$. It is a $(p_1, \ldots, p_k)$-graph if all these weak inequalities hold as equalities.

We begin this section with two families of examples of $(p_1, \ldots, p_k)$-graphs which will be used later on to show that the bounds of Theorems 1 and 2 are tight.

Example 2. Let $r \geq 2$ be fixed. For every positive integer n define an r-partite hypergraph $H_n = (V, E)$ as follows. For every $1 \leq i \leq r$ let $V_i = \{x_{ij} : \quad 1 \leq j \leq n\} \cup \{y_{ij} : \quad 1 \leq j \leq nr\}$; let $V = \bigcup_{i=1}^r V_i$. Define a weight function $h : V \to \{0, \ldots, n\}$ by

$$\begin{aligned} h(x_{ij}) &= j, \quad 1 \leq i \leq r, 1 \leq j \leq n; \\ h(y_{ij}) &= 0, \quad 1 \leq i \leq r, 1 \leq j \leq nr. \end{aligned}$$

Now a set $e \subseteq V$ is an edge of H_n if and only if the following holds:

(i) $|e \cap V_i| = 1$ for every $1 \leq i \leq r$;

(ii) $\sum_{v \in e} h(v) \geq \frac{nr}{2}$.

Obviously, H_n is an r-partite hypergraph. To estimate from above its fractional covering number, define a fractional cover $g : V \to \mathbb{R}^+$ by putting $g(v) = h(v) \big/ \frac{nr}{2}$ for every $v \in V$. Then

$$\tau^*(H_n) \leq |g| = \frac{\sum_{v \in V} h(v)}{\frac{nr}{2}} = n + 1 \ .$$

Now we have to evaluate the covering number of H_n. Since the set $T_0 = \big\{v \in V : h(v) \geq \lceil n/2 \rceil\big\}$ is obviously a cover we obtain that $\tau(H_n) \leq |T_0| = r\big(\lfloor n/2 \rfloor + 1\big) \leq nr$. Let us prove that $\tau(H_n) > \frac{nr}{2}$. Let $T \subseteq V$ be an optimal cover of H_n, $|T| = \tau(H_n) \leq nr$. Then $V_i \setminus T \neq \emptyset$ for every $1 \leq i \leq r$. Define

$$l_i = \max\big\{h(v) : v \in V_i \setminus T\big\}, \quad 1 \leq i \leq r \ .$$

Then $|T \cap V_i| \geq n - l_i$. From the definition of H_n and the fact that T is a cover it follows that $\sum_{i=1}^r l_i < \frac{nr}{2}$. Therefore

$$|T| = \sum_{i=1}^r |T \cap V_i| \geq \sum_{i=1}^r (n - l_i) = nr - \sum_{i=1}^r l_i > nr - \frac{nr}{2} = \frac{nr}{2} \quad .$$

Now, when $n \to \infty$ the ratio $\frac{\tau(H_n)}{\tau^*(H_n)} > \frac{nr/2}{n+1}$ becomes larger than $(r/2 - \epsilon)$ for every fixed $\epsilon > 0$.

Example 3. For every k-tuple of positive integers $(p_1, \ldots, p_k)$ $(k \geq 2)$ and every integer $n \geq p_1$ we define a hypergraph $H^n = H^n(p_1, \ldots, p_k) = (V, E)$ in the following way. Let V be the union of the pairwise disjoint sets V_i, $1 \leq i \leq k$, of sizes $|V_1| = n$, $|V_i| = \binom{n}{p_1} p_i$, $2 \leq i \leq k$. For every $2 \leq i \leq k$ partition the set V_i into $\binom{n}{p_1}$ classes $U_i^1, \ldots, U_i^{\binom{n}{p_1}}$, each of size p_i. Let $A^1, \ldots, A^{\binom{n}{p_1}}$ be the edges of the hypergraph $\binom{V_1}{p_1}$. Define now for every $1 \leq j \leq \binom{n}{p_1}$ an edge $e_j \in E(H^n)$ as

$$e_j = A^j \cup \bigcup_{i=2}^{k} U_i^j \quad .$$

Then for every $e, e' \in E(H^n)$ one has $e \cap e' \subseteq V_1$, so every subset of V_1 of size p_1 has its own continuation in the sets $V_2, \ldots, V_k$. The function $g : V \to \mathbb{R}^+$, $g(v) := 1/p_1$ for every $v \in V_1$, $g(v) := 0$ for every $v \notin V_1$ is obviously a fractional cover of H^n, so $\tau^*(H^n) \leq |g| = n/p_1$. It is easy to see that there exists an optimal cover T of size $|T| = \tau(H^n)$ such that $T \subseteq V_1$ (if $v \in T \setminus V_1$, then there exists a unique edge $e \in E(H^n)$ such that $v \in e$, but then the set $T' = \big(T \setminus \{v\}\big) \cup \{u\}$ is an optimal cover too, where u is any vertex from $e \cap V_1$), so $\tau(H^n) = \tau(\binom{V_1}{p_1}) = n - p_1 + 1$. When $n \to \infty$, the ratio $\frac{\tau(H^n)}{\tau^*(H^n)} \geq \frac{n-p_1+1}{n/p_1}$ becomes larger than $(p_1 - \epsilon)$ for every fixed $\epsilon > 0$.

Now we are ready to prove Theorems 1 and 2.

Proof of Theorem 1. Let us first prove the case $k = 2$. In this case according to the theorem's conditions $p_1 = p_2 = r/2 > 1$. Let $H = (V, E)$ be a hypergraph with a vertex partition (V_1, V_2) such that $|e \cap V_i| \leq r/2$ for every $e \in E$, $\quad i = 1, 2$. Suppose $g : V \to \mathbb{R}^+$ is an optimal fractional cover of H with value $|g| = \tau^*(H)$. Define a set B and a measure μ as required in Lemma 1 in the following way: $B = \big\{(x_1, x_2) \in [0, 2/r]^2 : x_1 + x_2 = 2/r\big\}$ (see Fig.1), μ is the uniform probability measure on B $(\mu(B) = 1)$. It is easy to see that for every $\bar{x} = (x_1, x_2) \in B$ the set

$$T(\bar{x}) = \big\{v \in V_1 : g(v) \geq x_1\big\} \cup \big\{v \in V_2 : g(v) \geq x_2\big\}$$

is a cover of H (if there were an edge $e \in E(H)$ such that $e \cap T(\bar{x}) = \emptyset$ then we would have $g(v) < x_1$ for every $v \in e \cap V_1$ and $g(v) < x_2$ for every $v \in e \cap V_2$, so $\sum_{v \in e} g(v) < x_1|e \cap V_1| + x_2|e \cap V_2| \leq x_1 r/2 + x_2 r/2 = 1$ - a contradiction since g is a fractional cover). Also, it is clear that μ has marginal distributions μ_i, $i = 1, 2$, uniform on the interval $[0, 2/r]$. Hence as in Lemma 1 we obtain that if $\bar{x}$ is randomly chosen from B according to the measure μ then

$$\tau(H) \leq E(|T(\bar{x})|) = \sum_{v \in V} \min\big(1, g(v) r/2\big) \leq r/2 |g| = r/2\, \tau^*(H) \quad .$$

We want to show more: $\tau(H) < r/2\, \tau^*(H)$. Note that if there exists a vertex $v \in V$ with $g(v) > 2/r$, then $E(|T(\bar{x})|) < r/2\, \tau^*(H)$, so suppose in the sequel

FIG. 1. The set B and the points $\bar{x}^*$ and $\bar{y}^*$ from the proof of the case $k=2$ of Theorem 1.

that $g(v) \leq 2/r$ for every $v \in V$. Also, we may assume that every set T which is realized as $T(\bar{x})$ with positive probability has size $|T| = r/2\,\tau^*(H)$, since otherwise $\tau(H) \leq \min\{|T(\bar{x})| : \bar{x} \in B\} < E(|T(\bar{x})|) = r/2\,\tau^*(H)$.

Let $v_0 \in V$ be a vertex with positive weight $g(v_0) > 0$. Suppose without loss of generality that $v_0 \in V_1$. Let $\bar{x}^* = (x_1^*, x_2^*) \in B$ be such that $x_1^* = g(v_0)$. Since g attains only a finite number of values, there exists a point $\bar{y}^* = (y_1^*, y_2^*) \in B$ such that $y_1^* < x_1^*$ and there are no vertices in V_i with weight $g(v)$ strictly between x_i^* and y_i^*, $i = 1,2$ (see Fig. 1). Therefore the set $T(\bar{x}) = T$ remains unchanged in the open interval between the points $\bar{x}^*$ and $\bar{y}^*$. Note that every $v \in V_1$ with $g(v) \geq x_1^*$ and every $v \in V_2$ with $g(v) > x_2^*$ belongs to T. In particular, $v_0 \in T$. Since the open interval between $\bar{x}^*$ and $\bar{y}^*$ has positive measure we have $|T| = r/2\,\tau^*(H)$. We claim that $T' = T \setminus \{v_0\}$ is also a cover of H. Indeed, suppose on the contrary that there exists an edge $e \in E$ such that $e \cap T' = \emptyset$. Since T is a cover we have $e \cap T = \{v_0\}$. But then $g(v) < x_1^*$ for every $v \in e \cap V_1 \setminus \{v_0\}$ and $g(v) \leq x_2^*$ for every $v \in e \cap V_2$, so

$$\sum_{v \in e} g(v) = g(v_0) + \sum_{v \in e \cap V_1 \setminus \{v_0\}} g(v) + \sum_{v \in e \cap V_2} g(v) < x_1^* + (r/2 - 1)x_1^* + r/2\,x_2^* = 1$$

- a contradiction. Hence we have $\tau(H) \leq |T'| = |T| - 1 < r/2\,\tau^*(H)$ and the case $k = 2$ of the inequality has been established.

Now we are about to prove our theorem for the case $k \geq 3$. First, we reduce this general case to the case $k = 3$. The reduction is based on the following simple

Observation. *Let $p_1, \ldots, p_k$, $k \geq 3$, be positive integers with $\sum_{i=1}^k p_i = r$ and suppose $p_i \leq r/2$ for every $1 \leq i \leq k$. Then there exists a partition of the set $[k]$ into three non-empty subsets I_1, I_2, I_3 such that $\sum_{i \in I_j} p_i \leq r/2$ for $j = 1,2,3$.*

Using this observation, we can prove

Proposition 1. *Let $p_1,\dots,p_k$ be as above. If $H=(V,E)$ is a $(p_1,\dots,p_k)$-bounded hypergraph then there exist positive integers s_1,s_2,s_3 with $s_1+s_2+s_3=r$ and $s_1,s_2,s_3\le r/2$ such that H is an (s_1,s_2,s_3)-bounded hypergraph.*

It follows immediately from the above proposition that it suffices to prove the bound for the case $k=3$. Moreover, the case when $k=3$ and one of the s_j equals $r/2$ may be further reduced, in a similar way, to the case $k=2$ which was already handled above. So it remains to treat the case when $k=3$ and $s_j<r/2,\ j=1,2,3$.

Suppose $H=(V,E)$ is an (s_1,s_2,s_3)-bounded hypergraph with $s_1+s_2+s_3=r$ and suppose without loss of generality that $0<s_3\le s_2\le s_1<r/2$. Let (V_1,V_2,V_3) be a partition of the vertex set V such that $|e\cap V_i|\le s_i,\quad i=1,2,3$, for every $e\in E$. Let $g:V\to\mathbb{R}^+$ be an optimal fractional cover of H and $f:E\to\mathbb{R}^+$ be an optimal fractional matching in H with value $|g|=|f|=\tau^*(H)$.

Define now a set $B\subseteq[0,2/r]^3$. Fix four points

$$\begin{aligned}
Q_1&=\left(\frac{s_1+s_2-s_3}{s_1 r},\ 0,\ \frac{2}{r}\right),\\
Q_2&=\left(\frac{2}{r},\ \frac{s_2+s_3-s_1}{s_2 r},\ 0\right),\\
Q_3&=\left(\frac{s_1+s_3-s_2}{s_1 r},\ \frac{2}{r},\ 0\right),\\
Q_4&=\left(0,\frac{s_1+s_2-s_3}{s_2 r},\frac{2}{r}\right)
\end{aligned}$$

in $[0,2/r]^3$ and denote

$$\begin{aligned}
B^1&=[Q_1Q_2],\\
B^2&=[Q_3Q_4],\\
B^3&=[Q_1Q_3],\\
B^4&=[Q_2Q_4],
\end{aligned}$$

where $[Q_{i_1}Q_{i_2}]$ denotes the closed interval between the points Q_{i_1} and Q_{i_2} (see Fig. 2). Now let

$$B=B^1\cup B^2\cup B^3\cup B^4\quad .$$

It is easy to check that the coordinates of the points Q_1,Q_2,Q_3,Q_4 satisfy the equation $s_1x_1+s_2x_2+s_3x_3=1$, and therefore this equation is satisfied by every point $\bar{x}=(x_1,x_2,x_3)\in B$. Then it follows that the set

$$T(\bar{x})=\big\{v\in V_1:g(v)\ge x_1\big\}\cup\big\{v\in V_2:g(v)\ge x_2\big\}\cup\big\{v\in V_3:g(v)\ge x_3\big\}$$

is a cover for every $\bar{x}=(x_1,x_2,x_3)\in B$.

Define now a probability measure μ on B. Let $\mu^i,\ 1\le i\le 4$, be the uniform measures on the intervals B^i such that

$$\begin{aligned}
\mu^1(B^1)=\mu^2(B^2)&=\frac{(s_1+s_3-s_2)(s_2+s_3-s_1)}{2s_3(s_1+s_2-s_3)},\\
\mu^3(B^3)&=\frac{(s_2-s_3)(s_2+s_3-s_1)}{s_3(s_1+s_2-s_3)},\\
\mu^4(B^4)&=\frac{(s_1-s_3)(s_1+s_3-s_2)}{s_3(s_1+s_2-s_3)},
\end{aligned}$$

FIG. 2. The set B and the points $\bar{x}^*$ and $\bar{y}^*$ from the proof of the case $k = 3$ of Theorem 1.

and let $\mu = \mu^1 + \mu^2 + \mu^3 + \mu^4$. (Note that μ^3 vanishes when $s_2 = s_3$, in this case $x_1 = 1/r$ along the interval B^3. Also, μ^4 vanishes when $s_1 = s_2 = s_3$, in this case $x_2 = 1/r$ along the interval B^4.) Since $\sum_{i=1}^4 \mu^i(B^i) = 1$, we have $\mu(B) = 1$. Now we have to check that μ indeed has marginal distributions μ_i, $1 \leq i \leq 3$, uniform on the interval $[0, 2/r]$. For μ_3 this is quite clear from Fig. 2. Since μ_1 is obviously uniform on each of the intervals $\left[0, \frac{s_1+s_3-s_2}{s_1 r}\right]$, $\left[\frac{s_1+s_3-s_2}{s_1 r}, \frac{s_1+s_2-s_3}{s_1 r}\right]$, $\left[\frac{s_1+s_2-s_3}{s_1 r}, \frac{2}{r}\right]$, and since in the first and the last of these three intervals the situation is the same (recall $\mu^1(B^1) = \mu^2(B^2)$) we have only to check that

$$\frac{\mu^3(B^3)}{\mu^2(B^2)} = \frac{\frac{s_1+s_2-s_3}{s_1 r} - \frac{s_1+s_3-s_2}{s_1 r}}{\frac{s_1+s_3-s_2}{s_1 r}} ,$$

which indeed holds. In a similar way one can check that the marginal distribution μ_2 is uniform on the interval $[0, 2/r]$, too.

At this moment, we have a set $B \subseteq [0, 2/r]^3$ such that $T(\bar{x})$ is a cover for every $\bar{x} \in B$ and a probability measure μ, defined on B and having marginal distributions uniform on the interval $[0, 2/r]$. Hence if $\bar{x} \in B$ is randomly chosen according to the measure μ then, as shown in Lemma 1,

$$\tau(H) \leq E(|T(\bar{x})|) = \sum_{v \in V} \min\left(1, g(v)r/2\right) \leq r/2\, \tau^*(H) \quad .$$

Recall that our aim is to prove that $\tau(H) < r/2\,\tau^*(H)$. Again, if there exists a vertex $v \in V$ with $g(v) > 2/r$ then $E(|T(\bar{x})|) < r/2\,\tau^*(H)$, so suppose that $g(v) \le 2/r$ for every $v \in V$. Also, we may assume that every set T which is realized as $T(\bar{x})$ with positive probability has size $|T| = r/2\,\tau^*(H)$. If all vertices $v \in V$ with $g(v) > 0$ belong to V_3, then for the set $T_0 = \{v : g(v) > 0\} \subseteq V_3$ we obtain using the complementary slackness conditions (1)

$$|T_0| = \sum_{v\in T_0} 1 = \sum_{v\in T_0}\sum_{e\ni v} f(e) = \sum_{e\in E} f(e)|e\cap T_0| \le \sum_{e\in E} f(e)s_3 = s_3\tau^*(H) \quad ,$$

and thus $\tau(H) \le |T_0| \le s_3\tau^*(H) < r/2\,\tau^*(H)$. So suppose that there exists a vertex $v_0 \in V_1 \cup V_2$ with $g(v) > 0$, say, $v_0 \in V_1$ (the argument is similar in case $v_0 \in V_2$). It is easily verified that we can always choose one of the intervals B^i with $\mu^i(B^i) > 0$ and a point $\bar{x}^* = (x_1^*, x_2^*, x_3^*) \in B^i$ in such a way that $x_1^* = g(v_0)$ and one of the variables x_2 or x_3 varies in the same direction as x_1 along B^i (that is, if we move along B^i in the direction of decreasing x_1, then x_2 or x_3 decreases too) (see Fig. 2); suppose this variable is x_2 (the argument is similar if it is x_3). Since g attains only a finite number of values, there exists a point $\bar{y}^* = (y_1^*, y_2^*, y_3^*) \in B^i$ with $y_1^* < x_1^*$ and $y_2^* < x_2^*$ such that there are no vertices v in V_i with weight $g(v)$ strictly between x_i^* and y_i^*, $i = 1,2,3$ (see Fig. 2). Therefore the set $T(\bar{x}) = T$ remains unchanged in the open interval between the points $\bar{x}^*$ and $\bar{y}^*$. Note that every $v \in V_1$ with $g(v) \ge x_1^*$ and every $v \in V_2$ with $g(v) \ge x_2^*$ and every $v \in V_3$ with $g(v) > x_3^*$ belongs to T. Since $\mu([\bar{x}\bar{y}]) > 0$, we have $|T| = r/2\,\tau^*(H)$. We claim that $T' = T\setminus\{v_0\}$ is also a cover. If it is not, then there exists an edge $e \in E$ such that $e \cap T' = \emptyset$, and so $e \cap T = \{v_0\}$. Hence $g(v) \le x_1^*$ for every $v \in e\cap V_1$, $g(v) < x_2^*$ for every $v \in e \cap V_2$, $g(v) \le x_3^*$ for every $v \in e \cap V_3$, but then

$$\sum_{v\in e} g(v) = \sum_{v\in e\cap V_1} g(v) + \sum_{v\in e\cap V_2} g(v) + \sum_{v\in e\cap V_3} g(v) < s_1x_1^* + s_2x_2^* + s_3x_3^* = 1 \quad ,$$

and we reach a contradiction. So we have $\tau(H) \le |T'| = |T| - 1 < r/2\,\tau^*(H)$.

The tightness of the proven bound follows from Example 2. This example shows also that the bound remains tight even for the class of r-partite hypergraphs. □

Remark. Using the observation and the proof of the theorem we can prove the following general proposition, which will be used in the sequel.

Proposition 2. *Let $p_1,\dots,p_k$ be positive integers with $\sum_{i=1}^k p_i = r$ and suppose $p_i \le r/2$ for every $1 \le i \le k$. Then there exists a k-dimensional random variable $\bar{\xi} = (\xi_1,\dots,\xi_k)$ whose values lie in the set $\{\bar{x} = (x_1,\dots,x_k) \in [0,2/r]^k : \sum_{i=1}^k p_ix_i = 1\}$, such that each ξ_i, $i = 1,\dots,k$, is uniformly distributed on $[0,2/r]$.*

Proof. If $k = 2$ (and then $p_1 = p_2 = r/2$), let $\bar{\xi}$ be distributed uniformly on the interval joining $(0,2/r)$ and $(2/r,0)$. If $k \ge 3$ then, according to the observation, there exists a partition of the set $[k]$ into three non-empty subsets I_1, I_2, I_3 such that $\sum_{i\in I_j} p_i = s_j \le r/2$, $j = 1,2,3$. Let $\bar{\zeta} = (\zeta_1,\zeta_2,\zeta_3)$ be a 3-dimensional random variable whose distribution is given by the measure μ defined in the course of the proof of Theorem 1. Now, define $\xi_i = \zeta_j$ if $i \in I_j$. The resulting k-dimensional random variable $\bar{\xi} = (\xi_1,\dots,\xi_k)$ is easily seen to satisfy the requirements. □

Proof of Theorem 2. Theorem 2 is a straightforward consequence of Theorem 1.

Let H be a $(p_1,\ldots,p_k)$-bounded hypergraph with $\sum_{i=1}^k p_i = r$ and suppose $p_1 \geq r/2$. Denote $p_1' = p_1$, $p_2' = p_2+2p_1-r$, $p_i' = p_i$, $3 \leq i \leq k$, $r' = \sum_{i=1}^k p_i' = 2p_1$. Then H is obviously a $(p_1',\ldots,p_k')$-bounded hypergraph with $p_i' \leq r'/2$ for every $1 \leq i \leq k$. So it follows from Theorem 1 that $\tau(H) < r'/2\,\tau^*(H) = p_1\tau^*(H)$.

The tightness of the bound follows from Example 3. $\square$

3. The ratio τ/τ^* in k-colourable hypergraphs

In this section we prove Theorems 3 and 4. First we show that the bounds given in these theorems cannot be improved. For Theorem 3, consider the hypergraphs $H^n = H^n(r-1,1)$ - as constructed in Example 3. Clearly, H^n is 2-colourable (V_1 and V_2 may be taken as the colour classes) and hence k-colourable for every $k \geq 2$. As shown in the analysis of Example 3, when $n \to \infty$ we have $\tau(H^n)/\tau^*(H^n) > r - 1 - \epsilon$ for every fixed $\epsilon > 0$. For Theorem 4, consider the hypergraph $H = \binom{[(r-1)k]}{r}$. A k-colouring of H is obtained by partitioning $V(H)$ into k classes of size $r-1$ each. As shown in Example 1,

$$\frac{\tau(H)}{\tau^*(H)} = \frac{(r-1)k-r+1}{(r-1)k/r} = \frac{k-1}{k}r \quad .$$

Proof of Theorems 3 and 4. The proofs of the two theorems go along the same lines, so we present them jointly, indicating the differences where relevant.

Suppose that the theorem fails, and let $H = (V,E)$ be a counterexample with smallest number of vertices. Then we must have $\bigcup_{e\in E} e = V(H)$. Let $g : V(H) \to \mathbb{R}^+$ be a minimal fractional cover of H and $f : E(H) \to \mathbb{R}^+$ be a maximal fractional matching in H with value $|g| = |f| = \tau^*(H)$. Consider two possible cases.

Case 1: $g(v) > 0$ for every $v \in V$, and $|e| \geq 2$ for every $e \in E$.

Then according to the complementary slackness conditions (1)

$$|V| = \sum_{v\in V} 1 = \sum_{v\in V}\sum_{e\ni v} f(e) = \sum_{e\in E} f(e)|e| \leq r\sum_{e\in E} f(e) = r\tau^*(H) \quad ,$$

so

$$\tau^*(H) \geq \frac{|V|}{r} \quad . \tag{2}$$

On the other hand, the union of every $(k-1)$ colour classes of H is obviously a cover of H, so

$$\tau(H) \leq \frac{k-1}{k}|V| \quad . \tag{3}$$

(2) and (3) yield

$$\frac{\tau(H)}{\tau^*(H)} \leq \frac{k-1}{k}r \quad ,$$

and since $\frac{k-1}{k}r < r-1$ if $k < r$ we obtain a contradiction to the choice of H.

Case 2: either there exists a vertex u_0 with $g(u_0) = 0$, or there exists an edge e_0 with $|e_0| = 1$.

We establish first the existence of a vertex v_0 with $g(v_0) \geq 1/(r-1)$. If the first subcase holds, let e_0 be an arbitrary edge containing u_0. Since $|e_0| \leq r$ and

$\sum_{v\in e_0} g(v) \geq 1$, there exists a vertex $v_0 \in e_0$ such that $g(v_0) \geq 1/(r-1)$. If the second subcase holds with $e_0 = \{v_0\}$, then $g(v_0) = 1 \geq 1/(r-1)$.

Let v_0 satisfy $g(v_0) \geq 1/(r-1)$. If $\{v_0\}$ is a cover of H then $\tau(H) = \tau^*(H) = 1$ and H is not a counterexample. So we may consider the hypergraph $H' = H - v_0$ obtained by deleting v_0 and the edges going through it. Since H' is also k-colourable and of rank at most r, it follows from the choice of H that the theorem's bound holds true for H'. Obviously,

$$\tau(H) \leq \tau(H') + 1 \tag{4}$$

(if $T \subseteq V(H')$ is a cover of H', then $T \cup \{v_0\}$ is a cover of H). On the other hand, the function $g' : V(H') \to \mathbb{R}^+$ defined by $g'(v) := g(v)$ for every $v \in V(H')$ is a fractional cover of H', so

$$\tau^*(H') \leq |g'| = |g| - g(v_0) \leq \tau^*(H) - \frac{1}{r-1} \quad . \tag{5}$$

In the case $k < r$ we have $\tau(H') < (r-1)\tau^*(H')$, so it follows from (4) and (5) that

$$\tau(H) \leq \tau(H')+1 < (r-1)\tau^*(H')+1 \leq (r-1)\left(\tau^*(H) - \frac{1}{r-1}\right)+1 = (r-1)\tau^*(H)\,,$$

while in the case $k \geq r$ we have $\tau(H') \leq \frac{k-1}{k}\, r\, \tau^*(H')$ and so

$$\begin{aligned}\tau(H) &\leq \tau(H') + 1 \leq \frac{k-1}{k} r\, \tau^*(H') + 1 \leq \frac{k-1}{k} r \left(\tau^*(H) - \frac{1}{r-1}\right) + 1 \\ &= \frac{k-1}{k} r\, \tau^*(H) - \frac{k-1}{k}\,\frac{r}{r-1} + 1 \leq \frac{k-1}{k} r\, \tau^*(H) \quad ,\end{aligned}$$

in both cases obtaining a contradiction to the choice of H. $\square$

4. The ratio τ/τ^* in strongly k-colourable hypergraphs

The hypergraph $H = \binom{[k]}{r}$ is strongly k-colourable and satisfies, as we have seen, $\tau(H)/\tau^*(H) = \frac{k-r+1}{k} r$. In the case $k \geq (r-1)r$ this example is extremal, as asserted in Theorem 5.

Proof of Theorem 5. Suppose that the theorem is false, and let $H = (V, E)$ be a counterexample with smallest number of vertices. Then we must have $\bigcup_{e\in E} e = V$. Let $g : V \to \mathbb{R}^+$ be an optimal fractional cover of H and $f : E \to \mathbb{R}^+$ be an optimal fractional matching in H with value $|g| = |f| = \tau^*(H)$. Consider two possible cases.

Case 1: $g(v) > 0$ for every $v \in V$, and $|e| = r$ for every $e \in E$.

Then it follows by applying (1) to g and f (see Case 1 in the proof of Theorems 3 and 4) that

$$\tau^*(H) \geq \frac{|V|}{r} \quad . \tag{6}$$

But the union of every $(k-r+1)$ colour classes is obviously a cover of H, so

$$\tau(H) \leq \frac{k-r+1}{k} |V| \quad . \tag{7}$$

Comparison of (6) and (7) gives

$$\frac{\tau(H)}{\tau^*(H)} \leq \frac{k-r+1}{k} r$$

- a contradiction to our assumption about H.

Case2: either there exists a vertex u_0 with $g(u_0) = 0$, or there exists an edge e_0 with $|e_0| \leq r-1$.

The argument in this case follows closely the one of Case 2 in the proof of Theorems 3 and 4, so we omit the details. To get the argument started, observe that in either one of the two subcases there is an edge e_0 such that $|e_0 \cap \{v \in V : g(v) > 0\}| \leq r-1$, and therefore there is a vertex $v_0 \in e_0$ with $g(v_0) \geq 1/(r-1)$. $\square$

Proof of Theorem 6. Let $H = (V, E)$ be a strongly k-colourable hypergraph of rank at most r, with colour classes $C_1, \dots, C_k$. Let $g : V \to \mathbb{R}^+$ be an optimal fractional cover with value $|g| = \tau^*(H)$.

Let σ be the cyclic permutation on $[k]$. We shall refer below to the action of σ on $\mathbb{R}^k$, which takes the form $\sigma(x_1, x_2, \dots, x_k) = (x_2, x_3, \dots, x_k, x_1)$.

We shall prove the theorem by establishing the conditions required in Lemma 1. Since the general case involves some messy details, we present first the argument in the case when $u = k^2/(k+r)$ is an integer. Then $t = kr/(k+r)$ is also an integer, since $u + t = k$. We have to prove, in this case, that $\frac{\tau(H)}{\tau^*(H)} \leq t$. Let

$$Q_1 = (\underbrace{0, \dots, 0}_{u \text{ times}}, \underbrace{\frac{1}{t}, \dots, \frac{1}{t}}_{t \text{ times}}) ,$$

$$Q_2 = (\underbrace{\frac{1}{r}, \dots, \frac{1}{r}}_{k \text{ times}}) ,$$

and let $B^0 = [Q_1 Q_2]$, that is, B^0 is the closed interval in $\mathbb{R}^k$ joining Q_1 and Q_2. For $i = 1, \dots, k-1$, define $B^i = \sigma^i(B^0)$, and let

$$B = \bigcup_{i=0}^{k-1} B^i .$$

Clearly $B \subseteq [0, 1/t]^k$. It can be checked that for every $\bar{x} = (x_1, \dots, x_k) \in B$ the sum of the largest r components equals 1, and hence the set

$$T(\bar{x}) = \bigcup_{i=1}^{k} \{v \in C_i : g(v) \geq x_i\}$$

is a cover of H.

Now, let μ^i be the uniform measure on B^i with $\mu^i(B^i) = 1/k$, $i = 0, 1, \dots, k-1$, and let $\mu = \sum_{i=0}^{k-1} \mu^i$. Then μ is a probability measure on B. For a given j, $1 \leq j \leq k$, there are u values of i for which the marginal distribution μ^i_j is uniform on $[0, \frac{1}{r}]$ and t values of i for which μ^i_j is uniform on $[\frac{1}{r}, \frac{1}{t}]$. It follows that the marginal distribution μ_j is uniform on $[0, \frac{1}{t}]$ for every $1 \leq j \leq k$. All the conditions of Lemma 1 are satisfied, and we conclude that $\frac{\tau(H)}{\tau^*(H)} \leq t$. We remark that this inequality can be shown to be strict except if $k = (r-1)r$ (by an argument similar to that given for strict inequality in Theorem 1).

In the case when $u = k^2/(k+r)$ and $t = kr/(k+r)$ are not integers, we establish the upper bound on $\tau(H)/\tau^*(H)$ by giving two constructions satisfying the conditions of Lemma 1, corresponding to two values of δ, namely δ_1 and δ_2, where:

$$\frac{1}{\delta_1} = \frac{kr}{k+r} + \frac{k-r}{2k}\{u\} = \frac{k^2-(k-r)\lfloor u \rfloor}{2k} ,$$
$$\frac{1}{\delta_2} = \frac{kr}{k+r} + \frac{r}{k}(1-\{u\}) = \frac{r\lceil u \rceil}{k} .$$

The two constructions represent two different adaptations of the construction for integral u described above, in which the role of u is played by the lower and upper integer parts of u respectively.

First construction. The $(\lceil t \rceil + 1)$-tuple $(p_0, p_1, \ldots, p_{\lceil t \rceil})$, where $p_0 = (r - \lceil t \rceil)\lfloor u \rfloor$ and $p_1 = \cdots = p_{\lceil t \rceil} = \lceil t \rceil$ satisfies the conditions of Proposition 2. Indeed, let us denote by s the sum of the p_i, i.e.,

$$s = \sum_{i=0}^{\lceil t \rceil} p_i = (r - \lceil t \rceil)\lfloor u \rfloor + \lceil t \rceil^2 .$$

Then, in order to check that each $p_i \leq s/2$, it suffices to check that:

(*i*) $(r - \lceil t \rceil)\lfloor u \rfloor \leq \lceil t \rceil^2$,

(*ii*) $\lceil t \rceil \geq 2$.

The first condition is equivalent (using $\lfloor u \rfloor = k - \lceil t \rceil$) to $\lceil t \rceil \geq kr/(k+r)$, which of course holds. If the second condition failed, it would mean that $t < 1$ (since we assume that t is not an integer), so $kr < k + r$, but this cannot be the case when $2 \leq r \leq k$.

Thus, according to Proposition 2, there exists a $(\lceil t \rceil + 1)$- dimensional random variable $\bar{\eta} = (\eta_0, \eta_1, \ldots, \eta_{\lceil t \rceil})$ whose values lie in the set

$$\left\{\bar{y} = (y_0, y_1, \ldots, y_{\lceil t \rceil}) \in [0, 2/s]^{\lceil t \rceil + 1} : \sum_{i=0}^{\lceil t \rceil} p_i y_i = 1\right\} ,$$

such that each η_i, $0 \leq i \leq \lceil t \rceil$, is uniformly distributed on $[0, 2/s]$. Now, define a k-dimensional random variable $\bar{\xi} = (\xi_1, \ldots, \xi_k)$ by:

$$\xi_i = \frac{s\lfloor u \rfloor}{k^2 - (k-r)\lfloor u \rfloor}\eta_0 \quad \text{for } i = 1, \ldots, \lfloor u \rfloor ,$$
$$\xi_{\lfloor u \rfloor + i} = \frac{s\lceil t \rceil}{k^2 - (k-r)\lfloor u \rfloor}\eta_i + \frac{2\lfloor u \rfloor}{k^-(k-r)\lfloor u \rfloor} \quad \text{for } i = 1, \ldots, \lceil t \rceil .$$

The parameters of this transformation were chosen so as to make ξ_i, $1 \leq i \leq \lfloor u \rfloor$, uniformly distributed on $[0, \frac{\lfloor u \rfloor}{k}\delta_1]$ and $\xi_{\lfloor u \rfloor + i}$, $1 \leq i \leq \lceil t \rceil$, uniformly distributed on $[\frac{\lfloor u\rfloor}{k}\delta_1, \delta_1]$. The sum of the largest r components of $\bar{\xi}$ can be computed as:

$$\sum_{i=1}^{\lceil t\rceil}\left(\frac{s\lceil t\rceil}{k^2-(k-r)\lfloor u\rfloor}\eta_i+\frac{2\lfloor u\rfloor}{k^2-(k-r)\lfloor u\rfloor}\right)+(r-\lceil t\rceil)\frac{s\lfloor u\rfloor}{k^2-(k-r)\lfloor u\rfloor}\eta_0$$
$$=\frac{s}{k^2-(k-r)\lfloor u\rfloor}\sum_{i=0}^{\lceil t\rceil}p_i\eta_i+\frac{2\lfloor u\rfloor\lceil t\rceil}{k^2-(k-r)\lfloor u\rfloor}=\frac{s+2\lfloor u\rfloor\lceil t\rceil}{k^2-(k-r)\lfloor u\rfloor}=1\ .$$

The same holds true for each of the permuted random variables $\sigma^i(\bar{\xi})$, $i=1,\dots,k-1$. Hence, denoting by B^i the range of $\sigma^i(\bar{\xi})$, we know that for every $0\le i\le k-1$ and for every $\bar{x}=(x_1,\dots,x_k)\in B^i$ the set

$$T(\bar{x})=\bigcup_{j=1}^{k}\{v\in C_j : g(v)\ge x_j\}$$

is a cover of H. Let the measure μ^i on B^i be the distribution of $\sigma^i(\bar{\xi})$. Let $B=\bigcup_{i=0}^{k-1}B^i$ and let $\mu=\frac{1}{k}\sum_{i=0}^{k-1}\mu^i$. Then it can be seen that the set B and the probability measure μ on it satisfy all the conditions of Lemma 1, enabling us to conclude that $\tau(H)/\tau^*(H)\le 1/\delta_1$.

Second construction. Let

$$Q_1=(\underbrace{0,\dots,0}_{\lceil u\rceil\text{ times}},\underbrace{\frac{k}{r\lceil u\rceil},\dots,\frac{k}{r\lceil u\rceil}}_{\lfloor t\rfloor\text{ times}})\ ,$$
$$Q_2=(\underbrace{\frac{1}{r},\dots,\frac{1}{r}}_{k\text{ times}})\ ,$$

and let $B^0=[Q_1Q_2]$. The sum of the largest r components is $\frac{k\lfloor t\rfloor}{r\lceil u\rceil}\le\frac{kt}{ru}=1$ at Q_1 and equals 1 at Q_2, and therefore is at most 1 at every point of B^0. We may proceed as in the integral case, taking B^0 and its cyclic shifts and the uniform measure on them, verifying that the conditions of Lemma 1 are satisfied, and concluding that $\tau(H)/\tau^*(H)\le 1/\delta_2$. $\Box$

A few words about the bound in Theorem 6. As one can see, the deviation of this bound from the conjectured bound $\frac{\tau(H)}{\tau^*(H)}<\frac{kr}{k+r}$ is caused only by indivisibility (of kr and k^2 by $k+r$). As we have already mentioned in the introduction, this deviation does not exceed the constant $3-2\sqrt{2}$.

What about examples? Consider the first non-trivial case $r=3, k=4$. The hypergraph $H=\binom{[4]}{3}$ has $\frac{\tau(H)}{\tau^*(H)}=\frac{3}{2}$. In Example 2 (with $r=3$) the ratio $\frac{\tau(H)}{\tau^*(H)}\to\frac{3}{2}$. Using a variation of the construction in Example 2, we succeeded to build an example with the ratio $\frac{\tau(H)}{\tau^*(H)}\approx 1.7$, which is not so far from the conjectured bound 12/7. But we do not have examples with $\frac{\tau(H)}{\tau^*(H)}\to\frac{12}{7}$. The best upper bound we know for this case is 7/4, given by Theorem 6.

5. The ratio τ/τ^* in k-partitionable hypergraphs

Recall that a hypergraph $H = (V, E)$ is called k-partitionable if the vertex set V can be partitioned into k covers $T_1, \dots, T_k$.

In order to see that the bound of Theorem 7 cannot be improved, consider the hypergraphs

$$H^n = H^n(r-k+1, \underbrace{1, \dots, 1}_{k-1 \text{ times}})$$

as constructed in Example 3. Clearly, H^n is k-partitionable and r-uniform. As shown in the analysis of Example 3, when $n \to \infty$ we have $\tau(H^n)/\tau^*(H^n) > r - k + 1 - \epsilon$ for every fixed $\epsilon > 0$.

Proof of Theorem 7. Let $H = (V, E)$ be a k-partitionable hypergraph of rank at most r, and let $(T_1, \dots, T_k)$ be a partition of the vertex set V into k covers. Suppose $g : V \to \mathbb{R}^+$ is an optimal fractional cover with value $|g| = \tau^*(H)$.

Define a set $B \subseteq [0, 1/(r-k+1)]^k$. First define $(k+1)$ points $Q_1^1, \dots, Q_1^k, Q_2$ in $[0, 1/(r-k+1)]^k$ as follows.

$$Q_1^i = (0, \dots, 0, \underbrace{1/(r-k+1)}_{i}, 0, \dots, 0), \quad 1 \le i \le k ,$$

$$Q_2 = \left(\frac{k-1}{k(r-k+1)}, \frac{k-1}{k(r-k+1)}, \dots, \frac{k-1}{k(r-k+1)} \right) .$$

Now define k intervals $B^1, \dots, B^k$ by

$$B^i = [Q_1^i Q_2], \quad 1 \le i \le k$$

and let

$$B = B^1 \cup \dots \cup B^k .$$

It is easy to check that under the theorem's conditions ($r \ge (k-1)k$) for every point $\bar{x} = (x_1, \dots, x_k) \in B$ and for every $1 \le j \le k$ we have $(r-k)x_j + \sum_{i=1}^k x_i \le 1$. Therefore the set

$$T(\bar{x}) = \bigcup_{i=1}^k \{v \in T_i : g(v) \ge x_i\}$$

is a cover of H for every $\bar{x} \in B$. (Indeed, suppose on the contrary that there exists a point $\bar{x} = (x_1, \dots, x_k) \in B$ and an edge $e \in E(H)$ such that $e \cap T(\bar{x}) = \emptyset$. This means $g(v) < x_i$ for every $v \in e \cap T_i$, $1 \le i \le k$. Let $x_j = \max\{x_i : 1 \le i \le k\}$. Then since $|e \cap T_i| \ge 1$ for every $1 \le i \le k$ and $|e| \le r$, we have

$$\sum_{v \in e} g(v) < \sum_{i=1}^k |e \cap T_i| x_i \le (r-k)x_j + \sum_{i=1}^k x_i \le 1$$

-a contradiction since g is a fractional cover).

Define now a probability measure μ on B. Let μ^i, $1 \le i \le k$, be the uniform measures on the intervals B^i such that $\mu^i(B^i) = 1/k$, $1 \le i \le k$, and let

$$\mu = \mu^1 + \dots + \mu^k .$$

The marginal distributions of μ are uniform on each of the two intervals $\left[0, \frac{k-1}{k}\frac{1}{r-k+1}\right]$ and $\left[\frac{k-1}{k}\frac{1}{r-k+1}, \frac{1}{r-k+1}\right]$. Note that the first interval is $(k-1)$ times longer than the second one. Since every coordinate x_i runs through the first interval in every B^j, $j \neq i$, and through the second one in B^i, we obtain that all marginal distributions μ_i, $1 \leq i \leq k$, are uniform on the interval $[0, 1/(r-k+1)]$.

Now, if $\bar{x} \in B$ is randomly chosen from B according to the measure μ we have as in Lemma 1

$$\tau(H) \leq E(|T(\bar{x})|) \leq (r-k+1)\tau^*(H) .$$

Using ideas similar to those in the proof of Theorem 1, one can show that

$$\tau(H) < (r-k+1)\tau^*(H) .$$

We omit the details. □

Proof of Theorem 8. Let $H = (V, e)$ be a k-partitionable hypergraph of rank at most r, and let $(T_1, \ldots, T_k)$ be a partition of V into k covers. Let $g : V \to \mathbb{R}^+$ be an optimal fractional cover with value $|g| = \tau^*(H)$.

The proof has a similar structure to that of Theorem 6. We prove the upper bound on $\tau(H)/\tau^*(H)$ by establishing the conditions required in Lemma 1. We present first the argument in the case when $u = k^2/(k+r)$ is an integer. In this case $t = kr/(k+r)$ is also an integer ($u + t = k$), and so is $w = r^2/(k+r)$ (note that $w = r - k + u$). We have to prove that $\tau(H)/\tau^*(H) \leq w$. Let

$$Q_1 = (\underbrace{\frac{1}{w}, \ldots, \frac{1}{w}}_{u \text{ times}}, \underbrace{0, \ldots, 0}_{t \text{ times}}) ,$$
$$Q_2 = (\underbrace{\frac{1}{r}, \ldots, \frac{1}{r}}_{k \text{ times}}) ,$$

and let $B^0 = [Q_1 Q_2]$. For every point $\bar{x} = (x_1, \ldots, x_k) \in B^0$ and for every $1 \leq j \leq k$ we have $(r-k)x_j + \sum_{i=1}^k x_i \leq 1$ (the left-hand side is largest when $1 \leq j \leq u$, and then it equals 1). As explained in the proof of Theorem 7, this implies that the set

$$T(\bar{x}) = \bigcup_{i=1}^{k} \{v \in T_i : g(v) \geq x_i\}$$

is a cover of H for every $\bar{x} \in B^0$. We may proceed as in the proof of Theorem 6, taking $B \subseteq [0, 1/w]^k$ to be the union of B^0 and its cyclic shifts and using uniform measures, verifying that the marginals are uniform on $[0, 1/w]$, and concluding that $\tau(H)/\tau^*(H) \leq w$. We remark that this inequality can be shown to be strict except if $k = r = 2$.

In the case when u, t and w are not integers, we give two constructions satisfying the conditions of Lemma 1, corresponding to two values of δ, namely δ_1 and δ_2, where:

$$\frac{1}{\delta_1} = \frac{r^2}{k+r} + \frac{r-k}{2k}\{u\} = \frac{kr + (r-k)\lceil t \rceil}{2k} ,$$
$$\frac{1}{\delta_2} = \frac{r^2}{k+r} + 1 - \{u\} = \lceil w \rceil .$$

First construction. The $(\lceil t\rceil+1)$-tuple $(p_0,p_1,\ldots,p_{\lceil t\rceil})$, where $p_0=\lfloor w\rfloor\lfloor u\rfloor$ and $p_1=\cdots=p_{\lceil t\rceil}=\lceil t\rceil$ satisfies the conditions of Proposition 2. Indeed, let us denote by s the sum of the p_i, i.e.,

$$s=\sum_{i=0}^{\lceil t\rceil} p_i=\lfloor w\rfloor\lfloor u\rfloor+\lceil t\rceil^2 .$$

Then in order to check that each $p_i\le s/2$, it suffices to check that

(*i*) $\lfloor w\rfloor\lfloor u\rfloor\le\lceil t\rceil^2$,
(*ii*) $\lceil t\rceil\ge 2$.

The first condition holds because $wu=t^2$, and the second one is easy to check, too.

Thus, according to Proposition 2, there exists a $(\lceil t\rceil+1)$- dimensional random variable $\bar\eta=(\eta_0,\eta_1,\ldots,\eta_{\lceil t\rceil})$ whose values lie in the set

$$\Big\{\bar y=(y_0,y_1,\ldots,y_{\lceil t\rceil})\in[0,2/s]^{\lceil t\rceil+1}:\sum_{i=0}^{\lceil t\rceil}p_iy_i=1\Big\}\ ,$$

such that each $\eta_i,\ 0\le i\le\lceil t\rceil$, is uniformly distributed on $[0,2/s]$. Now, define a k-dimensional random variable $\bar\xi=(\xi_1,\ldots,\xi_k)$ by:

$$\xi_i=\frac{s\lfloor u\rfloor}{kr+(r-k)\lceil t\rceil}\eta_0+\frac{2\lceil t\rceil}{kr+(r-k)\lceil t\rceil}\quad\text{for } i=1,\ldots,\lfloor u\rfloor\ ,$$
$$\xi_{\lfloor u\rfloor+i}=\frac{s\lceil t\rceil}{kr+(r-k)\lceil t\rceil}\eta_i\quad\text{for } i=1,\ldots,\lceil t\rceil\ .$$

The parameters of this transformation were chosen so as to make $\xi_i,\ 1\le i\le\lfloor u\rfloor$, uniformly distributed on $[\frac{\lceil t\rceil}{k}\delta_1,\delta_1]$ and $\xi_{\lfloor u\rfloor+i},\ 1\le i\le\lceil t\rceil$, uniformly distributed on $[0,\frac{\lceil t\rceil}{k}\delta_1]$. The maximum (over $1\le j\le k$) of $(r-k)\xi_j+\sum_{i=1}^k\xi_i$ is attained when $1\le j\le\lfloor u\rfloor$, and then its value can be computed as:

$$(r-k+\lfloor u\rfloor)\left(\frac{s\lfloor u\rfloor}{kr+(r-k)\lceil t\rceil}\eta_0+\frac{2\lceil t\rceil}{kr+(r-k)\lceil t\rceil}\right)+\sum_{i=1}^{\lceil t\rceil}\frac{s\lceil t\rceil}{kr+(r-k)\lceil t\rceil}\eta_i$$
$$=\frac{s}{kr+(r-k)\lceil t\rceil}\sum_{i=0}^{\lceil t\rceil}p_i\eta_i+\frac{2\lfloor w\rfloor\lceil t\rceil}{kr+(r-k)\lceil t\rceil}=\frac{s+2\lfloor w\rfloor\lceil t\rceil}{kr+(r-k)\lceil t\rceil}=1\ .$$

This guarantees that for every $\bar x=(x_1,\ldots,x_k)$ in B^0 (the range of $\bar\xi$) the set

$$T(\bar x)=\bigcup_{i=1}^k\{v\in T_i: g(v)\ge x_i\}$$

is a cover of H. The construction is completed in the usual way (taking cyclic shifts) to conclude that $\tau(H)/\tau^*(H)\le 1/\delta_1$.

Second construction. Let

$$Q_1=(\underbrace{\frac{1}{\lceil w\rceil},\ldots,\frac{1}{\lceil w\rceil}}_{\lceil u\rceil\text{ times}},\underbrace{0,\ldots,0}_{\lfloor t\rfloor\text{ times}})\ ,$$

$$Q_2=(\underbrace{\frac{\lfloor t\rfloor}{k\lceil w\rceil},\ldots,\frac{\lfloor t\rfloor}{k\lceil w\rceil}}_{k\text{ times}})$$

and let $B^0 = [Q_1 Q_2]$. For $\bar{x} = (x_1, \ldots, x_k) \in B^0$, the maximum (over $1 \leq j \leq k$) of $(r-k)x_j + \sum_{i=1}^{k} x_i$ is attained when $1 \leq j \leq \lceil u \rceil$, and then its value is 1 at Q_1 and is $\frac{r\lfloor t \rfloor}{k\lceil w \rceil} \leq \frac{rt}{kw} = 1$ at Q_2, and hence is at most 1 at every point of B^0. The construction is completed as above, leading to the conclusion that $\tau(H)/\tau^*(H) \leq 1/\delta_2$. $\square$

The difference between the theorem's bound and the bound in Conjecture 2 stems from the indivisibility of k^2,r^2 and kr by $k+r$. This difference is not more than 1 for all values of k, r.

In the first interesting case, namely, $k = 3, r = 4$, Example 2 (with $r = 4$) provides the ratio $\frac{\tau(H)}{\tau^*(H)} \to 2$ and so does Example 3 with parameters (2,1,1). We managed to build a 3-partitionable, 4-uniform hypergraph H, for which $\frac{\tau(H)}{\tau^*(H)} \approx$ 2.22. This is still smaller than the conjectured bound 16/7. The best upper bound we know for this case is 7/3, given by Theorem 8.

References

[1] Z. Füredi, *Matchings and covers in hypergraphs*, Graphs and Combinatorics **4** (1988), 115–206.

[2] M. Krivelevich, *Approximate set covering in uniform hypergraphs*, preprint.

[3] L. Lovász, *On the ratio of optimal integral and fractional covers*, Discrete Math. **13** (1975), 383–390.

[4] L. Lovász, *On minimax theorems of combinatorics*, Doctoral Thesis, Mathematikai Lapok **26** (1975), 209–264. (Hungarian)

[5] L. Lovász, *Covers, packings and some heuristic algorithms*, Proc. 5th British Comb. Conf., Congressus Numerantium 15, Utilitas Math., Winnipeg, 1976, pp. 417–429.

[6] A. A. Sapozhenko, *On the complexity of disjunctive normal forms obtained by greedy algorithm*, Discret. Analyz **21** (1972), 62–71. (Russian)

[7] S. K. Stein, *Two combinatorial covering theorems*, J. Comb. Theory(A) **16** (1974), 391–397.

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