On k-saturated graphs with restrictions on the degrees

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Abstract

A graph G is called k-saturated, where $k \geq 3$ is an integer, if G is K^k-free but the addition of any edge produces a K^k (we denote by K^k a complete graph on k vertices). We investigate k-saturated graphs, and in particular the function $F_k(n, D)$ defined as the minimal number of edges in a k-saturated graph on n vertices having maximal degree at most D. This investigation was suggested by Hajnal, and the case $k = 3$ was studied by Füredi and Seress. The following are some of our results. For $k = 4$, we prove that $F_4(n, D) = 4n - 15$ for $n > n_0$ and $\lfloor \frac{2n-1}{3} \rfloor \leq D \leq n - 2$. For arbitrary k, we show that the limit $\lim_{n\to\infty} F_k(n, cn)/n$ exists for all $0 < c \leq 1$, except maybe for some values of c contained in a sequence $c_i \to 0$. We also determine the asymptotic behaviour of this limit for $c \to 0$. We construct, for all k and all sufficiently large n, a k-saturated graph on n vertices with maximal degree at most $2k\sqrt{n}$, significantly improving an upper bound due to Hanson and Seyffarth.

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1 Introduction

A graph $G = (V, E)$ is called *k-saturated* for an integer $k \geq 3$ if G does not contain a complete graph on k vertices K^k, but the addition of any edge to G yields a K^k. The theorem of Erdős, Hajnal and Moon ([2]) states that if G is a k-saturated graph on $n \geq k-2$ vertices, then $|E(G)| \geq (k-2)n - \binom{k-1}{2}$. However, for every k the extremal example for this theorem contains a vertex of degree $n-1$ (we call such a vertex *conical*). Hajnal ([6]) asked what is the minimal number of edges in a k-saturated graph on n vertices with no conical vertex, or, more generally, what is the minimal number of edges in a k-saturated graph on n vertices with all vertex degrees at most D. The case $k = 3$ was treated by Füredi and Seress in [5]. Some additional results were obtained in [3]. Both papers used a linear programming method introduced by Pach and Surányi ([11]) for the study of the problem of determining the minimal number of edges in a graph of diameter two and all degrees at most D. In this paper we study the case $k \geq 4$. Our methods are similar to those of Füredi and Seress, but contain several new ingredients.

The following related problem was considered by Duffus and Hanson in [1]: what is the minimal number of edges in a k-saturated graph on n vertices with minimum degree δ? Some results on this problem are presented as well.

We also address the problem of the lowest possible maximal degree in a k-saturated graph on n vertices. Clearly, every k-saturated graph has diameter two, therefore it can easily be deduced that the maximal degree in a k-saturated graph is at least $(n-1)^{1/2}$. Hanson and Seyffarth ([7]) constructed k-saturated graphs on n vertices with maximal degree $O(n^{\alpha_k})$, where $\alpha_k < 1$, but $\alpha_k \to 1$ as $k \to \infty$. They also conjectured that the correct value of the lowest possible maximal degree is asymptotically $c_k n^{1/2}$ as $n \to \infty$, where c_k is a constant depending only on k. In this paper we build k-saturated graphs with maximal degree $O(n^{1/2})$ for each k, thus matching the lower bound up to a constant (depending on k) factor. The case $k = 3$ has already been done by Hanson and Seyffarth, and a better value for the constant was obtained by Füredi and Seress.

We end this section with some notation. For a graph G we denote by $\bar{G}$ the complement of G. For a subset $U \subseteq V(G)$ we denote by $G[U]$ the induced subgraph of G on U. We also write $G \setminus U$ instead of $G[V(G) \setminus U]$. The degree of a vertex x is denoted by $d(x)$. We denote by $\Delta(G)$ and $\delta(G)$ the maximal and the minimal degree of G, respectively. Let

$$
\begin{aligned}
F_k(n, D) &= \min\{|E(G)| : G \text{ is } k\text{-saturated}, |V(G)| = n, \Delta(G) \leq D\} , \\
F_k^*(n, D) &= \min\{|E(G)| : G \text{ is } k\text{-saturated}, |V(G)| = n, \Delta(G) = D\}
\end{aligned}
$$

(for triples (k, n, D), for which the corresponding graphs do not exist we set $F_k(n, D) =$

∞ or $F_k^*(n, D) = \infty$). The above definitions clearly imply

$$F_k(n, D') \leq F_k(n, D) \leq F_k^*(n, D)$$

for every $D' \geq D$. Using this notation, Hajnal's question is to determine $F_k(n, D)$ and in particular $F_k(n, n-2)$.

The rest of the paper is organized as follows. In Section 2 we treat the values of $F_k(n, D)$ and $F_k^*(n, D)$ for $D = n - 2$ and $D = n - 3$. In Section 3 we obtain some structural results for k-saturated graphs which are used to treat the case $D = cn, 0 < c < 1$. In Section 4 we consider the case $D = o(n)$. Some additional results on 4-saturated graphs and k-saturated graphs, $k > 4$, are presented in Sections 5 and 6, respectively.

2 Graphs with maximal degree $n - 2$ or $n - 3$

In this section we study the values of $F_k^*(n, n-2)$ and $F_k^*(n, n-3)$. The results obtained supply some information about $F_k(n, n-2)$ and $F_k(n, n-3)$ as well. Later we will obtain additional results about these functions.

The following two propositions appear essentially (in dual form) in [10] (p. 447).

Proposition 1 *Let $k \geq 4$ and $G = (V, E)$ be a k-saturated graph on n vertices with $\Delta(G) = n - 2$. If $d(x) = n - 2$ and $(x, y) \notin E(G)$, then $(y, z) \in E(G)$ for every vertex $z \in V \setminus \{x, y\}$ and the graph $G' = G \setminus \{x, y\}$ is $(k-1)$-saturated with no conical vertex. Conversely, given a $(k-1)$-saturated graph G' on $n-2$ vertices with no conical vertex, one can add two non-adjacent vertices x and y and join them to all other vertices, thus obtaining a k-saturated graph G on n vertices with $\Delta(G) = n - 2$.*

Proposition 2 *Let $k \geq 4$ and $G = (V, E)$ be a k-saturated graph on n vertices with $\Delta(G) = n - 3$. If $d(x) = n - 3$ and $(x, u), (x, v) \notin E(G)$ then either*

1. *$(u, v) \notin E(G)$ and then $(u, z), (v, z) \in E(G)$ for every vertex $z \in V \setminus \{x, u, v\}$ and $G' = G \setminus \{x, u, v\}$ is $(k-1)$-saturated with $\Delta(G') \leq n - 6$. Conversely, given a $(k-1)$-saturated graph G' on $n - 3$ vertices with $\Delta(G') \leq n - 6$, one can add three independent vertices x, u and v and join them to all other vertices, thus obtaining a k-saturated graph G on n vertices with $\Delta(G) = n - 3$,*

or

2. *$(u,v) \in E(G)$ and then for each $z \in V \setminus \{x,u,v\}$ at least one of the edges $(u,z),(v,z)$ belongs to $E(G)$ and the graph G', obtained from $G \setminus \{x,u,v\}$ by adding a new vertex w and joining it to the vertices that are joined in G to both u and v, is $(k-1)$-saturated with $\Delta(G') \leq n-5$. Conversely, given a $(k-1)$-saturated graph G' on $n-2$ vertices with $\Delta(G') \leq n-5$, one can replace any vertex $w \in V(G')$ by two new vertices u and v, join u and v, join both u and v to all vertices of $V(G')$ to which w was joined, also join one of u, v to every other vertex of $V(G') \setminus \{w\}$ so that both u and v are chosen at least once, and add a new vertex x joined to all other vertices but u and v, thus obtaining a k-saturated graph G on n vertices with $\Delta(G) = n-3$.*

Turning to our notation, we can easily see that the above propositions imply:

Proposition 3 *For $k \geq 4$ one has*

1. $F_k^*(n, n-2) = F_{k-1}(n-2, n-4) + 2n - 4$;
2. $F_k^*(n, n-3) = F_{k-1}(n-2, n-5) + 2n - 5$.

Proof. 1. Follows immediately from Proposition 1.

2. Suppose G is a k-saturated graph on n vertices with $\Delta(G) = n-3$. Let $d(x) = n-3$, $(x,u),(x,v) \notin E(G)$. If $(u,v) \notin E(G)$, then according to part 1 of Proposition 2 the graph $G' = G \setminus \{x,u,v\}$ is $(k-1)$-saturated with $\Delta(G') \leq n-6$ and $|E(G')| = |E(G)| - d(x) - d(u) - d(v) = |E(G)| - 3(n-3)$, therefore

$$|E(G)| \geq F_{k-1}(n-3, n-6) + 3n - 9 . \tag{1}$$

In case $(u,v) \in E(G)$, consider the graph G' described in part 2 of Proposition 2. Let $V_1 = \{y \in V(G) \setminus \{x,u,v\} : (y,u) \in E(G), (y,v) \notin E(G)\}$, $V_2 = \{y \in V(G) \setminus \{x,u,v\} : (y,v) \in E(G), (y,u) \notin E(G)\}$, and $V_3 = \{y \in V(G) \setminus \{x,u,v\} : (y,u) \in E(G), (y,v) \in E(G)\}$. Then $V_1 \cup V_2 \cup V_3 = V(G) \setminus \{x,u,v\}$, and

$$\begin{aligned}
|E(G')| &= |E(G)| - d(x) - 1 - |V_1| - |V_3| - |V_2| - |V_3| + |V_3| \\
&= |E(G)| - d(x) - 1 - |V_1| - |V_2| - |V_3| \\
&= |E(G)| - (n-3) - 1 - (n-3) \\
&= |E(G)| - (2n-5) .
\end{aligned}$$

Recalling that G' is $(k-1)$-saturated on $n-2$ vertices we obtain

$$|E(G)| \geq F_{k-1}(n-2, n-5) + 2n - 5 . \tag{2}$$

Proposition 2 and inequalities (1), (2) imply that

$$F_k^*(n, n-3) = \min\{F_{k-1}(n-3, n-6) + 3n - 9, F_{k-1}(n-2, n-5) + 2n - 5\} \ .$$

Since one can obtain from a $(k-1)$-saturated graph G_1 on $n-3$ vertices with $\Delta(G_1) \leq n-6$ and $|E(G_1)| = F_{k-1}(n-3, n-6)$ a $(k-1)$-saturated graph G_2 on $n-2$ vertices with $\Delta(G_2) \leq n-5$ and $|E(G_2)| \leq |E(G_1)| + \Delta(G_1)$ just by fixing any vertex $v \in V(G_1)$, adding a new vertex u and joining it to all vertices of $V(G_1)$ to which v is joined, we have

$$F_{k-1}(n-2, n-5) \leq F_{k-1}(n-3, n-6) + n - 6 \ .$$

Therefore

$$F_k^*(n, n-3) = F_{k-1}(n-2, n-5) + 2n - 5 \ . \qquad \square$$

It follows from the results of Duffus and Hanson ([1], see also [5]) that $F_3(n, n-2) = F_3(n, n-3) = 2n - 5$ for $n \geq 5$. Hence we derive:

Corollary 1 *1.* $F_4^*(n, n-2) = 4n - 13$ *for* $n \geq 7$ *;*

2. $F_4^*(n, n-3) = 4n - 14$ *for* $n \geq 7$ *.*

An extremal graph for $F_4^*(n, n-2)$ can be obtained from the cycle C^5 (where C^r denotes a cycle on r vertices) by replicating one vertex, adding two new non-adjacent vertices x and y and joining them to all other vertices of the graph. As for $F_4^*(n, n-3)$, an extremal graph can be obtained by replicating any vertex of the following graph G on seven vertices: $V(G) = \{0, 1, \ldots, 6\}$, $E(G) = \{(i, i+1) \bmod 7 : 0 \leq i \leq 6\} \cup \{(i, i+3) \bmod 7 : 0 \leq i \leq 6\}$. In the subsequent sections we will show that $F_4(n, n-2) \leq 4n - 15$ for $n \geq 9$ (and a construction exists with maximal degree $n-4$), and $F_4(n, n-2) = 4n - 15$ for sufficiently large n.

Proposition 4 *1.* $F_k^*(n, n-2) = F_k^*(n, n-3) + 1$ *for* $n \geq 2k - 1$ *;*

2. $F_k(n, n-2) = F_k(n, n-3)$ *for* $n \geq 2k - 1$.

Proof. By induction on $k \geq 3$. For $k = 3$ it was proved by Duffus and Hanson, and Füredi and Seress that $F_3^*(n, n-2) = 2n - 4$ and $F_3^*(n, n-3) = F_3(n, n-3) = 2n - 5$ for $n \geq 5$. Assuming that the proposition holds true for $k - 1$, we obtain from Proposition 3

1. $F_k^*(n, n-2) = F_{k-1}(n-2, n-4) + 2n - 4 = F_{k-1}(n-2, n-5) + 2n - 4 = F_k^*(n, n-3) - (2n-5) + (2n-4) = F_k^*(n, n-3) + 1$;

2. If G is a k-saturated graph with $\Delta(G) \leq n-2$, then either $\Delta(G) = n-2$, and then $|E(G)| \geq F_k^*(n, n-2) = F_k^*(n, n-3)+1 \geq F_k(n, n-3)+1$, or $\Delta(G) \leq n-3$, and then $|E(G)| \geq F_k(n, n-3)$. $\square$

An upper bound for $F_k^*(n, n-2)$ can be obtained by considering a complete $(k-1)$-partite graph $K^{n-2(k-2),2,\ldots,2}$ $(n \geq 2k-2)$, yielding

$$F_k^*(n, n-2) \leq 2(k-2)n - (2k^2 - 6k + 4) \ .$$

In Section 6 we will improve this bound slightly.

3 The structure of k-saturated graphs

This section extends the proofs of [5] for the case of general k. It contains some new ideas as well.

A *hypergraph* (set system) is a pair $\mathcal{H} = (V, \mathcal{E})$, where V is a finite ground set (the *vertex set*) and $\mathcal{E}$ is a family of distinct subsets of V (the *edge set*). We will occasionally identify a hypergraph with its edge set.

Suppose $G = (V, E)$ is a k-saturated graph and suppose $V_0 \subseteq V$ is such that $V \setminus V_0$ is independent in G. Then the number of edges in G can be computed using the following description of G:

(i) a graph $G_0 = G[V_0]$;

(ii) a hypergraph $\mathcal{H}$ on V_0, whose edges $H_1, \ldots, H_m$ are the neighbourhoods of the vertices in $V \setminus V_0$, listed without repetitions;

(iii) an assignment of weights $y_1, \ldots, y_m$ where y_i is the fraction of vertices of $V \setminus V_0$ with neighbourhood H_i, thus, $y_i \geq 0$ and $\sum_{i=1}^m y_i = 1$.

Then

$$|E(G)| = |E(G_0)| + |V \setminus V_0| \sum_{i=1}^{m} y_i |H_i| \ .$$

Moreover, it can be easily checked, using the fact that G is k-saturated, that the pair $(G_0, \mathcal{H})$ satisfies the following conditions.

1. G_0 is K^k-free;

2. $G_0[H_i]$ is K^{k-1}-free for every $H_i \in \mathcal{H}$;

3. $H_i \cap H_j$ contains a K^{k-2} for every pair $H_i, H_j \in \mathcal{H}$;

4. for every edge $H_i \in \mathcal{H}$ and every vertex $x \in V_0 \setminus H_i$ the subset $H_i \cup \{x\}$ contains a K^{k-1};

5. if $(x, y) \notin E(G_0)$ then either there exists in G_0 a copy of K^{k-2} completely joined to x and y, or there exists a copy of K^{k-3} on the vertices $v_1, \ldots, v_{k-3} \in V_0$, completely joined to x and y, and an edge $H_i \in \mathcal{H}$ such that $\{x, y, v_1, \ldots, v_{k-3}\} \subseteq H_i$.

It turns out that such pairs $(G_0, \mathcal{H})$ play a crucial role in determining the functions $F_k(n, D)$, and therefore the following definition is very useful.

Definition 1 *Let $G_0 = (V_0, E)$ be a graph and $\mathcal{H} = (V_0, \mathcal{E})$ be a hypergraph on some set V_0. The pair $(G_0, \mathcal{H})$ is a k-*core *if it satisfies the above conditions (1)–(5).*

Definition 2 *$(G_0, \mathcal{H})$ is a k-*pre-core *if it satisfies conditions (1)–(4).*

Note that the definition of a k-core generalizes that of a core ($k = 3$) given by Füredi and Seress. Observe also that if $(G_0, \mathcal{H})$ is a k-pre-core, then one can add, if necessary, edges to $E(G_0)$, obtaining a new graph G_0' such that $(G_0', \mathcal{H})$ is a k-core.

Given a k-core $(G_0, \mathcal{H})$ and weights $y_i \geq 0$, $1 \leq i \leq m$, such that $\sum_{i=1}^m y_i = 1$, one can construct for n large enough a k-saturated graph G on n vertices as follows. Choose sets $V_1, \ldots, V_m$ disjoint from each other and from V_0 such that $\lfloor y_i(n - |V_0|) \rfloor \leq |V_i| \leq \lceil y_i(n - |V_0|) \rceil$ and $\sum_{i=0}^m |V_i| = n$, and define $V = \bigcup_{i=0}^m V_i$. Two vertices $x, y \in V_0$ are adjacent in G if and only if they are adjacent in G_0. The set $\bigcup_{i=1}^m V_i$ is independent in G. Finally, two vertices $x \in V_0$ and $y \in V_i, 1 \leq i \leq m$, are adjacent in G if and only if $x \in H_i$.

The degree of a vertex $x \in V_0$ in G is $n \sum_{i: x \in H_i} y_i + O(1)$, the number of edges in G is $n \sum_{i=1}^m y_i |H_i| + O(1)$. These observations lead us to the following linear programming formulation.

Definition 3 *Given a hypergraph $\mathcal{H} = \{H_1, \ldots, H_m\}$ on a set V_0 and a real number $c > 0$, let $A(\mathcal{H}, c) = \min \sum_{i=1}^m |H_i| y_i$, under the restrictions*

$$\sum_{x \in H_i} y_i \leq c \qquad \text{for all } x \in V_0 \;, \tag{3}$$

$$y_i \geq 0 \qquad \text{for all } 1 \leq i \leq m \;, \tag{4}$$

$$\sum_{i=1}^m y_i = 1 \;. \tag{5}$$

Recall that the *fractional matching number* $\nu^*(\mathcal{H})$ of a hypergraph $\mathcal{H} = (V_0, \mathcal{E})$, where $\mathcal{E} = \{H_1, \dots, H_m\}$, is defined as $\nu^*(\mathcal{H}) = \max \sum_{i=1}^m f_i$ under the restrictions

$$\begin{aligned} \sum_{x \in H_i} f_i &\le 1 \qquad \text{for all } x \in V_0 \ , \\ f_i &\ge 0 \qquad \text{for all } 1 \le i \le m \ . \end{aligned}$$

Clearly, $c \ge 1/\nu^*(\mathcal{H})$ is a necessary and sufficient condition for the feasibility of the conditions (3)-(5). Note also that if $\mathcal{H}$ is r-uniform (that is, $|H| = r$ for every $H \in \mathcal{H}$) and $c \ge 1/\nu^*(\mathcal{H})$ then $A(\mathcal{H}, c) = r$.

The following example shows that for every $c > 0$ there exists a k-core $(G_0, \mathcal{H})$ such that the conditions (3)–(5) are feasible for $\mathcal{H}$.

Example 1. Let q be a prime power. Define for every $k \ge 3$ a graph G_0^k and a hypergraph $\mathcal{H}^k$ on a set V_0^k. The set V_0^k consists of $k-2$ copies of two disjoint sets of size q^2+q+1 each, denoted by $A_1, B_1, \dots, A_{k-2}, B_{k-2}$. Elements of A_i are identified with the points of a projective plane $PG(2,q)$, those of B_i with the lines of $PG(2,q)$. For each line l, there is an edge $H \in \mathcal{H}^k$ consisting of all points of l in $A_1, \dots, A_{k-2}$ and all singletons $\{l\}$ in $B_1, \dots, B_{k-2}$. Thus, $\mathcal{H}^k$ has q^2+q+1 edges of size $|H| = (k-2)(q+1)+k-2 = (k-2)(q+2)$. In the graph G_0^k the union $\bigcup_{i=1}^{k-2} B_i$ is independent and each of the sets A_i is independent. Each A_i is completely joined to all A_j, B_j, $i \ne j$, and within (A_i, B_i) a vertex in B_i corresponding to a line l is joined to all vertices in A_i corresponding to the points of $PG(2,q)$ not lying on l. It can rather easily be checked that the pair $(G_0^k, \mathcal{H}^k)$ is a k-core for all $q \ge 2$, while for $q = 1$ (in this case $PG(2,1)$ denotes a triangle) it is a k-pre-core. Assigning $y_i = 1/(q^2+q+1)$ we get a feasible solution of (3)–(5) for every c satisfying $c \ge (q+1)/(q^2+q+1)$. This assignment implies also that $A(\mathcal{H}^k, c) = (k-2)(q+2)$ for these values of c.

We note that the above example is in fact a generalization of Example 3.1 of [5].

Definition 4 *For every* $0 < c \le 1$ *define* $K_k(c) = \inf A(\mathcal{H}, c)$, *where* $\mathcal{H}$ *ranges over all hypergraphs with* $\nu^*(\mathcal{H}) \ge 1/c$ *such that, for a suitable graph* G_0, *the pair* $(G_0, \mathcal{H})$ *is a* k*-core (or equivalently, a* k*-pre-core).*

Claim 1 $K_k(c) \le 2(k-2)(1+1/c)$.

Proof. Let q be a prime satisfying $1/c \le q \le 2/c$. Then Example 1 gives a k-core $(G_0^k, \mathcal{H}^k)$ for which $A(\mathcal{H}^k, c) = (k-2)(q+2) \le (k-2)(2/c+2) = 2(k-2)(1+1/c)$. $\square$

Claim 2 *In the definition of $K_k(c)$, the infimum may be taken over hypergraphs with at most $2(k-2)(1/c+1/c^2)+1$ edges.*

Proof. Let $(G_0, \mathcal{H})$ be a k-core such that $A(\mathcal{H}, c) \leq 2(k-2)(1+1/c)$. By Claim 1, it suffices to consider such k-cores in determining $K_k(c)$. Let $\mathcal{H} = \{H_1, \ldots, H_m\}$. The system of inequalities (3)–(5) defines a convex polytope P in R^m. P is bounded and non-empty, and therefore the function $\sum_{i=1}^m |H_i| y_i$ attains its minimum at a vertex p of P. But every vertex of P is the intersection of at least m hyperplanes of the type: $\sum_{x \in H_i} y_i = c$ for some $x \in V_0$, or $y_i = 0$ for some $1 \leq i \leq m$, or $\sum_{i=1}^m y_i = 1$. Since

$$\sum_{x \in V_0} \sum_{x \in H_i} y_i = \sum_{i=1}^m |H_i| y_i = A(\mathcal{H}, c) \leq 2(k-2)(1+\frac{1}{c}) \ ,$$

at most $2(k-2)(1/c+1/c^2)$ hyperplanes of the first type contain p. Therefore, for at least $m - 2(k-2)(1/c+1/c^2) - 1$ values of i, the equation $y_i = 0$ occurs at p. Denote by $\mathcal{H}_1$ the set of those edges H_i of $\mathcal{H}$ for which $y_i \neq 0$ at p. Clearly, $A(\mathcal{H}_1, c) = A(\mathcal{H}, c)$ and $|\mathcal{H}_1| \leq 2(k-2)(1/c+1/c^2)+1$. Also, one can easily check that $(G_0, \mathcal{H}_1)$ is a k-pre-core. Adding edges to $E(G_0)$, if necessary, we obtain a k-core $(G_1, \mathcal{H}_1)$, thus proving the assertion of the claim. □

The next step is to show that the number of vertices in the hypergraph for the definition of $K_k(c)$ can be bounded from above by a function of c as well. It seems that the corresponding proof of Füredi and Seress cannot be extended for the case of general k, therefore we present a different proof. Let us call a hypergraph $\mathcal{H} = (V, \mathcal{E})$ *separated* if for all $x \neq y \in V$ there exists an edge $H \in \mathcal{E}$ such that $|H \cap \{x, y\}| = 1$. This definition implies that for every pair $x \neq y \in V$, the sets of edges containing x and y, respectively, are different, and therefore the number of vertices in a separated hypergraph can be bounded from above by $|V| \leq 2^{|\mathcal{E}|}$. By identifying vertices, if necessary, we can obtain from every hypergraph $\mathcal{H}$ a separated hypergraph $\mathcal{H}_0$ with the same number of edges and the same fractional matching number: $\nu^*(\mathcal{H}) = \nu^*(\mathcal{H}_0)$. If $V(\mathcal{H}_0) = \{x_1, \ldots, x_p\}$ and x_i is obtained by identifying a_i vertices of $\mathcal{H}$, we say that $\mathcal{H}$ is an $(a_1, \ldots, a_p)$ *blow-up* of $\mathcal{H}_0$. We let

$$\begin{aligned} B(\mathcal{H}_0) \ &= \{(a_1, \ldots, a_p) \in N^p : \text{ the } (a_1, \ldots, a_p) \text{ blow-up of } \mathcal{H}_0 \\ &\qquad \text{forms a } k\text{-core with a suitable graph } G_0\} \ . \end{aligned}$$

For a given $c > 0$, define a family of hypergraphs $\mathbf{H}(c)$ by

$$\begin{aligned} \mathbf{H}(c) = \quad & \{\mathcal{H}_0 : \mathcal{H}_0 \text{ is separated, } \nu^*(\mathcal{H}_0) \geq 1/c, \ V(\mathcal{H}_0) = \{x_1, \ldots, x_p\}, \\ & \mathcal{E}(\mathcal{H}_0) = \{H_1, \ldots, H_m\}, m \leq 2(k-2)(1/c+1/c^2)+1, B(\mathcal{H}_0) \neq \emptyset\} \ . \end{aligned}$$

The above observations imply that $\mathbf{H}(c)$ is non-empty and finite. Using the above definitions and Claim 2, the problem of determining $K_k(c)$ can be rewritten as

$$K_k(c) = \min_{\mathcal{H}_0 \in \mathbf{H}(c)} \inf_{(a_1,\dots,a_p)\in B(\mathcal{H}_0)} \min \sum_{i=1}^{m} \left(\sum_{x_j \in H_i} a_j \right) y_i$$

$$\begin{array}{lrcll} \text{s.t.} & \sum_{x_j \in H_i} y_i & \leq & c, & j = 1, \dots, p\ , \\ & y_i & \geq & 0, & i = 1, \dots, m\ , \\ & \sum_{i=1}^{m} y_i & = & 1\ . & \end{array}$$

In the infimum of the above expression for $K_k(c)$ it suffices to consider only those $(a_1, \dots, a_p)$ that are minimal elements of $B(\mathcal{H}_0)$ in the natural partial order $\prec$ of N^p ($(a_1, \dots, a_p) \prec (a_1', \dots, a_p')$ iff $a_i \leq a_i'$ for every $1 \leq i \leq p$). Since the poset $(N^p, \prec)$ has no infinite antichain, this enables us to restrict the choice of $(a_1, \dots, a_p)$ to a finite set. Hence we obtain the following result.

Claim 3 *The infimum in the definition of $K_k(c)$ is attained.* □

Theorem 1 *The above defined function $K_k(c)$ is monotone nonincreasing, piecewise linear and right-continuous. The points of discontinuity are all rational and contained in a sequence $c_1 > c_2 > \dots \to 0$.*

Proof. We use Lemma 3.6 of [5], which states that for an arbitrary hypergraph $\mathcal{H}$ the function $A(\mathcal{H}, c)$ is continuous, piecewise linear and monotone nonincreasing on the interval $[1/\nu^*(\mathcal{H}), \infty)$. It follows from the proof of Claim 3 that for every fixed $\gamma > 0$ the value of $K_k(c)$ is determined on $[\gamma, 1]$ by a finite number of blow-ups of separated hypergraphs whose number in turn can be bounded from above by a function of γ. Therefore $K_k(c)$ on $[\gamma, 1]$ is the minimum of finitely many functions $A(\mathcal{H}, c)$, and hence $K_k(c)$ is also monotone nonincreasing and piecewise linear. The only possible discontinuities are left-discontinuities at points of the form $1/\nu^*(\mathcal{H})$ for some hypergraph $\mathcal{H}$ from this finite collection; in particular, there are finitely many discontinuities in $[\gamma, 1]$ and they are all rational. □

The following theorem, whose proof is shaped after Lemma 4.2 of [5], shows that every k-saturated graph with few edges is built on a k-core with a small number of vertices. (All logarithms are base two.)

Theorem 2 *Let an integer $k \geq 3$ and a real C be fixed. Then there exists an integer n_0 such that for every $n > n_0$, if $G = (V, E)$ is a k-saturated graph on n vertices with $\leq Cn$ edges, then there exists a subset $V_0 \subset V$ such that*

(a) $|V_0| \leq (2C+1)n/\log\log n$;

(b) $V \setminus V_0$ *is independent in* G;

(c) *For every $x \in V \setminus V_0$ let $H(x) = \{y \in V_0 : (x, y) \in E(G)\}$. Let $\mathcal{H}$ be a hypergraph on V_0 with edge set $\{H(x) : x \in V \setminus V_0\}$ and let $G_0 = G[V_0]$. Then $(G_0, \mathcal{H})$ is a k-core.*

Proof. Let $G = (V, E)$ be a k-saturated graph on n vertices with at most Cn edges. Let $X = \{x \in V : d(x) \geq \log\log n\}$. Then $|X| \log\log n \leq \sum_{x \in V} d(x) \leq 2Cn$, and therefore

$$|X| \leq \frac{2Cn}{\log\log n} . \tag{6}$$

For every $y \in V \setminus X$ let $H(y) = \{x \in X : (x, y) \in E(G)\}$. Clearly, $|H(y)| < \log\log n$ for all $y \in V \setminus X$. Define $Y = \{y \in V \setminus X : \exists z \in V \setminus X$ such that $H(y) \cap H(z)$ does not contain a copy of $K^{k-2}\}$. We claim that the set $V_0 = X \cup Y$ satisfies the requirements of the theorem. Indeed, if $u_1, u_2 \in V \setminus V_0$, then $K^{k-2} \subseteq H(u_1) \cap H(u_2)$, but G is K^k-free, therefore $(u_1, u_2) \notin E(G)$, and hence $V \setminus V_0$ is independent and (b) holds. As observed earlier in this section, in a k-saturated graph (b) implies (c). Therefore, in view of (6) it remains to prove that $|Y| \leq n/\log\log n$, provided that n is sufficiently large.

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a *sunflower* if $H_i \cap H_j = \bigcap_{H \in \mathcal{E}} H$ for all $H_i \neq H_j \in \mathcal{E}$. The sets $H_i \setminus \bigcap_{H \in \mathcal{E}} H$ are called *petals*. Let us prove now that the hypergraph $\{H(y) : y \in Y\}$ does not contain a sunflower with more than $\log\log^2 n + \log\log n$ petals. To show this, suppose that $\{H(y_i) : 0 \leq i \leq \lfloor \log\log^2 n + \log\log n \rfloor\}$ is a sunflower for some $y_i \in Y$, and let $U = \bigcap_i H(y_i)$. By the definition of Y, there exists a vertex $z \in Y$ such that $K^{k-2} \not\subseteq H(y_0) \cap H(z)$. Since all vertices in $V \setminus X$ have degree $< \log\log n$, less than $\log\log^2 n$ of the y_i are of distance at most two from z in $G \setminus X$. Therefore for more than $\log\log n$ of the vertices y_i the distance between y_i and z in $G \setminus X$ is more than two. Hence (recalling that G is k-saturated), there exists a copy of K^{k-2} (we denote it by T_i), contained in X and completely joined to both y_i and z. Clearly, $V(T_i) \not\subseteq U$ for every such y_i (otherwise $K^{k-2} \subseteq H(y_0) \cap H(z)$), and hence there exists a point $x_i \in V(T_i)$ such that $x_i \notin H(y_j)$ for every $j \neq i$. All x_i are different and belong to $H(z)$, thus yielding $|H(z)| > \log\log n$, a contradiction.

Now, by a theorem of Erdős and Rado ([4]) if a hypergraph has more than $r!m^r$ edges of size at most r, then some subhypergraph is a sunflower with $m+1$ petals. This implies that the set system $\{H(y) : y \in Y\}$ has at most $(\log\log n)!(\log\log^2 n + \log\log n)^{\log\log n} < n/\log\log^3 n$ members (here we use the assumption that n is sufficiently large). Finally, for each $H \subseteq X$ we have $|\{y \in Y : H(y) = H\}| \leq \log\log^2 n$, because these y must be of distance at most two in $G \setminus X$ from the vertex $z \in Y$ for which $H(z) \cap H$ does not contain a K^{k-2} . $\square$

Theorem 3 *If $K_k(c)$ is continuous at c, then* $\lim_{n\to\infty} F_k(n, cn)/n = K_k(c)$.

Proof. Let us prove first that $\limsup_{n\to\infty} F_k(n, cn)/n \leq K_k(c)$. Suppose to the contrary that $\limsup_{n\to\infty} F_k(n, cn)/n \geq K_k(c) + \epsilon$ for some positive constant ϵ. Since $K_k(c)$ is continuous at c, there exists a constant $\delta > 0$ such that $K_k(c-\delta) < K_k(c) + \epsilon$. It follows from Claim 3, that there exists a k-core $(G_0, \mathcal{H})$ on a set V_0 and a weight function w on the edges of $\mathcal{H}$ such that w is a feasible solution of (3)-(5) for $c - \delta$ and $A(\mathcal{H}, c-\delta) = \sum_{H\in\mathcal{H}} |H|w(H) = K_k(c-\delta)$. As explained in the beginning of this section, we can use this k-core and weight function to construct, for sufficiently large n, a k-saturated graph G on n vertices with $\Delta(G) \leq (c-\delta)n + O(1)$ and $|E(G)| \leq K_k(c-\delta)n + O(1)$. Therefore for sufficiently large n we have

$$\frac{F_k(n, cn)}{n} \leq \frac{F_k(n, (c-\delta)n + O(1))}{n} \leq K_k(c-\delta) + o(1) < K_k(c) + \epsilon \ ,$$

a contradiction.

Now we prove that $\liminf_{n\to\infty} F_k(n, cn)/n \geq K_k(c)$. Suppose to the contrary that $\liminf_{n\to\infty} F_k(n, cn)/n \leq K_k(c) - \epsilon$ for some constant $\epsilon > 0$. This means that there exists an infinite increasing sequence $\{n_i\}$ such that $F_k(n_i, cn_i)/n_i \leq K_k(c) - \epsilon$; that is, there exists a sequence of graphs $\{G^i\}$ such that every G^i is k-saturated with $|V(G^i)| = n_i$, $\Delta(G^i) \leq cn_i$, $|E(G^i)| \leq (K_k(c) - \epsilon)n_i$. The function $K_k(c)$ is right-continuous (the argument in this direction does not depend on the assumption of continuity at c), and therefore there exists a positive constant δ such that $K_k(c+\delta) > K_k(c) - \epsilon$. For i sufficiently large, according to Theorem 2, there exists a subset $V_0^i \subset V(G^i)$ and a k-core $(G_0^i, \mathcal{H}^i)$ satisfying (a)-(c). Recalling the notation of Theorem 2, we define the weight function w on $\mathcal{E}(\mathcal{H}^i)$ by

$$w(H) = \frac{|\{x \in V(G^i) \setminus V_0^i : H(x) = H\}|}{|V(G^i) \setminus V_0^i|} \ .$$

Clearly, $\sum_{H\in\mathcal{E}(\mathcal{H}^i)} w(H) = 1$. For $z \in V_0^i$, using the fact that $d_{G^i}(z) \leq cn_i$, we obtain

$$\sum_{z\in H} w(H) \leq \frac{cn_i}{|V(G^i) \setminus V_0^i|} \leq c + \delta$$

for sufficiently large i (the last inequality holds because $|V_0^i| = o(n_i)$). This implies that w is a feasible solution of the problem (3)–(5) for $c+\delta$, and then according to the definition of $K_k(c)$ we have $\sum_{H \in \mathcal{E}(\mathcal{H}^i)} |H| w(H) \geq K_k(c+\delta)$. But $|E(G^i)| = |E(G_0^i)| + |V(G^i) \setminus V_0^i| \sum_{H \in \mathcal{E}(\mathcal{H}^i)} w(H)|H|$ and we obtain

$$|E(G^i)| \geq |V(G^i) \setminus V_0^i| K_k(c+\delta) > (K_k(c) - \epsilon) n_i$$

for i sufficiently large, thus obtaining a contradiction. $\square$

The exact determination of $K_k(c)$ seems to be hopeless in general. However, we can determine its asymptotic behaviour for $c \to 0$.

Theorem 4 *Let G be a k-saturated graph on n vertices with $\delta(G) = \delta$ and $\Delta(G) = \Delta$. Then*

$$\delta \geq \frac{(k-2)(n-1)}{\Delta + k - 3} \ .$$

Proof. Let x be a vertex with $d(x) = \delta$. Denote $A = \{y : (x,y) \in E(G)\}$, $B = V \setminus (A \cup \{x\})$, then $|A| = \delta$, $|B| = n - \delta - 1$. Since the addition of the edge (x,z) for $z \in B$ yields a copy of K^k in G, every $z \in B$ has at least $k-2$ neighbours in A, and therefore the number of edges between A and B is at least $(k-2)|B| = (k-2)(n-\delta-1)$. On the other hand, this number of edges does not exceed $|A|(\Delta - 1) = \delta(\Delta - 1)$, and we conclude that $(k-2)(n-\delta-1) \leq \delta(\Delta-1)$, or

$$\delta \geq \frac{(k-2)(n-1)}{\Delta + k - 3} \ . \qquad \square$$

Theorem 5 $\frac{k-2}{c} \leq K_k(c) \leq \frac{k-2+o(1)}{c}$ *(here the $o(1)$ term tends to 0 as c tends to 0).*

Proof. The lower bound can be deduced from Theorem 4 (with the help of the previous theorems and some technicalities), but we give here a direct proof. Let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be a hypergraph which forms a k-core with a suitable graph, and let $y_1, \ldots, y_m$ be a feasible solution of (3)–(5). Then for each $H_i \in \mathcal{H}$ we have

$$|H_i| c \geq \sum_{x \in H_i} \sum_{x \in H_j} y_j = \sum_{j=1}^{m} |H_i \cap H_j| y_j \geq (k-2) \sum_{j=1}^{m} y_j = k-2$$

(using the third condition in the definition of a k-core). It follows that $|H_i| \geq (k-2)/c$, and therefore $A(\mathcal{H}, c) \geq (k-2)/c$, which proves the lower bound.

To prove the upper bound, we return to Example 1 and take a prime q satisfying $1/c \leq q \leq 1/c + (1/c)^{7/12}$. (Such a prime exists for all sufficiently small c, since by a theorem of Huxley [9], there always exists a prime between n and $n + n^{7/12}$ for n sufficiently large.) Then we obtain a k-core $(G_0^k, \mathcal{H}^k)$ for which $A(\mathcal{H}^k, c) = (k-2)(q+2) \leq (k-2)(1/c + (1/c)^{7/12} + 2)$. Therefore it follows from the definition of $K_k(c)$ that

$$K_k(c) \leq A(\mathcal{H}^k, c) \leq \frac{k-2}{c}(1 + c^{5/12} + 2c) = \frac{k-2}{c}(1 + o(1)) . \qquad \square$$

4 Graphs with maximal degree $o(n)$

To construct k-saturated graphs with maximal degree $o(n)$ we use the following k-core (which for $k = 3$ coincides with Example 2.2 of [5]).

Example 2. Let $q \geq k-1$ ($q \geq 3$ for the case $k = 3$) be a prime power. Enumerate the points $p_0, \ldots, p_{q^2+q}$ and the lines $l_0, \ldots, l_{q^2+q}$ of a projective plane $PG(2,q)$ in such a way that $p_{q^2+q} \in l_0, \ldots, l_q$, and $p_{iq+j} \in l_i$ for every $0 \leq i \leq q$, $0 \leq j \leq q-1$. For a point $p = p_{iq+j}$ we call i the *level* of p and j the *place* of p. Deleting the point p_{q^2+q} and the lines $l_0, \ldots, l_q$ we obtain a truncated projective plane of order q. We describe now a set V_0^k and a k-core $(G_0^k, \mathcal{H}^k)$ on it. V_0^k consists of $k-1$ copies of T^k, where T^k is obtained from a truncated projective plane of order q by replacing each point p by $k-2$ points $x^0, \ldots, x^{k-3}$, where we refer to t as the *type* of x^t. Thus, each point of V_0^k has four coordinates: its level $0 \leq i \leq q$, its place $0 \leq j \leq q-1$, its type $0 \leq t \leq k-3$ and the copy $0 \leq s \leq k-2$ of T^k it belongs to. For each line l_r in the truncated plane, there is an edge $H_{r-q} \in \mathcal{E}(\mathcal{H}^k)$, consisting of all points of l_r (in all $k-1$ copies, of all $k-2$ types). The edges of G_0^k are as follows. Within each level, two vertices are joined if and only if they are in distinct copies and have either distinct places or distinct types. In the case $k \geq 4$, a point x in level i is joined to a point y in level i', where $i < i'$, if and only if the type of y succeeds that of x (in Z_{k-2}) and the place of y is one of the $k-2$ successors of the place of x (in Z_q). Then $(G_0^k, \mathcal{H}^k)$ is a k-core. The verification of this assertion is technical and rather tedious. Let us prove, for example, that G_0^k is K^k-free. Suppose to the contrary that $G_0^k[\{v_1, \ldots, v_k\}] \cong K^k$. It is easy to see that if x, y, z form a triangle in G_0^k, then the points x, y, z belong to at most two different levels. Therefore the points $v_1, \ldots, v_k$ belong to at most two different levels i_1 and i_2. Suppose $i_1 < i_2$. Let $v_1, \ldots, v_r$ belong to level i_1 and $v_{r+1}, \ldots, v_k$ belong to level i_2. Since there are $k-1$ copies and two vertices from the same copy and the same level are non-adjacent, we obtain that $1 \leq r \leq k-1$. Therefore the type of each of the points $v_{r+1}, \ldots, v_k$ succeeds the type of each of the points $v_1, \ldots, v_r$. Hence $v_1, \ldots, v_r$ have the same type and also $v_{r+1}, \ldots, v_k$ have the same type. Thus the places of $v_1, \ldots, v_r$ are all distinct, and the same holds for $v_{r+1}, \ldots, v_k$. The place of each v_h, $r+1 \leq h \leq k$, is among the $k-2$ successors of the place of each $v_{h'}$, $1 \leq h' \leq r$. But now one can easily check that r distinct intervals of length $k-2$ in Z_q (recall $q \geq k-1$) have at most $k-1-r$ points in common, and we obtain a contradiction.

Based on the above described k-core, $(G_0^k, \mathcal{H}^k)$, we can build a k-saturated graph G^k as follows. Let $n \geq (k-1)(k-2)(q^2+q)+q^2$. For $1 \leq i \leq q^2$, we choose sets V_i disjoint from each other and from V_0^k such that $\lfloor (n-(k-1)(k-2)(q^2+q))/q^2 \rfloor \leq |V_i| \leq \lceil (n-(k-1)(k-2)(q^2+q))/q^2 \rceil$ and $|V_0^k| + \sum_{i=1}^{q^2} |V_i| = n$. Note that, by our assumption about n, all V_i are non-empty. Define $V(G^k) = V_0^k \cup \bigcup_{i=1}^{q^2} V_i$. Two vertices $x, y \in V_0^k$ are adjacent in G^k if and only if they are adjacent in G_0^k. The set $\bigcup_{i=1}^{q^2} V_i$ is independent in G^k. Finally, $x \in V_0^k$ and $y \in V_i$ are adjacent if and only if $x \in H_i$. Then G^k is k-saturated. If $x \in \bigcup_{i=1}^{q^2} V_i$, then $d(x) = (k-1)(k-2)(q+1)$. If $x \in V_0^k$ then

$$\begin{aligned} d(x) &\leq q(\lfloor (n-(k-1)(k-2)(q^2+q))/q^2 \rfloor + 1) \\ &+ (k-2)((k-2)(q-1)+(k-3)) + (k-1)(k-2)q \\ &\leq n/q + ((k-2)^2+1)q \ . \end{aligned}$$

Finally,

$$\begin{aligned} |E(G^k)| &\leq (k-1)(k-2)(q+1)(n-(k-1)(k-2)(q^2+q)) \\ &+ (k-2)(2k-3)q(k-1)(k-2)(q^2+q)/2 \\ &< (k-1)(k-2)(q+1)n \ . \end{aligned}$$

Theorem 6 *For all* $1/2 < \epsilon < 1$ *and all* $c > 0$

$$\left(\frac{k-2}{2c} - o(1)\right) n^{2-\epsilon} \leq F_k(n, cn^\epsilon) \leq \left(\frac{(k-1)(k-2)}{c} + o(1)\right) n^{2-\epsilon} \ .$$

(Here k is fixed and $o(1)$ tends to zero as n tends to infinity.)

Proof. If G is k-saturated with $\Delta(G) \leq cn^\epsilon$, then according to Theorem 4

$$|E(G)| \geq n\delta(G)/2 \geq \frac{(k-2)(n-1)n}{2(\Delta+k-3)} \geq \frac{(k-2)(n-1)n}{2(cn^\epsilon+k-3)} = \frac{k-2}{2c} n^{2-\epsilon}(1-o(1)) \ ,$$

thus proving the lower bound for $F_k(n, cn^\epsilon)$. To prove the upper bound, choose a constant b such that $b > 2((k-2)^2+1)/c^3$ and let $a = n^{1-\epsilon}/c + bn^{2-3\epsilon}$. Let q be a prime satisfying $a \leq q \leq a + a^{7/12}$ (such a prime exists if n is large enough). Observe that $q = (1/c)n^{1-\epsilon}(1 + o(1))$. Now, Example 2 gives a k-saturated graph, G^k, on n vertices with

$$\begin{aligned} \Delta(G^k) &\leq \frac{n}{q} + ((k-2)^2+1)q \leq \frac{n}{a} + ((k-2)^2+1)q \\ &\leq cn^\epsilon - \frac{bc^2}{2} n^{1-\epsilon} + ((k-2)^2+1)q < cn^\epsilon \end{aligned}$$

for n sufficiently large. Also,

$$|E(G^k)| < (k-1)(k-2)(q+1)n = \frac{(k-1)(k-2)}{c} n^{2-\epsilon}(1+o(1)) . \qquad \square$$

Theorem 7 *For every $k \geq 3$ there exists a k-saturated graph G^k on n vertices with*

$$\Delta(G^k) \leq \left(\frac{(k-2)(2k-3)+1}{\sqrt{(k-1)(k-2)+1}} + o(1) \right) \sqrt{n} .$$

(Here k is fixed and $o(1)$ tends to zero as n tends to infinity.)

Proof. Turning again to Example 2, we denote $a = (n/((k-1)(k-2)+1))^{1/2} - n^{1/3}$ and choose a prime q satisfying $a - n^{1/3} \leq a - a^{7/12} \leq q \leq a$. Then

$$\begin{aligned} & ((k-1)(k-2)+1)q^2 + (k-1)(k-2)q \\ \leq\ & ((k-1)(k-2)+1)\left(\left(\frac{n}{(k-1)(k-2)+1} \right)^{1/2} - n^{1/3} \right)^2 + (k-1)(k-2)n^{1/2} \\ \leq\ & n - \frac{2n^{5/6}}{((k-1)(k-2)+1)^{1/2}} + n^{2/3} + (k-1)(k-2)n^{1/2} \\ \leq\ & n , \end{aligned}$$

and therefore we can substitute q in Example 2. Also,

$$\begin{aligned} r &:= n - ((k-1)(k-2)+1)q^2 - (k-1)(k-2)q \\ &\leq n - ((k-1)(k-2)+1)\left(\frac{n^{1/2}}{((k-1)(k-2)+1)^{1/2}} - 2n^{1/3} \right)^2 \\ &\quad -(k-1)(k-2)\left(\frac{n^{1/2}}{((k-1)(k-2)+1)^{1/2}} - 2n^{1/3} \right) \\ &= O(n^{5/6}) . \end{aligned}$$

Now, as in [5] we use the fact (see, e.g., [8]) that the sizes of the sets V_i can be chosen in such a way that $1 + \lfloor r/q^2 \rfloor \leq |V_i| \leq 1 + \lceil r/q^2 \rceil$, and each vertex from V_0^k has degree at most $((k-2)(2k-3)+1)q + 2\lceil r/q \rceil$. Then

$$\begin{aligned} \Delta(G^k) &\leq ((k-2)(2k-3)+1)q + 2\left\lceil \frac{r}{q} \right\rceil \\ &= \left(\frac{(k-2)(2k-3)+1}{\sqrt{(k-1)(k-2)+1}} + o(1) \right) \sqrt{n} \,. \end{aligned} \qquad \square$$

This result improves significantly an upper bound, given by Hanson and Seyffarth ([7]). Our coefficient is asymptotically $2k$ as $k \to \infty$. Hanson and Seyffarth proved a lower bound of $\sqrt{(k-2)n} - O(1)$ for the lowest possible maximal degree (this can be deduced immediately from our Theorem 4). The existence of a constant c_k such that the lowest possible maximal degree in a k-saturated graph on n vertices is asymptotically $c_k\sqrt{n}$ as $n \to \infty$, conjectured by Hanson and Seyffarth, remains open (but we know that such c_k, if it exists, must satisfy $\sqrt{k-2} \leq c_k \leq 2k$).

5 More on 4-saturated graphs

We begin by noting the following construction of 4-saturated graphs.

Example 3. Let $n \geq 9$ and let $\lfloor \frac{2n-1}{3} \rfloor \leq D \leq n-4$. Let G_0 be the graph $\overline{C^6}$. Let $\mathcal{H}$ be the hypergraph with edges H_1, H_2, H_3 of size four, each obtained by deleting a pair of antipodal vertices of the cycle. Then $(G_0, \mathcal{H})$ is a 4-core. We add $n-6$ vertices, split into non-empty blocks V_1, V_2, V_3, and join every vertex in V_i to each vertex in H_i, $i = 1, 2, 3$. We obtain a 4-saturated graph G on n vertices with $4n - 15$ edges. This graph has $\delta(G) = 4$, and the sizes of the blocks V_i can be chosen so as to have $\Delta(G) = D$, for any D in the indicated range.

The main result of this section is the optimality of this construction. The fact that every 4-saturated graph on n vertices with no conical vertex has at least $4n - o(n)$ edges can be shown as follows. Hajnal [6] proved that if G is k-saturated and has no conical vertex then $\delta(G) \geq 2(k-2)$. (The case $k = 4$ of this is easy to prove.) Thus, every vertex in our graph has degree at least four. By Theorem 2 we may assume that the graph contains an independent set of vertices of size $n - o(n)$. These vertices are incident to at least $4n - o(n)$ edges.

However, in order to replace $o(n)$ by a sharp estimate we have to work harder. The following definition and lemma will be required. A graph $G = (V, E)$ is *4-partite*

4-saturated with respect to the partition V_1, V_2, V_3, V_4 of V, if each V_i is independent in G, no copy of K^4 is contained in G, but adding any legal edge (with endpoints in distinct V_i 's) will create a K^4.

Lemma 1 *If G is 4-partite 4-saturated with respect to the partition V_1, V_2, V_3, V_4 of $V(G)$, where $|V(G)| = n$, and at most one of the V_i 's is empty, then $|E(G)| \geq 2n - 3$.*

Proof. We proceed by induction on n. If one of the V_i 's , say V_4, is empty, then G must be a complete tripartite graph with three non-empty parts V_1, V_2, V_3. The number of edges is minimum when two parts consist of one vertex each, in which case $|E(G)| = 2n - 3$. Thus, we may assume that all parts are non-empty.

We may also assume that $\delta(G)$ is 2 or 3. Indeed, it is easy to check that there cannot be vertices of degree zero or one. If $\delta(G) \geq 4$ then $|E(G)| \geq 2n$.

Let x be a vertex with $d(x) = \delta(G)$. Then the graph $G \setminus \{x\}$ satisfies the assumptions of the lemma, except that it might be possible to add a legal edge to $G \setminus \{x\}$ without creating a K^4. This may happen only if adding the same edge to G creates a K^4 containing x. We distinguish two cases.

Case 1. $d(x) = 2$.

In this case, x does not participate in a K^4 after adding an edge not containing x. Hence we may apply the induction hypothesis to $G \setminus \{x\}$. This yields $|E(G \setminus \{x\})| \geq 2n - 5$, and therefore $|E(G)| \geq 2n - 3$.

Case 2. $d(x) = 3$.

The only way to add an edge e to $G \setminus \{x\}$, which creates a K^4 in G containing x, is for e to join two neighbours of x, say y and z. Moreover, y and z must both be joined in G to the remaining neighbour of x, and hence e is unique. Thus, either $G \setminus \{x\}$ or $(G \setminus \{x\}) + e$ satisfies the assumptions of the lemma. In either case, induction yields $|E(G)| \geq 2n - 3$. □

Theorem 8 *If G is a 4-saturated graph with no conical vertex, $|V(G)| = n$ and $\delta(G) = 4$, then $|E(G)| \geq 4n - 15$.*

Proof. We proceed by induction on n. We may assume that $n \geq 8$, since $\delta(G) = 4$ and so for $n \leq 7$ we have $|E(G)| \geq 2n > 4n - 15$. Furthermore, by Corollary 1 we may assume that $\Delta(G) \leq n - 4$.

The following observation will be useful. Suppose that the vertices x and y of G are *twins*, i.e., they have the same neighbours. It is easy to see that in this case $G \setminus \{x\}$ is also 4-saturated and has no conical vertex. By Hajnal's result, $\delta(G \setminus \{x\}) \geq 4$. Since $d(x) = d(y)$, it follows that $\delta(G \setminus \{x\}) = 4$. Hence we can apply the induction hypothesis to get $|E(G \setminus \{x\})| \geq 4n - 19$ and therefore $|E(G)| \geq 4n - 15$. Thus we may assume that G has no pairs of twins.

For a vertex $x \in V(G)$, we denote by $N(x)$ the open neighbourhood of x and by $N[x]$ the closed neighbourhood of x (i.e., $N[x] = N(x) \cup \{x\}$). We shall make repeated use of the fact that in a 4-saturated graph two vertices x and y are adjacent if and only if $N(x) \cap N(y)$ contains no edge.

Let x be a vertex of degree four, fixed for the rest of the proof. Let $N(x) = \{x_1, x_2, x_3, x_4\}$. For every vertex $y \in V(G) \setminus N[x]$, since y is not adjacent to x, there must be an edge in $N(y) \cap N(x)$. It follows that we can write $V(G) \setminus N[x]$ as the disjoint union

$$V(G) \setminus N[x] = \bigcup_S V_S \ ,$$

where S varies over the subsets of $N(x)$ which contain an edge, and

$$V_S = \{y \in V(G) \setminus N[x] : N(y) \cap N(x) = S\} \ .$$

Each V_S is an independent set, because the neighbourhoods of any two vertices in V_S have an edge in common. Moreover, if $S \cap T$ contains an edge then, for the same reason, $V_S \cup V_T$ is independent. In particular, if $y \in V_{N(x)}$ then $N(y) = N(x)$, contradicting the absence of twins. Hence $V_{N(x)} = \emptyset$. To simplify notation, we write, for example, V_{12} for $V_{\{x_1, x_2\}}$. We also write $V_S \sim V_T$, meaning that every vertex in V_S is adjacent to every vertex in V_T, and $V_S \not\sim V_T$, meaning $V_S \cup V_T$ is independent.

The graph $G[N(x)]$ has the following property: for every vertex x_i there is an edge which does not contain x_i. Indeed, if all edges of $G[N(x)]$ contained x_i, the degree of x_i in G would be at least $n - 3$, contradicting $\Delta(G) \leq n - 4$. The graph $G[N(x)]$ is also triangle-free, because G is K^4-free. It follows that $G[N(x)]$ can be, up to isomorphism, one of three graphs: $2K^2$ (two disjoint edges), P^4 (a path on 4 vertices) or C^4 (a 4-cycle).

Case 1. $G[N(x)] = 2K^2$.

Without loss of generality, we assume that (x_1, x_2) and (x_3, x_4) are the two edges. By the above remarks, the graph $G \setminus N[x]$ is bipartite, with parts

$$\begin{aligned} A &= V_{12} \cup V_{123} \cup V_{124} \ , \\ B &= V_{34} \cup V_{134} \cup V_{234} \ . \end{aligned}$$

If $y \in V_{12}$ then, since y is not adjacent to x_4, $N(y) \cap N(x_4)$ must contain an edge. But $N(y) \cap N(x_4) \subseteq B$, so this is impossible. Thus $V_{12} = \emptyset$, and similarly $V_{34} = \emptyset$. Next, we claim that $G \setminus N[x]$ is a complete bipartite graph on A, B. Indeed if, for example, $y \in V_{123}$ were not adjacent to $z \in V_{134}$, then $N(y) \cap N(z)$ would have to contain an edge, which is not the case, since $N(y) \cap N(z) = \{x_1, x_3\}$. It follows that each of the sets V_{ijk} is a set of twins, and hence $|V_{ijk}| \leq 1$. Since $\Delta(G) \leq n-4$, each V_{ijk} is non-empty. By now, the graph G is fully determined. It has 9 vertices and 22 edges, so $|E(G)| > 4n - 15$.

Case 2. $G[N(x)] = P^4$.

We assume that (x_1, x_2), (x_2, x_3), (x_3, x_4) are the edges. Now, in addition to the sets of Case 1, we also have V_{23}. Arguments similar to those given in Case 1 show that $V_{12} = V_{34} = \emptyset$, and the following relations hold between V_{134} and the remaining sets V_S:

$$V_{134} \not\sim V_{234}, V_{134} \sim V_{123}, V_{134} \sim V_{124}, V_{134} \sim V_{23} \ .$$

Hence V_{134} is a set of twins, and therefore $|V_{134}| \leq 1$. But this means that $d(x_2) \geq n-3$, contradicting $\Delta(G) \leq n-4$.

Case 3. $G[N(x)] = C^4$.

We assume that (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1) are the edges. The sets V_S involved in this case are $V_{12}, V_{23}, V_{34}, V_{41}, V_{234}, V_{134}, V_{124}, V_{123}$. Arguments as above show that the following relations hold: if S is an edge and T is a triple, then $V_S \not\sim V_T$ or $V_S \sim V_T$ according as $S \subseteq T$ or $S \not\subseteq T$; if T and T' are triples, then $V_T \not\sim V_{T'}$ or $V_T \sim V_{T'}$ according as $T \cap T'$ is an edge or not. It follows that for each triple T, the set V_T consists of twins, and therefore $|V_T| \leq 1$.

Let $U = V_{12} \cup V_{23} \cup V_{34} \cup V_{41}$. Then $G[U]$ is 4-partite 4-saturated with respect to this partition. To see this, suppose for example that $y \in V_{12}, z \in V_{23}$ and $(y, z) \notin E(G)$. Then $N(y) \cap N(z)$ must contain an edge. But $N(y) \cap N(z) \subseteq \{x_2\} \cup V_{134} \cup V_{34} \cup V_{41}$, and since x_2 and V_{134} are isolated in the latter, the edge must be found in $V_{34} \cup V_{41}$. The argument is similar if y and z come from other pairs of sets.

Suppose that at most one of the sets V_S in the partition of U is empty. Then it follows from Lemma 1 that $|E(G[U])| \geq 2|U| - 3$. Letting $W = U \cup N[x]$ we obtain $|E(G[W])| \geq 4|W| - 15$. Each of the singletons V_T, $|T| = 3$, if present, adds at least four new edges (three joining it to the vertices in T and at least one to U). Thus, regardless of how many V_T are non-empty, we have $|E(G)| \geq 4n - 15$.

If, on the other hand, two or more of the sets V_S in the partition of U are empty, then those which are non-empty must be singletons and joined to each other. From the fact that $d(x_i) \leq n - 4$, it can be seen that we must have two singletons V_S and $V_{S'}$, where S and S' are disjoint, as well as all singletons V_T, $|T| = 3$. The graph G is fully determined, it has 11 vertices and 31 edges, so $|E(G)| > 4n - 15$. □.

Corollary 2 *There exists an integer n_0 such that if G is a 4-saturated graph with no conical vertex and $|V(G)| = n > n_0$, then $|E(G)| \geq 4n - 15$.*

Proof. By Hajnal's result, $\delta(G) \geq 4$. If $\delta(G) \geq 5$, then Theorem 2 gives at least $5n - o(n)$ edges, which exceeds $4n - 15$ for large n. If $\delta(G) = 4$, we apply Theorem 8. $\square$

Together with Example 3, the corollary establishes that $F_4(n, D) = 4n - 15$ for $n > n_0$ and $\lfloor \frac{2n-1}{3} \rfloor \leq D \leq n - 2$.

Corollary 3 *If G is a 4-saturated graph, $|V(G)| = n$ and $\delta(G) = 4$, then $|E(G)| \geq 4n - 19$. The lower bound is sharp for $n \geq 11$.*

Proof. If G has no conical vertex, we apply Theorem 8. Assume then that x is a conical vertex. Then G has the properties stated if and only if $G \setminus \{x\}$ is a 3-saturated graph, $|V(G \setminus \{x\})| = n - 1$ and $\delta(G \setminus \{x\}) = 3$. By a result of Duffus and Hanson [1], the graph $G \setminus \{x\}$ must have at least $3(n-1) - 15$ edges, and the lower bound is sharp for $n - 1 \geq 10$. Adding the $n - 1$ edges containing x, we get the desired result. $\square$

We recall that Duffus and Hanson ([1]) investigated the function $E(n, k, \delta)$, defined as the minimal number of edges in a k-saturated graph on n vertices having minimal degree δ. For the case $k = \delta = 4$, they showed that $E(n, 4, 4) \leq 4n - 14$ for $n \geq 7$, with equality for $n = 7$. Our Corollary 3 establishes that $E(n, 4, 4) = 4n - 19$ for $n \geq 11$. Example 3 shows that $E(n, 4, 4) \leq 4n - 15$ for $n \geq 9$. The proof of Corollary 3 and the fact that $E(8, 3, 3) = 12$, shown by Duffus and Hanson, imply that $E(9, 4, 4) = 20$.

When D goes below $\lfloor \frac{2n-1}{3} \rfloor$, we do not know the exact behaviour of $F_4(n, D)$, but we do have the following construction.

Example 4. Let V_0 consist of 12 vertices, denoted x_{ij}, $0 \leq i \leq 3$, $1 \leq j \leq 3$. Let $V^i = \{x_{ij} : 1 \leq j \leq 3\}$ for $0 \leq i \leq 3$. Let G_0 be the 4-partite graph on V_0 with partition V^0, V^1, V^2, V^3 obtained by joining each x_{ij} to all vertices of $V^{i+j(\bmod 4)}$. Let $\mathcal{H}$ be the hypergraph on V_0 with edges $V^l \cup \{x_{ij} : i + j \equiv l (\mathrm{mod} 4)\}$ for $0 \leq l \leq 3$. Then $(G_0, \mathcal{H})$ is a 4-pre-core. Assigning a weight of $1/4$ to each edge of $\mathcal{H}$, we get $A(\mathcal{H}, 1/2) = 6$.

This construction gives the estimate $K_4(c) \leq 6$ for $c \geq 1/2$. Below that, we have the estimate $K_4(c) \leq 8$ for $c \geq 3/7$, derived from the case $k = 4, q = 2$ of Example 1.

6 More on k-saturated graphs, $k > 4$

If G is a k -saturated graph on n vertices with no conical vertex, combining Hajnal's bound $\delta(G) \geq 2(k-2)$ and Theorem 2, we see that $|E(G)| \geq 2(k-2)n - o(n)$. In

Section 2 we showed that there exist such graphs with $2(k-2)n - O(1)$ edges and maximal degree $n-2$ and $n-3$. But does there exist a k-saturated graph G on n vertices with $|E(G)| = 2(k-2)n(1+o(1))$ and $\Delta(G) \leq cn$ for some constant $0 < c < 1$? We conjecture that the answer to this question is positive for every $k \geq 4$, that is,

Conjecture 1 *For every $k \geq 4$ there exists a constant $0 < c_k < 1$ such that $K_k(c) = 2(k-2)$ for every $c_k \leq c < 1$.*

Note that the above conjecture fails to be true for $k = 3$, as shown by Füredi and Seress. For $k \geq 4$, the case $q = 1$ of Example 1 yields $K_k(c) \leq 3(k-2)$ for $c \geq 2/3$, but we have better examples for infinitely many values of k, as shown by the following theorem.

Theorem 9 *Conjecture 1 holds true in the following cases (with the indicated values of c_k):*

(i) $k \equiv 0 (\mathrm{mod} 2), \quad c_k = \frac{k-2}{k-1}$;

(ii) $k \equiv 2 (\mathrm{mod} 3), \quad c_k = \frac{2k-4}{2k-1}$;

(iii) $k = 5, \quad c_5 = \frac{3}{5}$;

(iv) $k = 7, \quad c_7 = \frac{2}{3}$;

(v) $k = 17, \quad c_{17} = \frac{6}{7}$.

Proof. In each of the above cases we describe a k-core $(G_0, \mathcal{H})$, yielding the cited result for $K_k(c)$ with a uniform weight assignment. The verification of the required properties is left to the reader.

(i) $G_0 = \overline{C^{2(k-1)}}$, $H \in \mathcal{H}$ are obtained by omitting from $V(G_0)$ a pair of antipodal vertices (this generalizes the 4-core of Example 3);

(ii) $G_0 = \overline{C^{2k-1}}$, $H \in \mathcal{H}$ are obtained by omitting from $V(G_0)$ three equally spaced vertices;

(iii) $G_0 = \overline{P}$ (where P denotes the Petersen graph), $H \in \mathcal{H}$ are the complements of the 4-element independent sets of P;

(iv) G_0 is obtained from the graph $\overline{C^{15}}$ on the vertices $\{0, 1, \ldots, 14\}$ by deleting the edges $(0,7), (1,8), (5,12), (6,13), (10,2), (11,3)$; $H \in \mathcal{H}$ are obtained by omitting from $V(G_0)$ five equally spaced vertices;

(v) G_0 is obtained from $\overline{C^{35}}$ on the vertices $\{0, 1, \ldots, 34\}$ by deleting the edges $(0, 13), (5, 18), (10, 23), (15, 28), (20, 33), (25, 3), (30, 8)$; $H \in \mathcal{H}$ are obtained by omitting from $V(G_0)$ five equally spaced vertices. □

Note that the conjecture remains open for $k \equiv 1, 3 (\mathrm{mod} 6)$, $k \neq 7$. The values $k = 5, 17$ are covered already by (ii), but the corresponding values of c_k are improved in (iii), (v).

Finally, we return to the investigation of $F_k^*(n, n-2)$ and $F_k^*(n, n-3)$. We can now state sharper bounds, using the results of Sections 5 and 6. In view of Proposition 4 it suffices to state them for $F_k^*(n, n-2)$.

Theorem 10 *(a) $F_5^*(n, n-2) \leq 6n - 27$ for $n \geq 11$ and we have equality for $n > n_0$;*
(b) $F_k^*(n, n-2) \leq 2(k-2)n - (2k^2 - 5k + 4)$ for $k \geq 6$, $\quad n \geq 2k + 5$.

Proof. (a) Proposition 3 states that $F_5^*(n, n-2) = F_4(n-2, n-4) + 2n - 4$, while the results of Section 5 assert that $F_4(n, n-2) \leq 4n - 15$ for $n \geq 9$ and $F_4(n, n-2) = 4n - 15$ for sufficiently large n. Combining these two facts we obtain

$$F_5^*(n, n-2) = F_4(n-2, n-4) + 2n - 4 \leq 4(n-2) - 15 + 2n - 4 = 6n - 27$$

for $n \geq 11$, with equality for $n > n_0$.

(b) By induction on $k \geq 6$. For the case $k = 6$, we use the 5-core $(G_0, \mathcal{H})$ from the proof of case (iii) of the previous theorem to build a 5-saturated graph G on $n \geq 15$ vertices with no conical vertex and with $|E(G)| = 6n - 30$, thus obtaining $F_5(n, n-2) \leq 6n - 30$. Then Proposition 3 gives

$$F_6^*(n, n-2) = F_5(n-2, n-4) + 2n - 4 \leq 6(n-2) - 30 + 2n - 4 = 8n - 46$$

for $n \geq 17$. For $k > 6$, using induction and Proposition 4, we obtain

$$\begin{aligned} F_k^*(n, n-2) &= F_{k-1}(n-2, n-4) + 2n - 4 \leq F_{k-1}^*(n-2, n-5) + 2n - 4 \\ &= F_{k-1}^*(n-2, n-4) - 1 + 2n - 4 \\ &\leq 2(k-3)(n-2) - (2(k-1)^2 - 5(k-1) + 4) + 2n - 5 \\ &= 2(k-2)n - (2k^2 - 5k + 4)\ . \qquad \square \end{aligned}$$

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References

[1] D.A. Duffus and D. Hanson, Minimal k-saturated and color critical graphs of prescribed minimum degree. *J. Graph Theory* 10 (1986) 55–67.

[2] P. Erdős, A. Hajnal and J. W. Moon, A problem in graph theory, *Amer. Math. Monthly* 71 (1964) 1107–1110.

[3] P. Erdős and R. Holzman, On maximal triangle-free graphs, *J. Graph Theory* 18 (1994) 585–594.

[4] P. Erdős and R. Rado, Intersection theorems for systems of sets, *J. London Math. Soc.* 35 (1960) 85–90.

[5] Z. Füredi and Á. Seress, Maximal triangle-free graphs with restrictions on the degrees, *J. Graph Theory* 18 (1994) 11–24.

[6] A. Hajnal, A theorem on k-saturated graphs, *Canad. J. Math.* 17 (1965) 720–724.

[7] D. Hanson and K. Seyffarth, k-saturated graphs of prescribed maximum degree, *Congres. Numer.* 42 (1984) 169–182.

[8] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields.* Oxford Mathematical Monographs. Clarendon Press, Oxford (1979).

[9] M. Huxley, The difference between consecutive primes. Proceedings of the Symposium in Pure Mathematics, Vol. 24, American Mathematical Society (1973) 141–146.

[10] L. Lovász and M. D. Plummer, *Matching Theory*, North Holland, Amsterdam (1986).

[11] J. Pach and L. Surányi, Graphs of diameter 2 and linear programming. *Algebraic Methods in Graph Theory*, Colloquia Mathematica Societatis János Bolyai Vol. 25, North Holland (1981) 599–629.