Avoider-Enforcer games played on edge disjoint hypergraphs

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Abstract

We analyze Avoider-Enforcer games played on edge disjoint hypergraphs, providing an analog of the classic and well known game *Box*, due to Chvátal and Erdős. We consider both strict and monotone versions of Avoider-Enforcer games, and for each version we give a sufficient condition to win for each player. We also present applications of our results to several general Avoider-Enforcer games.

1 Introduction

Let p and q be two positive integers, let X be a set and let $\mathcal{F} \subseteq 2^X$ be a family of *target sets*. In a (p,q) Avoider-Enforcer game $\mathcal{F}$ two players, called Avoider and Enforcer, alternately claim p and q previously unclaimed elements of the *board* X per move, respectively. If the number of unclaimed elements is strictly less than p (respectively q) before Avoider's (respectively Enforcer's) move, then he claims all these elements. The definition of the game is complete by stating which player begins the game. The game ends when all the elements of the board have been claimed. Avoider loses the game if by the end of the game he has claimed all the elements of some target set. Otherwise, Avoider wins.

Avoider-Enforcer games are the misère version of the well studied Maker-Breaker games. In a (p,q) Maker-Breaker game $\mathcal{F}$ two players, called Maker and Breaker, alternately claim p and q previously unclaimed elements of the board X per move, respectively. Maker wins if by the end of the game he has claimed all the elements of some $F \in \mathcal{F}$.

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It turns out that Avoider-Enforcer games are much harder to analyze than Maker-Breaker games. One of the main reasons for this difficulty is the lack of bias monotonicity in this type of games (see e.g. [7, 6]). While in a Maker-Breaker game it is never a disadvantage to claim more elements per move (for either of the players), it is sometimes a disadvantage to claim less elements per move in an Avoider-Enforcer game (for either of the players).

In order to overcome this difficulty, Hefetz et al. proposed a bias monotone version for Avoider-Enforcer games [6]. In this version Avoider and Enforcer claim **at least** p and q board elements per move, respectively. Throughout the paper we refer to this set of rules as the *monotone* rules, as opposed to the *strict* rules, and to the games played by each set of rules as *monotone* and *strict* games, respectively. It is worth mentioning that these seemingly minor adjustments in the rules may completely change the game. For example, even in such a natural game as *the connectivity game* – where the board is $E(K_n)$ and Avoider's goal is to avoid having a spanning connected graph – the two versions of the game are essentially different. In the strict rules, Avoider wins the $(1, q)$ game if and only if at the end of the game he has at most $n-2$ edges [7] (i.e. $q \geq \lfloor \frac{n}{2} \rfloor$ or $q \geq \lfloor \frac{n}{2} \rfloor - 1$, depending on the parity of n and the identity of the first player). On the other hand, the asymptotic threshold for the property "Avoider wins the $(1, q)$ connectivity game played on $E(K_n)$ according to the monotone rules" is $\frac{n}{\ln n}$ (see [6, 10]).

One of the main tools in analyzing Avoider-Enforcer games is the following sufficient condition for Avoider's win which was proved by Hefetz et al. [7], and is motivated by the generalized Erdős-Selfridge's sufficient condition for Breaker's win due to Beck (see [1, 3]):

Theorem 1.1 *[Theorem 1.1 [7]] If Avoider is the last player (i.e., the player to make the last move) and*

$$\sum_{F \in \mathcal{F}} (1 + 1/p)^{-|F|} < (1 + 1/p)^{-p}$$

then Avoider wins the (p, q) game $\mathcal{F}$ for every $q \geq 1$.

If Enforcer is the last player then the above sufficient condition can be relaxed to

$$\sum_{F \in \mathcal{F}} (1 + 1/p)^{-|F|} < 1.$$

Note that this sufficient condition holds in both versions of Avoider-Enforcer games (the strict and the monotone rules). One major disadvantage of the condition in Theorem 1.1 is that q does not appear in it. This fact might indicate that, at least for large values of q, the condition is far from being tight.

In this paper, as another step towards understanding Avoider-Enforcer games we examine the misére version of the well known Maker-Breaker game *Box*, defined by Chvátal and Erdős in [2]. The game Box is a (p, q) Maker-Breaker game, where the target sets (referred to as *boxes*) are disjoint. Chvátal and

Erdős used this game as an auxiliary game to provide Breaker with a winning strategy in the biased connectivity game played on $E(K_n)$. They showed that for every $\varepsilon > 0$ and for every $q \geq (1+\varepsilon)n/\ln n$, in a $(1, q)$ Maker-Breaker game played on $E(K_n)$, Breaker has a strategy to isolate a vertex in Maker's graph (provided that n is large enough). Their result implies that Breaker wins various natural games played on $E(K_n)$ such as *the connectivity game*, *the perfect matching game* (where Maker is trying to build a perfect matching) and *the Hamiltonicity game* (where Maker is trying to build a Hamilton cycle), provided that $q \geq (1+\varepsilon)n/\ln n$. It turns out that this result is asymptotically tight for various games as later proved by Gebauer and Szabó [4] (the connectivity game) and by Krivelevich [9] (the perfect matching and the Hamiltonicity games). Since the paper of Chvátal and Erdős [2] is definitely a cornerstone in the theory of Maker-Breaker games, it is natural to investigate the misère version of the game *Box*, referred to as the *misére box game*.

Let p and q be two positive integers. Let $b_1 \leq \ldots \leq b_n$ be a non-decreasing sequence of positive integers and let $\mathcal{F} = \{B_1, \ldots, B_n\}$ be a hypergraph such that $|B_i| = b_i$ for every $1 \leq i \leq n$ and $B_i \cap B_j = \emptyset$ for every $1 \leq i \neq j \leq n$. The misére box game $mBox(b_1, \ldots, b_n, (p, q))$ is just the (p, q) Avoider-Enforcer game $\mathcal{F}$ (played according to the strict rules). If all boxes are of equal size $b_1 = \ldots = b_n = k$ (the uniform game), then we denote this game by $mBox(n \times k, (p, q))$. Analogously, we denote by *monotone-*$mBox(b_1, \ldots, b_n, (p, q))$ and *monotone-*$mBox(n \times k, (p, q))$ the corresponding misére box games played according to the set of monotone rules.

Note that even in this simple game the lack of monotonicity in the strict rules is noticeable. For example, consider the $mBox(2, 2, (p, q))$ game where Avoider is the first player to move. It is easy to verify that the case $(p, q) = (1, 1)$ is Avoider's win, the case $(p, q) = (1, 2)$ Enforcer's win, and the case $(p, q) = (2, 2)$ is Avoider's win again. Therefore, this game is monotone in neither p nor q.

Our main results are the following:

Theorem 1.2 *Let p, q, n be positive integers and let $b_1 \leq \ldots \leq b_n$. If there exists a positive integer k such that $k \leq b_1$ and $gcd(p+q, k) > p$, then Avoider wins the game $mBox(b_1, \ldots, b_n, (p, q))$ as a first or a second player.*

Theorem 1.3 *Let p, q, k be positive integers such that $gcd(p+q, \ell) \leq p$ for every $1 \leq \ell \leq k$. Then there exists an integer $N = N(p, q, k)$ such that for every integer $n \geq N$ and for every $b_1 \leq b_2 \leq \ldots \leq b_n$ such that $b_N \leq k$, Enforcer has a winning strategy in the game $mBox(b_1, \ldots, b_n, (p, q))$ as a first or a second player.*

Combining Theorems 1.2 and 1.3 we get the following necessary and sufficient condition for Enforcer's win in the uniform game, provided that n is large enough:

Corollary 1.4 *Let p, q, k be three integers. Then there exists an integer $N = N(p, q, k)$ such that for every $n \geq N$ Enforcer has a winning strategy in the game mBox$(n \times k, (p, q))$ as a first or a second player if and only if gcd$(p+q, \ell) \leq p$ for every $1 \leq \ell \leq k$.*

Although the above theorems are about Avoider-Enforcer games played on an edge-disjoint hypergraph, the following immediate corollary of Theorem 1.3 helps us to provide a winning strategy for Enforcer on a general (not necessarily edge-disjoint) hypergraph.

Corollary 1.5 *Let p, q, k be positive integers such that gcd$(p+q, \ell) \leq p$ for every $1 \leq \ell \leq k$. Then there exists an integer $N = N(p, q, k)$ such that for every hypergraph $\mathcal{F}$, if $\mathcal{F}$ contains a matching $\mathcal{M} \subseteq \mathcal{F}$ which satisfies:*

(1) $|\mathcal{M}| = N$*;*

(2) $\max\{|F| : F \in \mathcal{M}\} \leq k$*;*

then Enforcer has a winning strategy in the (p, q) Avoider-Enforcer game $\mathcal{F}$ as a first or a second player.

We now state our results about the misére box game played according to the set of monotone rules.

By Theorem 1.1 we have that for every hypergraph $\mathcal{F}$ with all edges of size at least k, if $|\mathcal{F}| < \left(1 + \frac{1}{p}\right)^{k-p}$, Avoider wins the (p, q) game $\mathcal{F}$ as a first or a second player for every q. In the following theorem we improve this result for the case where $q \geq 2kp$ and $\mathcal{F}$ is an edge-disjoint hypergraph, in particular providing a winning criterion depending on q.

Theorem 1.6 *Let p, q, k, n be positive integers such that $k > p$, $q \geq kp$ and $n \leq (q-p)\left(\frac{q}{kp} + 1\right)^{k-p-1}$. Then for every $k \leq b_1 \leq \ldots \leq b_n$, Avoider has a winning strategy in the game monotone-mBox$(b_1, \ldots, b_n, (p, q))$ as a first or a second player.*

Remark: The case $k \leq p$ (in fact $b_1 \leq p$) is trivial. Enforcer may fully claim all the boxes but B_1 and Avoider will lose in his next move. However this might be an illegal move for Enforcer, if there are less than q elements in these boxes. In any case, the second player — whoever that is — makes at most one move and this is a simple case study.

Theorem 1.7 *Let p, q be positive integers. For every positive integer k there exists an integer $N = N(p, q, k)$ such that for every $n \geq N$ and for every $b_1 \leq \ldots \leq b_n$ which satisfy $\frac{1}{N}\sum_{i=1}^{N} b_i \leq k$, Enforcer has a winning strategy in the game monotone-mBox$(b_1, \ldots, b_n, (p, q))$ as a first or a second player.*

The following immediate corollary of Theorem 1.7 can be used to provide a winning strategy for Enforcer on a general hypergraph.

Corollary 1.8 *Let p, q be positive integers. For every positive integer k there exists an integer $N = N(p, q, k)$ such that for every hypergraph $\mathcal{F}$, if $\mathcal{F}$ contains a matching $\mathcal{M} \subseteq \mathcal{F}$ which satisfies:*

(1) $|\mathcal{M}| = N$;

(2) $\frac{1}{N} \sum_{F \in \mathcal{M}} |F| \leq k$;

then Enforcer has a winning strategy in the (p, q) Avoider-Enforcer game $\mathcal{F}$ played according the set of monotone rules as a first or a second player.

Analogously to the Maker-Breaker variant, the misére box game might be useful in analyzing many other Avoider-Enforcer games which are much more involved. Therefore, it can be helpful to estimate the value of N from Theorem 1.7 for the cases $p = 1$ and $q = 1$.

Corollary 1.9 *The following two estimates hold:*

(i) $N(1, q, k) \leq (1 + q)^k$.

(ii) $N(p, 1, k) \leq 1 + e^{\frac{k}{p}}$.

Next, we present two examples for which the misére box game is used as an auxiliary game — one for the strict rules and one for the monotone rules.

Given positive integers p, q and a fixed graph H, the *H-game* is a (p, q) Avoider-Enforcer game where the board is the edge set of a graph G and the winning sets are all the edge-sets of subgraphs of G which are isomorphic to H. In the following corollary we show that given a fixed graph H and a large and dense enough graph G, for appropriate integers p and q, Enforcer has a winning strategy in the H-game played on $E(G)$ according to the strict rules.

Corollary 1.10 *Let p, q, k be positive integers for which $gcd(p + q, \ell) \leq p$ for every $1 \leq \ell \leq k$ and let $\varepsilon > 0$. Then there exists an integer $N = N(p, q, k, \varepsilon)$ such that for every $n \geq N$ the following holds:*
Suppose that

(i) H *is a graph with* $|E(H)| = k$;

(ii) G *is a graph with* $|V(G)| = n \geq N$ *vertices;*

(iii) $|E(G)| \geq \left(1 - \frac{1}{\chi(H)-1} + \varepsilon\right) \frac{n^2}{2}$;

then Enforcer has a winning strategy in the (p, q) H-game, played on $E(G)$ according to the strict rules.

In the following corollary we give a sufficient condition for Avoider to avoid touching a vertex while playing according to the set of monotone rules. This provides Avoider with a winning strategy in various natural games, such as the *the connectivity game*, avoiding a Hamilton cycle game, etc.

Corollary 1.11 *Let G be a graph with $|V(G)| = n$ and $\Delta(G) = d < \frac{n}{2} - 1$. Then for every $q \geq \frac{d}{\ln(n/(2d+2))}$, in the $(1, q)$ Avoider-Enforcer game played on $E(G)$ according to the set of monotone rules, Avoider has a strategy to isolate a vertex in his graph.*

The rest of this paper is organized as follows: In Subsection 1.1 we introduce some notation and terminology that will be used throughout this paper. In Section 2 we prove Theorems 1.2 and 1.3. In Section 3 we prove Theorems 1.6 and 1.7, and Corollary 1.9. In Section 4 we prove Corollaries 1.10 and 1.11. Finally, in Section 5 we present some concluding remarks and open problems.

1.1 Notation

The act of claiming one previously unclaimed element by one of the players is called a *step*. A *move* in the strict (p, q) game is a sequence of p steps by Avoider, or q steps by Enforcer. Similarly, in the monotone game, each move consists of at least p or q steps, respectively. A *round* in the game consists of one move of the first player, followed by one move of the second player. When one of the players claims an element in one of the boxes we say he *touches* that box.

A box B which hasn't been fully claimed yet is called a *surviving* box. A surviving box B which Enforcer hasn't touched yet is called *dangerous*, otherwise it is called *safe*. An unclaimed element in a safe box is called a *safe element*. A step in which Avoider claims a safe element is called a *safe step*. A move in which Avoider makes only safe steps is called a *safe move*, otherwise it is called a *dangerous move*.

The *size* of a box is the number of unclaimed elements remained in that box. We denote the boxes by $B_1, B_2, \ldots, B_n$. For every $1 \leq i \leq n$ we denote the size of the box B_i by b_i, and the average size of the first i boxes by $\bar{b}_i$. After every round we relabel the boxes so that $b_1 \leq \ldots \leq b_{n'}$, where n' is the number of the surviving boxes.

2 The strict rules

2.1 Avoider's win

In this subsection we prove Theorem 1.2.

Throughout this subsection, let p, q, k be three positive integers and let

$$d := gcd(p + q, k).$$

For proving Theorem 1.2 we need the following two lemmas:

Lemma 2.1 *Let n be a positive integer. If $d > p$, then Avoider, as a second player, can avoid making dangerous moves in the game $mBox(n \times k, (p, q))$.*

Proof Assume towards a contradiction that the claim is false, and that in his ith move Avoider cannot make a safe move for the first time. Let $0 \leq s < p$ be the number of safe elements on the board, immediately before Avoider's ith move. Since Avoider's jth move was safe for every $j < i$, it follows that all m boxes which have been touched so far during the game are safe. Therefore, exactly $mk - s$ elements have been claimed in these boxes by both Avoider and Enforcer. Since so far Enforcer has claimed exactly iq elements and Avoider has claimed exactly $(i-1)p$ elements, it follows that $iq + (i-1)p = mk - s$ which implies $i(p+q) - mk = p - s$. Since $d|(p+q)$ and $d|k$, it follows that $d|(p-s)$. Recall that $0 < p - s$, which implies $d \leq p - s$. But $d > p$, a contradiction. This completes the proof. $\square$

Lemma 2.2 *Let n be a positive integer and let $0 \leq r \leq p$ be an integer. Let $b_1 = k - r$ and $b_2 = \ldots = b_n = k$. If $d > p$, then Avoider, as a second player, has a winning strategy in the game $mBox(b_1, \ldots, b_n, (p, q))$.*

Proof Notice first that since $d = gcd(p + q, k)$, it follows that $d|k$. Therefore, $r \leq p < d \leq k$, which implies that b_1 is indeed a positive integer.

Now we describe a strategy for Avoider and then we prove it is a winning strategy. At any point during the game, if Avoider is unable to follow the proposed strategy then he forfeits the game. The strategy of Avoider is as follows:

(i) If there are at least p safe elements on the board, then Avoider claims arbitrarily p such elements.

(ii) Otherwise, let s be the number of safe elements on the board $(0 \leq s < p)$ and let B be an arbitrary dangerous box. Avoider claims all the safe elements and then he claims $p - s$ more elements from B.

We prove by induction on the number of boxes n that this is indeed a winning strategy for Avoider.

Let $n = 1$. In this case, since Enforcer is the first player, he must claim an element in the only box B_1. Hence, Avoider trivially wins this game.

Assume now that $n > 1$. Denote by B the box which is labeled B_1 at the beginning of the game. Notice that by Lemma 2.1, as long as Enforcer does not claim elements in B, Avoider can make safe moves, therefore he can play according to part (i) of the proposed strategy. It follows that if in his ith move Avoider has to play according to part (ii) for the first time, then all m boxes which have been touched so far by either of the players are safe, and B must be one of them. Moreover, there must be at least one dangerous box. Notice that immediately before Avoider's ith move there were $0 \leq s < p$ safe elements on the board. Therefore, we have that $iq + (i-1)p = (m-1)k + (k-r) - s$ which implies $i(p+q) - mk = p - (s+r)$. Since $d|(p+q)$ and $d|k$, it follows that $d|(p-(s+r))$. Recall that $-p \leq p-(s+r) \leq p$ and $d > p$, which implies that $s + r = p$. At his ith move, playing according to part (ii) of the proposed strategy, Avoider claims all s safe elements on the board and r more elements from an arbitrary dangerous box.

After Avoider's ith move, there is exactly one box of size $k - r$ and $n - m - 1$ boxes of size k. Since it is Enforcer's turn to move, it follows by the induction hypothesis that Avoider has a strategy to win this game. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2: First we describe a strategy for Avoider and then prove it is a winning strategy. At any point during the game, if Avoider is unable to follow the proposed strategy then he forfeits the game. Avoider's strategy is divided into the following two stages:

Stage I: This stage begins at the beginning of the game and ends at the first moment during Avoider's move in which all the dangerous boxes are of size at most k and there are no safe elements. At each step of this stage Avoider plays as follows:

(i) If there exists at least one element in a dangerous box of size greater than k prior to this step, then Avoider claims arbitrarily one of these elements.

(ii) Otherwise, Avoider claims an arbitrary safe element.

At the end of Stage I (which may be immediate, if $b_1 = \ldots = b_n = k$), Avoider proceeds to Stage II.

Stage II: Let $0 < r \leq p$ denote the number of Avoider's remaining steps in his move at the moment Stage I has ended. Let B be a box of size exactly k. Avoider claims r elements from B. From this point, Avoider plays according to the strategy proposed in Lemma 2.2.

It is evident that Avoider can follow Stage I of the proposed strategy without forfeiting the game and that if the game ends at this stage (that is, there are no more elements to claim), then Avoider wins the game. It thus suffices to prove that Avoider also wins even if the game ends at Stage II. Assume that in his ith move Avoider proceeds to Stage II. In particular, it means that all the surviving boxes are dangerous and each of them is of size exactly k. Let $0 \leq s < p$ be the number of steps Avoider can play in his ith move according to Stage I of the proposed strategy and let $r := p - s$. At the beginning of Stage

II, Avoider claims r elements of one dangerous box B. Thus, all the boxes but B are of size exactly k and $|B| = k - r$. Therefore, by Lemma 2.2 we conclude that by playing according to the proposed strategy at Stage II, Avoider wins the game. $\square$

2.2 Enforcer's win

In this subsection we prove Theorem 1.3.

Let p, q, k be three integers. Define:

$$N(p,q,k) := \begin{cases} (q+1)(\lceil \frac{q}{p} \rceil + 3)^{k-1} & k \leq p, \\ (2(p+q+1))^k & k > p. \end{cases}$$

Since p, q are fixed throughout the whole game and since k is the only parameter which we change during the proof, we denote $N(k) := N(p,q,k)$.

We prove the following theorem which trivially implies Theorem 1.3:

Theorem 2.3 *Let p, q, n, t be four integers, where $0 \leq t \leq q$, and let $b_1 \leq b_2 \leq \ldots \leq b_n$. Assume that there exists an integer k such that the following properties hold:*

(i) $gcd(p+q, \ell) \leq p$ for every $1 \leq \ell \leq k$;

(ii) $n \geq N(k)$;

(iii) $b_{N(k)} \leq k$;

then, Enforcer has a winning strategy in the game $mBox(b_1, \ldots, b_n, (p,q))$ as a first player even if in his first move he claims t elements.

Remark: Notice that the case $t = 0$ implies that Enforcer wins the game as a second player as well.

Proof of Theorem 2.3: First we describe a strategy for Enforcer and then prove it is a winning strategy.

Enforcer's strategy $\mathcal{S}$: At every step of the game, if there are safe elements, then Enforcer claims one such element arbitrarily. Otherwise, Enforcer claims an element in the largest box (ties are broken arbitrarily).

It is evident that Enforcer can play according to the proposed strategy. It thus suffices to prove that the proposed strategy is indeed a winning strategy for Enforcer.

Before proving it we first establish the following useful lemma:

Lemma 2.4 *Let n, ℓ be positive integers and let $b_1 \leq b_2 \leq \ldots \leq b_n = \ell$ be integers. Assume that Enforcer plays the game $mBox(b_1, \ldots, b_n, (p, q))$ according to the strategy $\mathcal{S}$. Then, as long as the size of the largest box is ℓ, Avoider cannot make ℓ consecutive safe moves.*

Proof Assume that Enforcer is the first player to move (otherwise, after Avoider's first move either Avoider had already lost or Enforcer is the first player in a new game $mBox(b'_1, \ldots, b'_n, (p, q))$, where $b'_i \leq b_i$ for all i. We may also assume that $b'_n = \ell$, otherwise the claim is trivial). By definition, playing according to the strategy $\mathcal{S}$, Enforcer ensures that at any point during the game there exists at most one safe box.

Denote $d := gcd(p + q, \ell)$ and $t := \frac{\ell}{d}$. Suppose that in his ith move Avoider starts a succession of safe moves, all of them in boxes of size ℓ. We prove that he cannot make ℓ such moves. Let $0 < r \leq \ell$ be the number of claimed elements in the safe box (recall that by $\mathcal{S}$ there is at most one safe box) at the beginning of round i. Note that the case $r = 0$ means that no elements have been claimed in the (new) safe box, which means that actually none of the boxes is safe. This case is covered by the case $r = \ell$ (all the elements have been claimed in the (previous) safe box).

Express r as $r = r_1 d + r_2$, where $0 \leq r_1 \leq t - 1$ and $0 < r_2 \leq d$. From number theory we know that there exists an $a \in \mathbb{Z}_\ell$ such that $a(p + q) \equiv (t - r_1)d \pmod{\ell}$. It follows that if Avoider keeps playing safe moves, then after a rounds (at the end of the $(i + a - 1)$st round) there are $(r + a(p + q)) \pmod{\ell} \equiv (r_1 d + r_2 + (t - r_1)d) \pmod{\ell} = r_2$ claimed elements in the safe box. Since $0 < r_2 \leq d \leq p$, and since Avoider has made the last p steps, it follows that all the elements in this box have been claimed by Avoider. Therefore, his $(i + a - 1)$st move is not safe.

Hence, Avoider cannot make $a \leq t \leq \ell$ consecutive safe moves. This completes the proof of the lemma. □

Now, by induction on k we prove that the strategy $\mathcal{S}$ is indeed a winning strategy.

Assume that $k = 1$. Playing according to the strategy $\mathcal{S}$, Enforcer always claims elements from a largest box. Since there are at least $N(1) = q + 1$ boxes of size 1, we conclude that, at some point, Avoider is forced to claim an element from a box of size 1 and then he loses the game.

Now, assume that $k > 1$ and that for every $\ell < k$, if $b_{N(\ell)} \leq \ell$, then $\mathcal{S}$ is indeed a winning strategy for Enforcer in the game $mBox(b_1, \ldots, b_n, (p, q))$ even if he claims t elements in his first move for some $0 \leq t \leq q$.

Notice that it suffices to prove the claim for $n = N(k)$. Indeed, for any larger n, by playing according to $\mathcal{S}$, in Enforcer's first step after $n - N(k)$ boxes are fully claimed, we have that $b_{N(k)} \leq k$ and that Enforcer has $0 \leq t \leq q$ more remaining steps to complete his move.

We prove that by playing according to $\mathcal{S}$, at some point during the game there exists an integer $1 \le \ell < k$ such that at least $N(\ell)$ boxes are still dangerous and $b_{N(\ell)} \le \ell$. Then, by the induction hypothesis we conclude that indeed, by playing according to $\mathcal{S}$, Enforcer wins the game.

Assume towards a contradiction that at any point during the game, for every $1 \le \ell < k$ we have that either there are less than $N(\ell)$ dangerous boxes or $b_{N(\ell)} > \ell$. In particular, it means that at the beginning of the game $b_{N(k-1)} = k$, and that while there are still boxes of size k Avoider cannot claim elements in more than $N(k-1)$ dangerous boxes. Otherwise, we would have $b_{N(k-1)} \le k-1$ (since by $\mathcal{S}$ Enforcer will not touch these boxes, as they are not the largest possible). Moreover, Avoider cannot claim two elements from more than $N(k-2)$ dangerous boxes, otherwise we would have $b_{N(k-2)} \le k-2$. In the same manner we get that Avoider can make at most $\sum_{i=1}^{k-1} N(i)$ steps in dangerous boxes while there are still boxes of size k.

Therefore, by the time that the largest dangerous box is of size at most $k-1$, at least $N(k)-N(k-1)$ boxes of size k are fully claimed. We distinguish between two cases:

Case 1: $k \le p$. In this case, Avoider claims at most $k-1$ safe elements and at least $p-k+1$ dangerous elements per move. All the dangerous boxes he touches become of size smaller than k. Therefore, in every round at most $q+k-1$ elements are claimed in boxes of size k which are not dangerous after that round. It follows that it takes at least $\frac{(N(k)-N(k-1))k}{q+k-1}$ rounds to fully claim all these boxes. Hence, by the time that dangerous boxes of size k no longer exist, the number of dangerous steps Avoider must have played is at least

$$\begin{aligned}
&\frac{(N(k)-N(k-1))k}{q+k-1}(p-k+1) = \\
&\frac{((q+1)(\lceil\frac{q}{p}\rceil+3)^{k-1}-(q+1)(\lceil\frac{q}{p}\rceil+3)^{k-2})k}{q+k-1}(p-k+1) = \\
&(q+1)(\lceil\frac{q}{p}\rceil+3)^{k-2}(\lceil\frac{q}{p}\rceil+3-1)\frac{k(p-k+1)}{q+k-1} \ge \\
&N(k-1)(\frac{q}{p}+2)\frac{p}{q+p-1} > \\
&N(k-1)\frac{q+2p}{q+p} = \\
&N(k-1)(1+\frac{p}{q+p})
\end{aligned}$$

where the first inequality follows from the fact that the quotient reaches minimum value at $k=p$.

On the other hand, we have that

$$\sum_{i=1}^{k-1} N(i) = (q+1)\frac{(\lceil\frac{q}{p}\rceil+3)^{k-1}-1}{(\lceil\frac{q}{p}\rceil+3)-1} <$$

$$(q+1)(\lceil\frac{q}{p}\rceil+3)^{k-2}\frac{\lceil\frac{q}{p}\rceil+3}{\lceil\frac{q}{p}\rceil+2} \le N(k-1)(1+\frac{p}{q+2p})$$

which is clearly a contradiction.

Case 2: $k > p$. Since $N(k) - N(k-1) > N(k)/2$ and since claiming all the elements in the boxes of size k takes at least $\frac{(N(k)-N(k-1))k}{p+q}$ rounds, Lemma 2.4 implies that Avoider must have made at least $\frac{N(k)-N(k-1)}{(p+q)}$ dangerous moves. The following inequality leads to a contradiction:

$$\sum_{i=1}^{k-1} N(i) \leq \frac{2(p+q+1)((2(p+q+1))^{k-1}-1)}{2(p+q+1)-1} <$$

$$< \frac{(2(p+q+1))^k}{2(p+q)} = \frac{N(k)}{2(p+q)} < \frac{N(k)-N(k-1)}{(p+q)}.$$

This completes the proof. □

3 The monotone rules

3.1 Avoider's win

In this section we prove Theorem 1.6. In order to simplify the proof, for every three integers p, q, k, we define:

$$N(p,q,k) := \begin{cases} q & k = p+1, \\ (q-p)\left(\frac{q}{kp}+1\right)^{k-p-1} & k > p+1. \end{cases}$$

In fact, we can use the slightly weaker but simpler general formula $N(p,q,k) = (q-p)\left(\frac{q}{kp}+1\right)^{k-p-1}$ for any $k \geq p+1$, but for the purposes of the proof it will be easier to use the above definition. Since p, q are fixed and k is the only parameter we change during the proof, we denote $N(k) := N(p,q,k)$. We show that Theorem 1.6 holds for every $n \leq N(k)$.

Proof First we make some assumptions to simplify the analysis. We may assume that Avoider is the first player to move since otherwise, after Enforcer's first move, Avoider can just claim all the safe elements (if there are any) and pretend he is the first player in a new game with fewer boxes. We may also assume that $b_1 = \ldots = b_n = k$. Indeed, if some of the boxes are of size larger than k, then in his first move Avoider can reduce the size of each box to exactly k and then pretend he starts a new game.

In addition, throughout the game we assume that whenever Enforcer touches a box, he claims all the elements in this box (in this case we simply say that Enforcer claims the box). If this is not the case, then at the beginning of every move Avoider can claim all the safe elements on the board and then pretend he has just started his move. Finally, if Enforcer claims a box B_i and at the end of his move there is still a dangerous box B_j such that $b_i < b_j$, Avoider in his next move can claim $b_j - b_i$ elements from B_j and pretend that Enforcer has claimed B_j instead. So we may assume that Enforcer only claims boxes of maximal size.

Now, under these assumptions, we present a strategy for Avoider and then prove it is a winning strategy. At any point throughout the game, if Avoider is unable to follow the proposed strategy, then he forfeits the game.

Avoider's strategy $\mathcal{S}$: In every move, Avoider plays as follows:

(i) If there are at most q dangerous boxes left, then Avoider claims all the elements but one in each of the boxes and finishes his move.

(ii) Otherwise, if there are at least p boxes of maximal size, then Avoider chooses p arbitrary such boxes, and from each box he claims one element.

(iii) Otherwise, there are $r < p$ boxes of maximal size. Avoider first claims exactly one element from each of them. Subsequently, Avoider chooses p arbitrary boxes and claims one element from each such box.

We prove by induction on k that by playing according to $\mathcal{S}$, Avoider wins the game. First, assume that $k = p + 1$. In this case, since $n \leq q$, Avoider plays according to (i) of $\mathcal{S}$ and wins after Enforcer's first move. Second, let $k > p + 1$, and assume that the claim is true for all $p + 1 \leq \ell < k$. Assume that Avoider follows $\mathcal{S}$ and that Enforcer follows some fixed strategy (with the above mentioned assumptions).

Denote by *Stage 1* all the rounds in the game in which only boxes of size k are being touched (that is, boxes which were of size k at the beginning of the round). If at any point during Stage 1 the number of dangerous boxes is reduced to q, Avoider plays according to (i) and wins, so assume that this not the case. Thus, at each round during Stage 1 Avoider claims exactly one element in exactly p boxes of size k. Enforcer then responds by claiming at least $\lceil \frac{q}{k} \rceil$ boxes. Hence, Stage 1 lasts at most

$$\left\lfloor \frac{n}{\lceil \frac{q}{k} \rceil + p} \right\rfloor \leq \frac{n}{\frac{q}{k} + p}$$

rounds, in which Avoider reduces the size of at most $p\frac{n}{\frac{q}{k}+p}$ boxes to $k - 1$. In his first move after Stage 1 Avoider touches at most p additional boxes of size k. Then, there are at most

$$p + p\frac{n}{\frac{q}{k} + p} \leq p + \frac{N(k)}{\frac{q}{kp} + 1}$$

dangerous boxes, each of size exactly $k - 1$. It remains to show that $p + \frac{N(k)}{\frac{q}{kp}+1} \leq N(k-1)$, and then, by the induction hypothesis, Avoider wins the game.

Indeed, for $k = p + 2$ we have

$$p + \frac{N(k)}{\frac{q}{kp} + 1} = p + (q - p) = q = N(k - 1).$$

For $k > p+2$, note that

$$\left(\frac{q}{(k-1)p}+1\right)^{k-p-2} - \left(\frac{q}{kp}+1\right)^{k-p-2} \geq$$

$$\left(\frac{q}{(k-1)p}+1\right) - \left(\frac{q}{kp}+1\right) = \frac{q}{k(k-1)p} \geq \frac{1}{k-1},$$

where the last inequality follows from the fact that $q \geq kp$. Using this fact again and the above calculation, we get

$$p + \frac{N(k)}{\frac{q}{kp}+1} = p + (q-p)\left(\frac{q}{kp}+1\right)^{k-p-2} \leq (q-p)\left(\frac{q}{(k-1)p}+1\right)^{k-p-2} = N(k-1)$$

as required. □

3.2 Enforcer's win

In this subsection we prove Theorem 1.7.

Proof of Theorem 1.7: Let p, q, k be three positive integers, define:

$$N(p,q,k) = \begin{cases} q+1 & k \leq p, \\ q+1+\lceil q/k \rceil & k = p+1, \\ \left\lceil \frac{1}{p} N(p,q,k-1) \right\rceil \left(p + \left\lceil \frac{q}{k} \right\rceil\right) & k > p+1. \end{cases}$$

Similarly to subsection 3.1, since p and q are fixed throughout the game and k is the only parameter we change during the proof, we denote $N(k) := N(p,q,k)$.

Let $n \geq N(k)$ be an integer and let $b_1 \leq \ldots \leq b_n$ be integers such that $\overline{b}_{N(k)} \leq k$. We prove that Enforcer has a winning strategy in the game $mBox(b_1, \ldots, b_n, (p,q))$. Clearly, it suffices to deal with the case where Enforcer is the first player, since Avoider's move can only decrease $\overline{b}_{N(k)}$.

First we describe a strategy for Enforcer and then prove it is a winning strategy.

Enforcer's strategy $\mathcal{S}$: At any point during the game if Enforcer is unable to follow the proposed strategy then he forfeits the game. Enforcer plays each move as follows:

(i) If $b_1 \leq p$, then Enforcer fully claims all the boxes but B_1, and finishes his move.

(ii) If there exists an integer $\ell \leq k$ such that at least $N(\ell)$ boxes are still dangerous and $\overline{b}_{N(\ell)} \leq \ell$, then for the minimal such ℓ, Enforcer fully claims all the boxes B_i, for all $i > N(\ell)$. Then, he pretends he starts a new move and proceeds to (iii).

(iii) Let m be the minimal integer for which the largest m boxes contain at least q elements. In his move, Enforcer fully claims the largest m boxes.

Now, we prove that Enforcer can follow the proposed strategy without forfeiting the game and that this is indeed a winning strategy for him.

Assume that Enforcer plays the game against some fixed strategy of Avoider.

If $k \leq p$, then since there are at least $q+1$ boxes, it follows that in his first move, Enforcer can claim all the elements in all the boxes except of B_1. In his next move, Avoider must claim all the elements of B_1 and thus loses the game.

Now we prove the theorem for every $k \geq p+1$ by induction on k. We may assume for simplicity that $n = N(k)$, since otherwise Enforcer fully claims all the boxes $B_{N(k)+1}, \ldots, B_n$ in his first move anyway, according to $\mathcal{S}$.

Assume $k = p+1$. If $b_1 \leq p$, then since there are at least $q+1+\lceil q/k \rceil \geq q+1$ boxes, Enforcer wins the game in a similar way to the case $k \leq p$. Otherwise, all the boxes must be of size $p+1$. Playing according to $\mathcal{S}$, Enforcer fully claims $\lceil \frac{q}{p+1} \rceil$ boxes in his first move, leaving $q+1$ dangerous boxes. Then, after Avoider's first move we must have $b_1 \leq p$ and once again, Enforcer wins the game.

Assume now that $k > p+1$ and that $\mathcal{S}$ is a winning strategy for Enforcer for every $\ell < k$, $n \geq N(\ell)$ and for every $b_1 \leq \ldots \leq b_n$ such that $\bar{b}_{N(\ell)} \leq \ell$. Notice that if at any move during the game Enforcer plays according to part (ii) of $\mathcal{S}$ for some $\ell < k$, then by the induction hypothesis he wins the game. Clearly, if at some move he plays according to part (i) of $\mathcal{S}$, he wins immediately. Therefore, we can assume that Enforcer plays only according to part (iii) of $\mathcal{S}$. For simplicity, denote $N_p(k) := p \lceil N(k)/p \rceil$.

Notice that Enforcer can play entirely in boxes of size at least k for his first $\left\lceil \frac{1}{p} N(k-1) \right\rceil$ moves (fully claiming at most $\lceil \frac{q}{k} \rceil$ of them per move). Otherwise, by the time that all the surviving boxes are of size at most $k-1$, there are more than $N(k) - \lceil \frac{q}{k} \rceil \left\lceil \frac{1}{p} N(k-1) \right\rceil = N_p(k-1) \geq N(k-1)$ of them, which will lead Enforcer to play according to part (ii) of $\mathcal{S}$, in contradiction to our assumption. For the same reason we may conclude that Enforcer has claimed exactly $\lceil \frac{q}{k} \rceil$ boxes per move (and no less).

Let us now examine the board after $\left\lceil \frac{1}{p} N(k-1) \right\rceil$ rounds. There are exactly $\lceil \frac{q}{k} \rceil \left\lceil \frac{1}{p} N(k-1) \right\rceil$ boxes that Enforcer has fully claimed and $N_p(k-1)$ surviving boxes. Avoider must have claimed at least $p \left\lceil \frac{1}{p} N(k-1) \right\rceil = N_p(k-1)$ elements during these rounds. Suppose that t of them were in boxes that were later claimed by Enforcer. Since Enforcer has only touched boxes of size at least k so far, it follows that at the beginning of the game there were at least $\lceil \frac{q}{k} \rceil \left\lceil \frac{1}{p} N(k-1) \right\rceil k + t$ elements in the boxes that Enforcer has fully claimed, and since the total number of elements at the beginning of the game was at most $kN(k)$ it follows that there were at most $kN(k) - k \lceil \frac{q}{k} \rceil \left\lceil \frac{1}{p} N(k-1) \right\rceil - t = kN_p(k-1) - t$ elements in the surviving boxes. Since Avoider has claimed at least $N_p(k-1) - t$ elements in the surviving boxes, there are at most $(k-1)N_p(k-1)$ unclaimed elements in them now, i.e. $\bar{b}_{N_p(k-1)} \leq k-1$ and in particular $\bar{b}_{N(k-1)} \leq k-1$, and so by the induction hypothesis Enforcer wins the game. $\square$

Proof of Corollary 1.9:

For (i), note that by the definition of $N(k)$ in the proof of Theorem 1.7, for $p=1$ we obtain

$$N(k) \leq \left\lceil \frac{1}{p} N(k-1) \right\rceil \left(p + \left\lceil \frac{q}{k} \right\rceil\right) = N(k-1)\left(1 + \left\lceil \frac{q}{k} \right\rceil\right) \leq N(k-1)(1+q).$$

Therefore, $N(k) \leq (1+q)^k$.

For (ii), we estimate $N(k)$ in the following way. Denote by n the number of boxes at the beginning of the game and by $\phi(i)$ the average size of the dangerous boxes just before the beginning of the ith round. In each round Enforcer claims the largest box so he does not increase the average size of the dangerous boxes. Avoider then claims at least p elements in the remaining $n-i$ boxes. Therefore, $\phi(i+1) \leq \phi(i) - \frac{p}{n-i}$ for all $1 < i \leq n$. We know that $\phi(1) \leq k$ and notice that if $\phi(n) < 1$ it means that Enforcer has won the game. We have that:

$\phi(n) \leq \phi(n-1) - p \leq \phi(n-2) - p(\frac{1}{2}+1) \leq \ldots \leq \phi(1) - p(\frac{1}{n-1} + \ldots + 1) \leq k - p\ln(n-1)$.

So if $n > 1 + e^{\frac{k-1}{p}}$ Enforcer wins the game. $\square$

4 Some applications

In this section we prove Corollaries 1.10 and 1.11.

Proof of Corollary 1.10: Let $\mathcal{F}$ be the hypergraph of the game. By the well known Counting Lemma (see e.g, Theorem 2.8 at [8]), we conclude that $\mathcal{F}$ contains a matching of size $\Theta(n^2)$ (since G contains $\Theta\left(n^{|V(H)|}\right)$ distinct copies of H). Now, applying Corollary 1.5 we get that for a large enough n (compared to p, q and $k = |E(H)|$), Enforcer wins the (p,q) game played on $\mathcal{F}$ and hence wins the H-game. $\square$

Proof of Corollary 1.11: Since $\Delta(G) = d$ we can find (greedily) an independent subset $S \subseteq V$ of size $|S| = s \geq n/(d+1)$. Denote by $b_1 \leq \ldots \leq b_s \leq d$ the degrees in G of the vertices in S. Clearly, if $b_1 = 0$ Avoider wins the game no matter how he plays, so assume $1 \leq b_1$. Assume for simplicity that Avoider is the first player to move (since otherwise we can remove the edges that Enforcer has claimed in his first move from G and Avoider can pretend he is the first player in a game on this new graph. Obviously the set S is still independent).

Avoider's strategy goes as follows:

(1) In his first move, Avoider claims all the edges $e \in E(G)$ for which $e \cap S = \emptyset$.

(2) From now on, Avoider pretends he is BoxEnforcer in the game monotone-$mBox(b_1, \ldots, b_s, (q, 1))$, where the boxes are the stars with centers in S, and enforces BoxAvoider (which is the real Enforcer in the original game) to claim all the edges which touch some vertex from S.

It is evident that if Avoider can follow the proposed strategy then he wins the game. It thus suffices to prove that he can win as BoxEnforcer the game monotone-$mBox(b_1, \ldots, b_s, (q, 1))$ for every $b_1 \leq \ldots \leq b_s \leq d$. By using the estimate from Corollary 1.9 and the following calculation:

$$N(q, 1, d) \leq 1 + e^{\frac{d-1}{q}} \leq 2e^{\frac{d}{q}} \leq n/(d+1) \leq s$$

(where the third inequality holds since $q \geq \frac{d}{\ln(n/(2d+2))}$), we get the desired result. □

5 Concluding remarks

Avoider-Enforcer games are more difficult to analyze than Maker-Breaker games and much less is known about them. In this paper we examined Avoider-Enforcer games which are played on edge-disjoint hypergraphs. We also showed that the misére box game is useful when one wants to provide Enforcer with a winning strategy in a game played on a general hypergraph with a large matching. In general, our arguments do not help Avoider to win on a hypergraph which is not edge-disjoint. However, in some cases Avoider can pretend he is playing another game as Enforcer in order to achieve his goals, and then one can use our arguments as we showed in Corollary 1.11.

We believe that it is natural and interesting to investigate Avoider-Enforcer games played on general hypergraphs. As a first step, we suggest to consider Avoider-Enforcer games played on almost disjoint hypergraphs (where every two hyperedges intersect in at most one vertex), or on hypergraphs with bounded maximum degree. Even for these relatively simple hypergraph classes not much is known.

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