Continuous Box game

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Abstract

The classical positional game *Box* was introduced by Chvátal and Erdős in 1978 in their study of the biased connectivity game on the complete graph. Their analysis was subsequently extended by Hamidoune and Las Vergnas. The board of the Box game consists of elements of n disjoint sets (boxes), which might vary in size. The game is played by two players, Maker and Breaker. Maker claims m board elements per move whereas Breaker claims just one. Maker wins this game if and only if he claims all elements of some box by the end of the game.

In this paper we introduce the game CBox, a continuous version of the Box game, where the sizes of the boxes need not be integral and in every move Maker puts a non-negative real weight into each box, such that the weights sum up to the real number m. This new game, while closely related to the original Box game, turns out to be more amenable to analysis – we derive explicit and easy to use criteria for determining the winner in every instance of the game. Consequently, establishing a connection between CBox and Box, we also obtain applicable criteria for the Box game.

1 Introduction

The classical biased $(m : 1)$ positional game Box was first defined by Chvátal and Erdős in [2]. In this game, there are n pairwise disjoint sets, labeled $1, \ldots, n$, that we call *boxes*. For every $1 \leq i \leq n$, we denote the size of box i by a_i. There are two players, Maker and Breaker, who alternately claim previously unclaimed box elements, until all elements of all boxes are

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claimed. In each round, Maker claims m elements, and then Breaker responds by claiming one element. Maker's goal is to claim *a whole box* by the end of the game. We denote this game by $\mathrm{Box}(m; a_1, \ldots, a_n)$.

Note that as soon as Breaker claims an element of some box, Maker can never claim all the elements of that box, and in this sense the box becomes irrelevant for the remainder of the game. Hence, whenever Breaker claims an element of some box, we will say that it is *destroyed*. At any point during the game, a box will be called *surviving* if it is not destroyed to that point. Whenever Maker claims an element of some box, he reduces the number of elements he still has to claim in order to fully claim this box. We will say that Maker is *reducing the size of the box*. Hence, at any point during the game, by the *current remaining size* (or simply *size* for brevity) of a surviving box, we refer to the number of unclaimed elements of this box.

Chvátal and Erdős were mostly interested in the nearly uniform case, where $|a_i - a_j| \leq 1$ for every $1 \leq i < j \leq n$. They proved the following result.

Theorem 1.1 *[2, Theorem 2.1] Let m and $a_1, \ldots, a_n$ be positive integers such that $|a_i - a_j| \leq 1$, for every $1 \leq i < j \leq n$. If Breaker is the first player, then Maker has a winning strategy for $Box(m; a_1, \ldots, a_n)$ if and only if $\sum_{i=1}^{n} a_i \leq f(n,m)$, where $f(n,m)$ is defined by the following recursion, $f(1,m) = 0$ and $f(n,m) = \left\lfloor \frac{n(f(n-1,m)+m)}{n-1} \right\rfloor$ for every $n \geq 2$.*

Chvátal and Erdős used the Box game as an auxiliary game as part of their strategy for Breaker in the so called biased connectivity game on the edge set of the complete graph. In the $(1 : b)$ *connectivity game*, two players, called Maker and Breaker, alternately claim previously unclaimed edges of the complete graph K_n. Maker claims one edge per move, whereas Breaker claims b edges. Maker wins the game if and only if he can claim the edges of a connected spanning subgraph of K_n. In [2] it is proved that the smallest bias b of Breaker for which he has a winning strategy for the connectivity game is of order $n/\ln n$. This result was the first instance of the now widely studied *probabilistic intuition* in positional games (see, e.g., [1]), asserting that the outcome of a positional game played by two clever players is often the same as the outcome of the same game if both players play randomly. In fact, Chvátal and Erdős proved that, for $b \geq (1+\varepsilon)n/\ln n$, Breaker has a strategy to isolate a vertex in Maker's graph. Hence, playing according to this strategy, not only does Breaker win the connectivity game, but many other games as well. These are games whose board is $E(K_n)$ and whose winning sets are subgraphs of K_n with positive minimum degree. This upper bound on Breaker's bias has become a benchmark for several important positional games on K_n and has recently been shown to be asymptotically tight, first for the connectivity game [3], and subsequently for the Hamiltonicity game (Maker wins this game if and only if the graph he builds by the end of the game admits a Hamilton cycle) [5]. The Chvátal-Erdős paper [2] is undoubtedly a cornerstone in the theory of Positional Games.

As it turns out, the proof of one of the directions of Theorem 1.1 given in [2] is wrong (though the statement itself is correct). This was observed and corrected by Hamidoune and Las Vergnas in [4]. In fact, they provided a complete description of winning strategies for both players in the Box game, for any sequence of box sizes a_i (and even for the more general case where Breaker destroys $q > 1$ boxes per move).

To circumvent the relative inconvenience of using the recursively defined criterion of Theorem 1.1, Chvátal and Erdős observed the following estimates

$$(m-1)n\sum_{i=1}^{n-1} 1/i \le f(n,m) \le mn\sum_{i=1}^{n-1} 1/i. \tag{1}$$

Hence, they obtained a slightly weaker but applicable criterion for determining the winner of the Box game in the nearly uniform case. Unfortunately, this ease of application is missing in the solution of the general case (that is, when some boxes vary in size by more than 1) given in [4].

Motivated partly by the implicit nature of the Hamidoune–Las Vergnas criterion, we introduce a continuous variation of the classical Box game, which is more amenable to analysis and has simplifying consequences for the original game. In this game Maker has the possibility to distribute the m elements in each move fractionally among the boxes. Formally, the game *CBox*, the continuous version of Box, is defined as follows. In the game CBox$(m; a_1, \ldots, a_n)$ there are again n boxes labeled $1, \ldots, n$. For every $1 \le i \le n$, box i has positive real size a_i (not necessarily an integral value). The game is played by two players, called CMaker and Breaker. In every move, CMaker puts a non-negative real weight into each box such that the weights sum up to the real number m. Breaker then responds by destroying one box. (We call the second player of the continuous game Breaker, exactly as in the integral game, in order to emphasize that his move of destroying a box is identical in both versions.)

If CMaker (respectively Maker) has a strategy to win CBox (respectively Box) against any strategy of Breaker, then we say this game is *CMaker's win* (respectively *Maker's win*). Otherwise, we say the game is *Breaker's win*. It will be more convenient to have Maker (respectively CMaker) start the game (unlike in [2]), and this is what we assume throughout the paper. Moreover, during the game Box (respectively CBox), we will often dynamically update box sizes, reducing them after each of Maker's (respectively CMaker's) moves.

In the present paper we establish an *explicit* necessary and sufficient condition for CMaker's win in CBox which is easy to verify. In order to state our criterion, we first need to introduce some notation. For every non-negative integer j we denote the *jth harmonic number* by H_j, that is, $H_0 = 0$, and $H_j = \sum_{i=1}^{j} 1/i$, for every $j \ge 1$. For any real numbers $m, a_1, \ldots, a_n$, let

$$A_{i,j} := \max\{a_i, H_j \cdot m\},$$

where $1 \le i \le n$ and $j \in \mathbb{N}$.

Theorem 1.2 *For any positive real numbers $m, a_1, \ldots, a_n$, CBox$(m; a_1, \ldots, a_n)$ is CMaker's win if and only if $\sum_{i=1}^{t}(A_{i,t-i} - H_{t-i} \cdot m) \le tm$ holds for some $1 \le t \le n$.*

It is clear that if the box sizes a_i and the bias m are integral and Breaker has a winning strategy for CBox$(m; a_1, \ldots, a_n)$ while playing against CMaker, then Breaker can win against (the less powerful) Maker in the integral version Box$(m; a_1, \ldots, a_n)$ as well. Thus, any sufficient condition for Breaker's win in CBox applies to Box as well. For the complementary direction we prove the following.

Theorem 1.3 *Let $m, a_1, \ldots, a_n$ be positive integers. If CMaker has a winning strategy for the game CBox$(m; a_1, \ldots, a_n)$, then Maker has a winning strategy for the game Box$(m + 1; a_1, \ldots, a_n)$.*

Theorem 1.3 and the observation preceding it show that the original Box game and its continuous version CBox behave similarly, and enable us to apply results for CBox, like the one in Theorem 1.2, to Box. Small discrepancy between Maker's and CMaker's bias cannot be avoided, as discussed in Section 6, but we can get pretty close. If the box sizes $a_1, \ldots, a_n$ are given, Theorem 1.3 and Theorem 1.2 together give an explicit criterion that narrows down the location of the threshold bias m_0 for Box (at which Maker's win turns into Breaker's win) to just two consecutive integer values.

Corollary 1.4 *Let $m, a_1, \ldots, a_n$ be positive integers, and let $\tilde{m}$ be the smallest real number satisfying $\sum_{i=1}^{t}(A_{i,t-i} - H_{t-i} \cdot \tilde{m}) \leq t\tilde{m}$ for some $1 \leq t \leq n$.*

(i) If $m < \tilde{m}$, then Box$(m; a_1, \ldots, a_n)$ is Breaker's win.

(ii) If $m \geq \tilde{m} + 1$, then Box$(m; a_1, \ldots, a_n)$ is Maker's win.

Finally, we give two different, simple and illustrative arguments, each implying the correctness of the winning strategy of Breaker in Box (the implication of the theorem with the erroneous proof from [2]). The first involves only one simple condition relying on an exponential potential function, a classical tool in the theory of positional games.

Theorem 1.5 *If*

$$\sum_{i=1}^{n} e^{-a_i/m} < \frac{1}{e},$$

then Breaker has a winning strategy for both Box$(m; a_1, \ldots, a_n)$ and CBox$(m; a_1, \ldots, a_n)$.

An even shorter proof of the correctness of Breaker's strategy for the uniform game is given in Section 5. This argument uses practically no calculation and is based on an approach similar to the one in [3].

The rest of this paper is organized as follows. In Subsection 1.1 we make some observations that will be used throughout the paper. In Section 2 we describe explicit winning strategies in CBox for both CMaker and Breaker, use these strategies to devise a polynomial time algorithm for determining the winner of CBox, and prove Theorem 1.2. In Section 3 we discuss the implications of the results of Section 2 on Maker's win in Box and prove Theorem 1.3. In Section 4 we prove Theorem 1.5. As noted above, in Section 5 we consider the uniform Box game. Finally, in Section 6 we present some concluding remarks and an open problem.

1.1 Preliminaries

We make two simple but quite useful observations, to be used later in our arguments.

Observation 1. If the current sequence of box sizes $\mathbf{a} = (a_1, \ldots, a_n)$ is monotone non-decreasing, and it is Maker's (respectively CMaker's) turn to play, then for every resulting sequence $\mathbf{a}' = (a'_1, \ldots, a'_n)$ with $\sum_{i=1}^n a'_i = \sum_{i=1}^n a_i - m$, the vector $\mathbf{a}'' = (a''_1, \ldots, a''_n)$, obtained by sorting the coordinates of $\mathbf{a}'$ in the non-decreasing order, can also be obtained directly from $\mathbf{a}$ by a move of Maker (respectively CMaker). Clearly, from the point of view of the game, there is no difference between $\mathbf{a}'$ and $\mathbf{a}''$. Moreover, Breaker's move (in which he destroys one of the boxes) clearly turns a non-decreasing sequence of box sizes into a non-decreasing sequence. Thus, we can assume that at any point in the course of the game, the sizes of surviving boxes form a non-decreasing sequence.

Observation 2. There is a natural partial ordering of the possible size sequences of boxes in $\mathrm{Box}(m; a_1, \ldots, a_n)$ (respectively $\mathrm{CBox}(m; a_1, \ldots, a_n)$). A collection of boxes with sizes $b_1 \leq b_2 \leq \ldots \leq b_n$ is said to be *smaller or equal* than a collection of boxes with sizes $c_1 \leq c_2 \leq \ldots \leq c_n$, if $b_i \leq c_i$ for every $1 \leq i \leq n$. It is easy to see that if Breaker can win $\mathrm{Box}(m; b_1, \ldots, b_n)$ (respectively $\mathrm{CBox}(m; b_1, \ldots, b_n)$), then he can also win $\mathrm{Box}(m; c_1, \ldots, c_n)$ (respectively $\mathrm{CBox}(m; c_1, \ldots, c_n)$), whenever $(b_1, \ldots, b_n)$ is smaller or equal than $(c_1, \ldots, c_n)$.

For a strategy $\mathcal{M}$ of Maker (respectively CMaker) and a strategy $\mathcal{B}$ of Breaker, the pair $(\mathcal{M}, \mathcal{B})$ stands for the game $\mathrm{Box}(m; a_1, \ldots, a_n)$ (respectively $\mathrm{CBox}(m; a_1, \ldots, a_n)$), where Maker (respectively CMaker) plays according to $\mathcal{M}$ and Breaker plays according to $\mathcal{B}$.

2 Determining the winner of CBox

We will first prove that the outcome of the game $\mathrm{CBox}(m; a_1, \ldots, a_n)$ can be decided essentially by pitting two explicit strategies, that of CMaker and that of Breaker, against each other. More precisely, one should try each of n explicit strategies of CMaker, against one explicit strategy of Breaker. The corresponding strategies are to be described next.

We note that an analogous statement is true, and can be proven similarly, for the integer Box game. However, at the moment we will consider only the continuous game, as this is what we will use in the subsequent sections. In the next section we will take a closer look at the analogous problem for the Box game.

An optimal strategy for Breaker. A very natural (and, as we will see, the most useful) strategy for Breaker is to destroy the most dangerous surviving box (breaking ties arbitrarily), which is clearly a surviving box with the smallest size, in every move. We denote this strategy by `Destroy_smallest` and prove that it is indeed the only strategy Breaker needs to consider.

Proposition 2.1 *Breaker has a winning strategy for CBox$(m; a_1, \ldots, a_n)$ if and only if* `Destroy_smallest` *is a winning strategy of Breaker for CBox$(m; a_1, \ldots, a_n)$.*

Proof We proceed by induction on n. If $n = 1$, there is only one strategy for Breaker, and the assertion of the proposition trivially follows.

Assume then that $n > 1$ and let $\mathcal{B}$ be some winning strategy of Breaker for the game CBox$(m; a_1, \ldots, a_n)$. Let $\mathcal{M}$ be an arbitrary strategy of CMaker. We will prove that strategy `Destroy_smallest` wins against $\mathcal{M}$. Since $\mathcal{M}$ was chosen arbitrarily, this will imply that `Destroy_smallest` is a winning strategy of Breaker for CBox$(m; a_1, \ldots, a_n)$.

Let $b_1, \ldots, b_n$ denote the sizes of the boxes after CMaker's first move. Following $\mathcal{B}$, Breaker responds by destroying a box with label i_0, for some $1 \leq i_0 \leq n$. Since $\mathcal{B}$ was assumed to be a winning strategy, it follows that CBox$(m; b_1, \ldots, b_{i_0-1}, b_{i_0+1}, \ldots, b_n)$ is Breaker's win. The sequence $b_1, \ldots, b_n$ is non-decreasing by Observation 1. Hence, the sequence of box sizes $b_2, \ldots, b_n$ is larger or equal than the sequence of box sizes $b_1, \ldots, b_{i_0-1}, b_{i_0+1}, \ldots, b_n$. Then by Observation 2 Breaker has a winning strategy for CBox$(m; b_2, \ldots, b_n)$ as well. It thus follows by the induction hypothesis that Breaker can also win CBox$(m, b_2, \ldots, b_n)$ using `Destroy_smallest`. Hence, Breaker's strategy `Destroy_smallest` wins CBox$(m, a_1, \ldots, a_n)$ against the strategy $\mathcal{M}$ of CMaker. $\Box$

An optimal family of strategies for CMaker. We define a family of strategies of CMaker for CBox$(m; a_1, \ldots, a_n)$. Informally speaking, CMaker decides before the game starts in how many moves he would like to win, say t, and then aims to play only on the boxes labeled $1, \ldots, t$ (recall that $a_1 \leq \ldots \leq a_n$ is assumed). During the game, CMaker tries to balance the sizes of these t boxes.

For any integer $t \in \{1, \ldots, n\}$, denote by `Balance`(t) the following strategy. In this strategy CMaker considers the t smallest boxes, with sizes $a_1, a_2, \ldots, a_t$ (and does not touch the other boxes at all). Each of CMaker's moves is divided into several stages. First, he puts real weight into the largest box t, until the size of that box is reduced to the size of the second largest box $t-1$. Then, in the second stage, he equally distributes real weight among the two largest boxes until their size is reduced to the size of the third largest box, etc. He continues playing this way until he has distributed all of his weight for that move (a total of m). Whenever one of the t smallest boxes is destroyed by Breaker, CMaker ignores that box for the remainder of the game, applying the described strategy only on the surviving boxes.

The following properties of `Balance`(t) follows immediately from its description.

(i) Throughout the game, CMaker does not touch boxes $t + 1 \leq i \leq n$.

(ii) Let $1 \leq i \neq j \leq t$ be two surviving boxes. If CMaker touches box j but does not touch box i, then the current remaining size of box i is not larger than the current remaining size of box j.

(iii) At any point during the game, all surviving boxes CMaker has touched have exactly the same remaining size.

Theorem 2.2 *Let $m, a_1, \ldots, a_n$ be positive real numbers. There exists a strategy $\mathcal{M}$ of CMaker for CBox$(m; a_1, \ldots, a_n)$ such that $(\mathcal{M},$ `Destroy_smallest`$)$ is CMaker's win if and only if there exists an integer $1 \leq t \leq n$ such that $($`Balance`$(t),$ `Destroy_smallest`$)$ is CMaker's win.*

Proof Let $\mathcal{M}$ be a strategy for CMaker such that $(\mathcal{M}, \texttt{Destroy\_smallest})$ is CMaker's win. Let $\mathcal{G}_1$ be the instance of CBox$(m; a_1, \ldots, a_n)$ in which CMaker follows $\mathcal{M}$ and Breaker follows `Destroy_smallest` and let t denote the number of moves it takes CMaker to win $\mathcal{G}_1$. During this game, we denote the dynamically maintained sizes of all boxes by $b_1, \ldots, b_n$.

Let $\mathcal{G}_2$ be the instance of CBox$(m; a_1, \ldots, a_n)$ in which CMaker follows $\texttt{Balance}(t)$ and Breaker follows `Destroy_smallest`. During this game, we denote the dynamically maintained sizes of surviving boxes by $c_1, \ldots, c_n$.

Note that the entire course of play in both games is uniquely determined. Moreover, since Breaker follows `Destroy_smallest` in both games, immediately after his jth move, the set of surviving boxes is $\{j+1, \ldots, n\}$ in both games.

We will prove that, for every $1 \leq k \leq t$, immediately before CMaker's kth move in $\mathcal{G}_1$ and in $\mathcal{G}_2$, we have

$$\sum_{i=k}^{t} b_i \geq \sum_{i=k}^{t} c_i. \tag{2}$$

Observe that (2) entails the assertion of the theorem. Indeed, it follows by Observation 1 and Breaker's strategy `Destroy_smallest` that in both games immediately before CMaker's tth move the smallest surviving box is labeled t. Since, by assumption, CMaker wins $\mathcal{G}_1$ in his tth move, it follows that he must do so by fully claiming a box in this move. Observation 1 implies that he can do so by claiming box t. It thus follows by (2) (which holds in particular immediately before CMaker's tth move), that CMaker can fully claim box t in his tth move in $\mathcal{G}_2$ as well.

We will prove (2) by induction on the number of moves played. Clearly, (2) holds before the game starts. Assume it holds immediately before CMaker's i_0th move, for some $1 \leq i_0 < t$. Immediately after CMaker's i_0th move in each of the two games, (2) still holds, as according to $\texttt{Balance}(t)$ CMaker distributes his entire bias m among the boxes of $\{i_0, \ldots, t\}$. This decreases the sum on the right hand side of (2) by m. Clearly, the sum on the left hand side of (2) cannot be decreased by more than that.

In his i_0th move, Breaker destroys box i_0 in both games. If $c_{i_0} \geq b_{i_0}$ holds immediately after CMaker's i_0th move, then clearly (2) holds immediately after Breaker's i_0th move. Assume then that $c_{i_0} < b_{i_0}$ holds immediately after CMaker's i_0th move. If the box i_0 was not touched by CMaker in $\mathcal{G}_2$, then his total weight of $i_0 m$ from his first i_0 moves has been distributed among the boxes of $\{i_0+1, \ldots, t\}$, clearly implying (2). Otherwise, since in $\mathcal{G}_2$ CMaker follows $\texttt{Balance}(t)$, it follows by Observation 1 that $c_i = c_{i_0} < b_{i_0} \leq b_i$ holds for every $i_0 \leq i \leq t$. Hence, (2) still holds after Breaker's i_0th move in this case as well. □

The following statement is an immediate corollary of Proposition 2.1 and Theorem 2.2.

Corollary 2.3 *For any positive real numbers $m, a_1, \ldots, a_n$, Breaker has a winning strategy in the game CBox$(m; a_1, \ldots, a_n)$ if and only if the instance* $(\texttt{Balance}(t), \texttt{Destroy\_smallest})$ *is Breaker's win for every $1 \leq t \leq n$. In particular, there exists a polynomial time algorithm to decide which player has winning a strategy for CBox$(m; a_1, \ldots, a_n)$.*

Proof The algorithm consists of playing an instance of ($\texttt{Balance}(t), \texttt{Destroy\_smallest}$) for every $1 \leq t \leq n$. If at least one of these games is won by CMaker, the algorithm returns *CMaker's win*, otherwise it returns *Breaker's win*.

If $\mathrm{CBox}(m; a_1, \ldots, a_n)$ is CMaker's win, then it follows by Theorem 2.2 that one of the games ($\texttt{Balance}(t), \texttt{Destroy\_smallest}$) will be won by CMaker. Hence, the algorithm outputs the correct answer. If on the other hand $\mathrm{CBox}(m; a_1, \ldots, a_n)$ is Breaker's win, then by Proposition 2.1 $\texttt{Destroy\_smallest}$ is a winning strategy of Breaker for this game. Hence, in particular, ($\texttt{Balance}(t), \texttt{Destroy\_smallest}$) will be won by Breaker for every $1 \leq t \leq n$. It follows that the algorithm outputs the correct answer in this case as well. $\square$

A characterization of the winner of CBox. Taking one more step forward in the analysis of CBox, we prove Theorem 1.2 which gives an explicit necessary and sufficient condition for CMaker's win. We will repeatedly use the following simple fact:

$$\sum_{i=0}^{k-1}(H_k - H_i) = \sum_{i=0}^{k-1}\sum_{j=i+1}^{k}\frac{1}{j} = \sum_{j=1}^{k}\sum_{i=0}^{j-1}\frac{1}{j} = k\,.$$

Given Corollary 2.3, Theorem 1.2 is an immediate consequence of the following lemma.

Lemma 2.4 *Let $m, a_1, \ldots, a_n$ be positive real numbers and let $1 \leq k \leq n$ be an integer. The instance* ($\texttt{Balance}(k), \texttt{Destroy\_smallest}$) *of the game $\mathrm{CBox}(m; a_1, \ldots, a_n)$ is CMaker's win if and only if*

$$\sum_{i=1}^{k}(A_{i,k-i} - H_{k-i}\cdot m) \leq km\,. \tag{3}$$

Proof We prove the lemma by induction on k. For $k = 1$ inequality (3) reduces to $A_{1,0} - H_0 \cdot m \leq m$, or equivalently $a_1 \leq m$, which is clearly a necessary and sufficient condition for CMaker's win in ($\texttt{Balance}(1), \texttt{Destroy\_smallest}$).

Assume that the assertion of the lemma holds for some $1 \leq k < n$; we will prove that it holds for $k+1$ as well. Consider the game ($\texttt{Balance}(k+1), \texttt{Destroy\_smallest}$). In his first move, CMaker distributes weight among the boxes with labels in $\{i_0, \ldots, k+1\}$, for some $i_0 \geq 1$. Let $\mathbf{a}' = (a'_1, \ldots, a'_{k+1}, \ldots, a'_n)$, where $a'_1 \leq \ldots \leq a'_{k+1} \leq \ldots \leq a'_n$, denote the sequence of box sizes immediately after CMaker's first move. Since CMaker plays according to the strategy $\texttt{Balance}(k+1)$, it follows that $a'_{i_0} = \ldots = a'_{k+1} = r$, for some $r \geq 0$, and that $a'_i = a_i \leq r$, for every $1 \leq i < i_0$. In his first move Breaker destroys the first box, resulting in the new size sequence $\mathbf{a}'' = (a''_1, \ldots, a''_k, \ldots, a''_{n-1})$, where $a''_i = a'_{i+1}$ for every $1 \leq i \leq k$, and $a''_1 \leq \ldots \leq a''_k$. Continuing the play according to CMaker's strategy $\texttt{Balance}(k+1)$ now amounts to playing according to the strategy $\texttt{Balance}(k)$ on the new sequence $\mathbf{a}''$.

For every $1 \leq i \leq n-1$ and $j \in \mathbb{N}$, let

$$A''_{i,j} = \max\{a''_i, H_j \cdot m\} = \max\{a'_{i+1}, H_j \cdot m\}\,.$$

Assume first that (3) holds for $k+1$, that is,

$$\sum_{i=1}^{k+1}(A_{i,k+1-i} - H_{k+1-i}\cdot m) \leq (k+1)m\,. \tag{4}$$

We will prove that CMaker wins the $(\texttt{Balance}(k+1), \texttt{Destroy\_smallest})$. We distinguish between two cases.

Case 1: $r \leq H_k \cdot m$. Recall that $a'_i \leq r \leq H_k \cdot m$, for every $1 \leq i \leq k+1$. It follows that $A''_{i,k-i} \leq H_k \cdot m$ holds for every $1 \leq i \leq k$. Hence, for the new position $\mathbf{a}''$ we have

$$\sum_{i=1}^{k}(A''_{i,k-i} - H_{k-i}\cdot m) \leq \sum_{i=1}^{k}(H_k\cdot m - H_{k-i}\cdot m) = m\sum_{i=1}^{k}(H_k - H_{k-i}) = km\,.$$

It follows by induction that CMaker wins $(\texttt{Balance}(k), \texttt{Destroy\_smallest})$ played on $\mathbf{a}''$. Hence, $(\texttt{Balance}(k+1), \texttt{Destroy\_smallest})$ is CMaker's win when played on $\mathbf{a}$.

Case 2: $r > H_k \cdot m$. If $i_0 = 1$, that is, if CMaker touches the smallest box in his first move, then $a'_1 = \ldots = a'_{k+1} = r$, implying $a_{k+1} \geq a_1 > r$. Hence $A_{i,k+1-i} = \max\{a_i, H_{k+1-i}\cdot m\} = a_i$ holds for every $1 \leq i \leq k+1$, and $a_1 + \ldots + a_{k+1} = a'_1 + \ldots + a'_{k+1} + m = r(k+1) + m$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{k+1}(A_{i,k+1-i} - H_{k+1-i}\cdot m) &= \sum_{i=1}^{k+1} A_{i,k+1-i} - \sum_{i=1}^{k+1} H_{k+1-i}\cdot m \\
&= \sum_{i=1}^{k+1} a_i - \sum_{i=1}^{k+1} H_{k+1-i}\cdot m \\
&= r(k+1) + m - \sum_{i=1}^{k+1} H_{k+1-i}\cdot m \\
&> H_k\cdot m(k+1) + m - \sum_{i=1}^{k+1} H_{k+1-i}\cdot m \\
&= m\left(1 + \sum_{j=0}^{k}(H_k - H_j)\right) = (k+1)m,
\end{aligned}$$

contrary to (4).

Assume then that $i_0 \geq 2$. Observe that for every $1 \leq i \leq i_0 - 2$ we have $a'_{i+1} = a_{i+1}$, and thus $A''_{i,k-i} = A_{i+1,k-i}$ for these values of i. Moreover, for every $i_0 - 1 \leq i \leq k$, we have $A''_{i,k-i} = \max\{a'_{i+1}, H_{k-i}\cdot m\} = a'_{i+1}$ and $A_{i+1,k-i} = \max\{a_{i+1}, H_{k-i}\cdot m\} = a_{i+1}$. Thus, it follows that $\sum_{i=i_0-1}^{k} A''_{i,k-i} = \sum_{j=i_0}^{k+1} a'_j = \left(\sum_{j=i_0}^{k+1} a_j\right) - m = \left(\sum_{j=i_0}^{k+1} A_{j,k+1-j}\right) - m$. Hence, from (4) we get

$$\begin{aligned}
\sum_{i=1}^{k}(A''_{i,k-i} - H_{k-i}\cdot m) &= \sum_{i=1}^{i_0-2} A''_{i,k-i} + \sum_{i=i_0-1}^{k} A''_{i,k-i} - \sum_{i=1}^{k} H_{k-i}\cdot m \\
&= \sum_{j=2}^{i_0-1} A_{j,k+1-j} + \left(\sum_{j=i_0}^{k+1} A_{j,k+1-j}\right) - m - \sum_{j=2}^{k+1} H_{k+1-j}\cdot m \\
&= \sum_{j=2}^{k+1}(A_{j,k+1-j} - H_{k+1-j}\cdot m) - m \\
&\leq (k+1)m - m = km\,,
\end{aligned}$$

where the last inequality follows by (4), and since $A_{1,k} - H_k\cdot m \geq 0$. It follows by the induction hypothesis that CMaker wins $(\texttt{Balance}(k), \texttt{Destroy\_smallest})$ when playing on $\mathbf{a}''$. Hence, $(\texttt{Balance}(k+1), \texttt{Destroy\_smallest})$ is CMaker's win when played on $\mathbf{a}$.

Next, to prove the "only if" part of the statement, assume that (3) does not hold for $k+1$, that is,

$$\sum_{i=1}^{k+1}(A_{i,k+1-i} - H_{k+1-i}\cdot m) > (k+1)m\,. \tag{5}$$

We will prove that $(\texttt{Balance}(k+1), \texttt{Destroy\_smallest})$ is Breaker's win. We distinguish between the same two cases as before.

Case 1: $r \leq H_k\cdot m$. Note that for $1 \leq i < i_0$ we have $a_i = a'_i \leq r \leq H_k\cdot m$, implying $A_{i,k+1-i} = \max\{a_i, H_{k+1-i}\cdot m\} \leq H_k\cdot m$ for these values of i. Moreover, for every $i_0 \leq i \leq k+1$ we have $a'_i = r$, implying that

$$\begin{aligned}
A_{i,k+1-i} &= \max\{a_i, H_{k+1-i}\cdot m\} \\
&\leq \max\{a'_i, H_{k+1-i}\cdot m\} + (a_i - a'_i) \\
&= \max\{r, H_{k+1-i}\cdot m\} + (a_i - a'_i) \\
&\leq H_k\cdot m + (a_i - a'_i)\,,
\end{aligned}$$

holds for every $i_0 \leq i \leq k+1$.

It thus follows that

$$\begin{aligned}
\sum_{i=1}^{k+1}(A_{i,k+1-i} - H_{k+1-i}\cdot m) &= \sum_{i=1}^{i_0-1}(A_{i,k+1-i} - H_{k+1-i}\cdot m) + \sum_{i=i_0}^{k+1}(A_{i,k+1-i} - H_{k+1-i}\cdot m) \\
&\leq \sum_{i=1}^{i_0-1}(H_k\cdot m - H_{k+1-i}\cdot m) \\
&\quad + \sum_{i=i_0}^{k+1}(H_k\cdot m + (a_i - a'_i)) - H_{k+1-i}\cdot m) \\
&= \sum_{i=1}^{k+1}(H_k\cdot m - H_{k+1-i}\cdot m) + \sum_{i=i_0}^{k+1}(a_i - a'_i) \\
&= km + m = (k+1)m\,,
\end{aligned}$$

contrary to our assumption (5).

Case 2: $r > H_k \cdot m$. For every $i_0 \leq i \leq k+1$ we have $a'_i = r > H_k \cdot m$. Assume first that $a_1 > H_k \cdot m$. It follows that $a'_i = a_i \geq a_1 > H_k \cdot m$ holds for every $1 \leq i < i_0$. Therefore, for every $1 \leq i \leq k$ we have $A''_{i,k-i} = \max\{a'_{i+1}, H_{k-i} \cdot m\} > H_k \cdot m$. It follows that

$$\sum_{i=1}^{k}(A''_{i,k-i} - H_{k-i} \cdot m) > \sum_{i=1}^{k}(H_k \cdot m - H_{k-i} \cdot m) = km\,,$$

and thus Breaker wins by the induction hypothesis.

If, on the other hand, $a_1 \leq H_k \cdot m$, then $A_{1,k} = \max\{a_1, H_k \cdot m\} = H_k \cdot m$, and thus $A_{1,k} - H_k \cdot m = 0$. Since $a'_{i+1} = a_{i+1}$ holds for every $1 \leq i \leq i_0 - 2$, it follows that $A''_{i,k-i} = \max\{a'_{i+1}, H_{k-i} \cdot m\} = \max\{a_{i+1}, H_{k-i} \cdot m\} = A_{i+1,k-i}$ for these values of i. Moreover, for every $i_0 - 1 \leq i \leq k$, we have $a'_{i+1} = r > H_k \cdot m$, and therefore $A''_{i,k-i} = \max\{a'_{i+1}, H_{k-i} \cdot m\} = a'_{i+1}$. Since $a_{i+1} > a'_{i+1} = r > H_k \cdot m$ holds for every $i_0 - 1 \leq i \leq k$, it follows that $A_{i+1,k-i} = \max\{a_{i+1}, H_{k-i} \cdot m\} = a_{i+1} = A''_{i,k-i} + (a_{i+1} - a'_{i+1})$ for these values of i. Altogether, we get

$$\begin{aligned}
\sum_{i=1}^{k}(A''_{i,k-i} - H_{k-i} \cdot m) &= \sum_{i=1}^{i_0-2}(A''_{i,k-i} - H_{k-i} \cdot m) + \sum_{i=i_0-1}^{k}(A''_{i,k-i} - H_{k-i} \cdot m) \\
&= \sum_{j=2}^{i_0-1}(A_{j,k+1-j} - H_{k+1-j} \cdot m) \\
&\quad + \sum_{j=i_0}^{k+1}(A_{j,k+1-j} - (a_j - a'_j) - H_{k+1-j} \cdot m) \\
&= 0 + \sum_{j=2}^{k+1}(A_{j,k+1-j} - H_{k+1-j} \cdot m) - \sum_{j=i_0}^{k+1}(a_j - a'_j) \\
&= \sum_{j=1}^{k+1}(A_{j,k+1-j} - H_{k+1-j} \cdot m) - m \\
&> (k+1)m - m = km\,,
\end{aligned}$$

where the last inequality follows by our assumption (5).

Hence, again, Breaker wins by the induction hypothesis. □

3 Maker's win in Box

In this section, we give sufficient conditions for Maker's win and for Breaker's win in the Box game, using our results for CBox from the previous section.

As already observed by Hamidoune and Las Vergnas in [4], the strategy `Balance`(k) can be used by Maker in the Box game in essentially the same way it is used by CMaker in CBox. However, due to the discrete nature of Box, Maker might not be able to fully balance the sizes of the boxes he touches. Two boxes that Maker touches might differ in size, though by at most 1. Therefore, the first two properties of `Balance`(k), as a strategy for CBox, which were derived immediately before stating Theorem 2.2, apply for `Balance`(k), as a strategy for Box as well. The third property now reads as follows.

(iii)' At any point during the game, any two surviving boxes Maker has touched differ in size by at most 1.

It turns out that an analogous statement to Corollary 2.3 for the Box game holds as well. The proof of the analogue of Proposition 2.1 goes through word by word. The proof of the analogue of Theorem 2.2 is very similar. The main difference occurs in the case $c_{i_0} < b_{i_0}$ at the end of the proof of Theorem 2.2. Namely, in Box$(m; a_1, \ldots, a_n)$, both c_{i_0} and b_{i_0} are integers, and thus $c_{i_0} \leq b_{i_0} - 1$ holds in this case. Since Maker follows `Balance`(t), it follows that $c_i \leq c_{i_0} + 1 \leq b_{i_0} \leq b_i$ holds for every $i \geq i_0$. Hence, (2) still holds after Breaker's i_0th move in this game as well.

Corollary 3.1 *For any positive integers $m, a_1, \ldots, a_n$, Breaker has a winning strategy in the game Box$(m; a_1, \ldots, a_n)$ if and only if the instance* (`Balance`(t), `Destroy_smallest`) *is Breaker's win for every $1 \leq t \leq n$. In particular, there exists a polynomial time algorithm to decide which player has winning a strategy for Box$(m; a_1, \ldots, a_n)$.*

We are now ready to prove the main result of this section.

Proof (of Theorem 1.3) We consider two games, $\mathcal{G}_1$ and $\mathcal{G}_2$, played at the same time. The game $\mathcal{G}_1$ is an instance of Box$(m + 1; a_1, \ldots, a_n)$, in which Breaker follows the strategy `Destroy_smallest`, and Maker follows a strategy that will be described in the course of the proof. We will prove that Maker wins $\mathcal{G}_1$. This will conclude the proof of Theorem 1.3 by applying the aforementioned Box game analogue of Proposition 2.1.

The game $\mathcal{G}_2$ is an instance of CBox$(m; a_1, \ldots, a_n)$ in which Breaker follows the strategy `Destroy_smallest` and CMaker follows the strategy `Balance`(t), where t is chosen such that CMaker wins the game. The existence of such a t is guaranteed by Corollary 2.3.

At any point during the game $\mathcal{G}_1$, let b_i denote the (dynamically changing) size of box i, for every $1 \leq i \leq n$. Similarly, at any point during the game $\mathcal{G}_2$, let c_i denote the (dynamically changing) size of box i, for every $1 \leq i \leq n$.

During the course of the game $\mathcal{G}_2$ let i_0 be the smallest box index i for which $c_i < a_i$ and box i is surviving, that is, the smallest integer i such that box i has already been touched by CMaker but has not yet been destroyed by Breaker. Note that, by definition of the strategy `Balance`(t), it follows that $c_i = c_{i_0}$ holds, for every $i \geq i_0$.

Maker's strategy for $\mathcal{G}_1$, which will be described below, will require Maker to claim strictly less than $m + 1$ elements in some moves. This is legitimate due to the bias monotonicity of Maker-Breaker games. Maker's goal in $\mathcal{G}_1$ will be to ensure that, for every $1 \leq j \leq t - 1$, the following two conditions hold immediately after his jth move.

(i) $\lfloor c_i \rfloor \leq b_i \leq \lceil c_i \rceil$, for every $j \leq i \leq t$.

(*ii*) $\sum_{i=i_0}^{t} b_i \leq \sum_{i=i_0}^{t} c_i$.

Note that if condition (*i*) holds immediately after Maker's $(t-1)$th move, then he wins $\mathcal{G}_1$ in his tth move. Indeed, it follows by Breaker's strategy `Destroy_smallest`, that, in both games, Breaker destroys box i in his ith move, for every $1 \leq i \leq t-1$. Since, by assumption, CMaker follows `Balance`(t) and wins $\mathcal{G}_2$ in his tth move, it follows that $c_t \leq m$ holds immediately after Breaker's $(t-1)$th move. It then follows, by condition (*i*) (which holds in particular immediately before Maker's tth move in $\mathcal{G}_1$), that $b_t \leq m+1$. Hence, Maker can win $\mathcal{G}_1$ in his tth move by fully claiming box t.

First, we prove that, playing $\mathcal{G}_1$, Maker can ensure that conditions (*i*) and (*ii*) will hold immediately after his first move. In his first move in $\mathcal{G}_2$, CMaker reduces the sizes of the boxes labeled i, for every $i_0 \leq i \leq t$, to some real value r. In his first move in $\mathcal{G}_1$, Maker will claim exactly m elements, while ensuring condition (*i*). Since $\sum_{i=i_0}^{t}(a_i - r) = m$, it follows that $\sum_{i=i_0}^{t}(a_i - \lceil r \rceil) \leq m$ and $\sum_{i=i_0}^{t}(a_i - \lfloor r \rfloor) \geq m$, and thus condition (*i*) can indeed be guaranteed. Moreover, clearly $\sum_{i=i_0}^{t}(a_i - b_i) = m = \sum_{i=i_0}^{t}(a_i - c_i)$, implying that $\sum_{i=i_0}^{t} b_i = \sum_{i=i_0}^{t} c_i$. Hence, condition (*ii*) holds as well.

Next, assume that conditions (*i*) and (*ii*) hold immediately after Maker's jth move, for some $1 \leq j \leq t-2$. We will prove that Maker can make sure that both conditions will hold immediately after his $(j+1)$st move as well.

Recall that i_0 denotes the index of the smallest box that CMaker touched in his jth move in $\mathcal{G}_2$, and let r be the size of all boxes touched by CMaker to that point. Since Breaker follows `Destroy_smallest` in both $\mathcal{G}_1$ and $\mathcal{G}_2$, in both games he destroys box j in his jth move. Clearly, Breaker's move does not violate condition (*i*). On the other hand, instead of condition (*ii*), we are now guaranteed to have

$$\sum_{i=i_0}^{t} b_i \leq \sum_{i=i_0}^{t} c_i + 1\,, \tag{6}$$

since $b_j \geq \lfloor c_j \rfloor > c_j - 1$ holds by condition (*i*). Note that if $i_0 = j$, then both sums in inequality (6) are extended over $\{i_0+1, \ldots, t\}$.

In his $(j+1)$st move in $\mathcal{G}_2$, CMaker reduces to $c'_i = r' < r$ the size of the boxes labeled i for every $i'_0 \leq i \leq t$, where either $i'_0 \leq i_0$ or $i_0 = j$ and $i'_0 = j+1$. First, assume the former. Clearly $\sum_{i=i'_0}^{t}(c_i - r') = m$. In his $(j+1)$st move in $\mathcal{G}_1$, Maker will first distribute some of his elements to ensure that the new intermediate sizes b'_i satisfy $b'_i \leq \lceil r' \rceil$, for every $i'_0 \leq i \leq t$. For every $i'_0 \leq i \leq i_0 - 1$, Maker claims $a_i - \lceil r' \rceil$ elements of box i. This is at most $a_i - r'$, which is the weight that CMaker put in the same box in his $(j+1)$st move in $\mathcal{G}_2$. If $b_{i_0} < \lceil r' \rceil$, then, as $b_{i_0} \geq \lfloor r \rfloor$ holds by condition (*i*), it follows that $\lceil r' \rceil = \lceil r \rceil$, and thus $b_i \leq \lceil r' \rceil$ for every $i \geq i_0$ already before Maker's $(j+1)$st move. Otherwise, $b_i \geq \lceil r' \rceil$ for every $i_0 \leq i \leq t$. In this case we have

$$\begin{aligned}
\sum_{i=i_0}^{t}(b_i - \lceil r' \rceil) &= \sum_{i=i_0}^{t}(b_i - c_i) + \sum_{i=i_0}^{t}(c_i - \lceil r' \rceil) \\
&\leq 1 + \sum_{i=i_0}^{t}(c_i - \lceil r' \rceil) \leq m + 1\,,
\end{aligned}$$

where the first inequality follows from (6). Hence, we conclude that, in his $(j+1)$st move, Maker can indeed claim $p \leq m+1$ box elements to ensure that $b'_i \leq \lceil r' \rceil$ will hold for every $i'_0 \leq i \leq t$.

Let q be the smallest non-negative integer such that

$$q \cdot \lfloor r' \rfloor + (t - i'_0 + 1 - q)\lceil r' \rceil \leq (t - i'_0 + 1)r' \,. \tag{7}$$

Since substituting $q = t - i'_0 + 1$ validates the above inequality, it follows that the minimum value of q which satisfies inequality (7) is at most $t - i'_0 + 1$. Maker completes his $(j+1)$st move in $\mathcal{G}_1$ by claiming one additional element from the q boxes $i'_0, \ldots, i'_0 + q - 1$. Clearly, condition (i) is still satisfied. Note that, for every $1 \leq i \leq i'_0 - 1$, the current remaining size of box i in $\mathcal{G}_1$ and the current remaining size of box i in $\mathcal{G}_2$ is a_i (note that $a_i \leq \lfloor r' \rfloor$ since $a_i \leq r'$ is an integer). Moreover, in $\mathcal{G}_1$, the current remaining size of box i is $\lfloor r' \rfloor$, for every $i'_0 \leq i \leq i'_0 + q - 1$, and $\lceil r' \rceil$, for every $i'_0 + q \leq i \leq t$. Similarly, in $\mathcal{G}_2$, the current remaining size of box i is r' for every $i'_0 \leq i \leq t$. Therefore, inequality (7) guarantees that condition (ii) is satisfied as well.

It remains to prove that $p + q \leq m + 1$, thus rendering Maker's move valid. The choice of q implies that it is the smallest non-negative integer such that

$$\sum_{i=i'_0}^{t} b_i - p - q \leq \sum_{i=i'_0}^{t} c_i - m \,. \tag{8}$$

It follows by (6) and by the definitions of i_0 and i'_0 that, immediately before Maker's $(j+1)$st move in $\mathcal{G}_1$ and CMaker's $(j+1)$st move in $\mathcal{G}_2$, we have

$$\begin{aligned}
\sum_{i=i'_0}^{t} b_i &= \sum_{i=i'_0}^{i_0-1} b_i + \sum_{i=i_0}^{t} b_i \\
&= \sum_{i=i'_0}^{i_0-1} a_i + \sum_{i=i_0}^{t} b_i \\
&\leq \sum_{i=i'_0}^{i_0-1} c_i + \sum_{i=i_0}^{t} c_i + 1 \\
&= \sum_{i=i'_0}^{t} c_i + 1 \,.
\end{aligned}$$

Hence (8) is satisfied when substituting $q = m + 1 - p$. It follows by the minimality of q that $p + q \leq m + 1$ as required.

The latter case, where $i_0 = j$ and $i'_0 = j + 1$ is somewhat simpler and can be handled similarly, using the remark following inequality (6). We omit the straightforward details. $\square$

4 A simple criterion for Breaker's win

Proof (of Theorem 1.5) We will prove the statement for CBox; the statement for Box will readily follow.

Breaker employs the strategy `Destroy_smallest`, that is, in every move he destroys a box i whose size is minimal among the set S of all surviving boxes (breaking ties arbitrarily).

Let $\phi(x) = e^{-x/m}$. Note that

$$\phi(x-\delta) - \phi(x) \leq \frac{\delta}{m} \cdot \phi(x-\delta) \ . \tag{9}$$

Indeed, it follows by the Mean Value Theorem that

$$\phi(x-\delta) - \phi(x) = -\delta \cdot \phi'(\theta) = \frac{\delta}{m} \cdot e^{-\theta/m},$$

for some $x - \delta \leq \theta \leq x$. However, $e^{-\theta/m} \leq e^{-(x-\delta)/m}$ by the monotonicity of ϕ.

At any point during the game, if, for every $i \in S$, c_i is the current remaining size of box i, then the *current potential of the game* is set to be

$$\Phi := \sum_{i \in S} \phi(c_i) = \sum_{i \in S} e^{-c_i/m}.$$

If CMaker wins the game in his kth move, for some $k \in \mathbb{N}$, then, immediately before his kth move, there must exist some box $i \in S$ whose size is at most m. Hence, the potential of the game at this point must be at least e^{-1}. It follows that in order to prove that Breaker wins the game, it suffices to prove that Breaker can maintain the inequality $\Phi < e^{-1}$ throughout the game.

This holds before CMaker's first move by the theorem's assumption. Assume it holds immediately before CMaker's jth move; we will prove it will also hold immediately before his $(j+1)$st move, assuming Breaker follows his suggested strategy. In his jth move, CMaker distributes a total weight of m among the boxes in S. Assume that he puts a weight of $\delta_i \geq 0$ in box i, for every $i \in S$, where $\sum_{i \in S} \delta_i = m$. It follows that the remaining size of box i is $c_i - \delta_i$, where c_i denotes its size immediately before this move. Denoting the potential of the game immediately after CMaker's jth move by Φ' we have

$$\begin{aligned}
\Phi' - \Phi &= \sum_{i \in S} e^{-(c_i - \delta_i)/m} - \sum_{i \in S} e^{-c_i/m} \\
&= \sum_{i \in S} \left(e^{-(c_i - \delta_i)/m} - e^{-c_i/m}\right) \\
&\leq \frac{1}{m} \sum_{i \in S} \delta_i e^{-(c_i - \delta_i)/m} \\
&\leq \frac{1}{m} \sum_{i \in S} \left(\delta_i \cdot \max_{i \in S} e^{-(c_i - \delta_i)/m}\right) \\
&= \max_{i \in S} e^{-(c_i - \delta_i)/m}, \qquad (10)
\end{aligned}$$

where the first inequality follows from (9).

In his jth move, Breaker destroys box i such that $c_i - \delta_i$ is minimal. Let Φ'' denote the potential of the game immediately after Breaker's jth move. Then

$$\Phi' - \Phi'' = \exp\{-\min_{i \in S}(c_i - \delta_i)/m\} = \max_{i \in S} e^{-(c_i - \delta_i)/m} . \tag{11}$$

Combining (10) and (11) we conclude that $\Phi'' \leq \Phi$. This completes the proof. □

5 Uniform CBox

The authors of [2] were mostly interested in the special case of the Box game in which the initial size of all the boxes is the same. Applying Theorem 1.2 in this special case yields the following result.

Theorem 5.1 *If $a_i = s$ for every $1 \leq i \leq n$, and $s > m \cdot H_n$, then Breaker has a winning strategy for both Box$(m; a_1, \ldots, a_n)$ and CBox$(m; a_1, \ldots, a_n)$.*

Recall that in [2] it is assumed that Breaker is the first player, whereas in this paper we assume he is the second player. Bringing this difference into account, the sufficient condition for Breaker's win in Box (and CBox) given in Theorem 5.1, is exactly the same as the condition given in Theorem 1.1, if one uses the upper bound (1) on the function $f(n, m)$, rather than using the function itself.

In this section we give a short direct proof of Theorem 5.1. The proof uses an idea from [3].

Proof We will prove the statement for CBox; the statement for Box will readily follow.

Breaker again employs the strategy `Destroy_smallest`, that is, in every move he destroys a box i whose size is minimal among the set S of all surviving boxes (breaking ties arbitrarily). We will prove that this is a winning strategy for Breaker.

Assume for the sake of contradiction that CMaker wins the game; assume further that he wins in his kth move, for some $1 \leq k \leq n$. Hence, for every $1 \leq i \leq k-1$, Breaker destroys box i in his ith move, and in his kth move, CMaker fully claims box k. At any point during the game let c_i denote the remaining size of box i for every $i \in S \cap \{1, \ldots, k\}$. For every $1 \leq j \leq k$, let

$$\Phi(j) := \frac{1}{k - j + 1} \sum_{i=j}^{k} c_i$$

denote the *potential of the game* immediately before CMaker's jth move. Note that $\Phi(k) = c_k \leq m$ by our assumption that CMaker wins the game in his kth move, and that $\Phi(1) = s$ since $a_i = s$ for every $1 \leq i \leq n$. Since Breaker always destroys the smallest box, he does not decrease the potential of the game in any of his moves. For every $1 \leq j \leq k-1$, in his jth move CMaker decreases the potential of the game by at most $m/(k - j + 1)$. It follows that

$$\Phi(k) \geq s - \left(\frac{m}{k} + \frac{m}{k-1} + \ldots + \frac{m}{2} \right) = s - m(H_k - 1) \geq s - m(H_n - 1) > m,$$

where the last inequality follows by the assumed lower bound on s. This contradicts our assumption that CMaker wins the game and concludes the proof of the theorem. □

6 Concluding remarks

As previously noted, we were unable to completely adjust our proof of Theorem 1.2 (which deals with the CBox game) to fit the Box game setting. The closest we get to answering the question of who wins the Box game is Theorem 1.3, which then can be further combined with any of the obtained criteria for the determination of the winner of the CBox game. Even though Theorem 1.3 may suggest that the outcome of Box and of CBox with the same parameters is "often" the same, we cannot hope to completely get rid of the discrepancy between Maker's and CMaker's bias which appears in Theorem 1.3 – for example, $\mathrm{Box}(7; 13, 15, 15, 15)$ is Breaker's win whereas $\mathrm{CBox}(7; 13, 15, 15, 15)$ is CMaker's win.

The following example shows that the threshold bias gap between Box and CBox (played on the same set of boxes) can in fact be arbitrarily close to one. Let $\varepsilon > 0$ be arbitrarily small. Let m be an arbitrary real number satisfying $0 < \lfloor m \rfloor < m < \lfloor m \rfloor + \varepsilon$, let n be large enough to ensure $(m - \lfloor m \rfloor)H_n > 1$, and let $s = \lfloor mH_n \rfloor$. For every $1 \leq i \leq n$, let $a_i = s$. Since $s \leq mH_n$, it follows from Theorem 1.2 that $\mathrm{CBox}(m; a_1, \ldots, a_n)$ is CMaker's win. On the other hand, since $s > \lfloor m \rfloor H_n$ holds by our choice of n and s, it follows by Theorem 5.1 that $\mathrm{Box}(\lfloor m \rfloor; a_1, \ldots, a_n)$ is Breaker's win. Hence, the smallest positive integer p for which $\mathrm{Box}(p; a_1, \ldots, a_n)$ is Maker's win is strictly larger than $m + (1 - \varepsilon)$.

We think that finding a set of easily checkable explicit conditions that gives a complete description of the integral Box games which are Maker's win would be of considerable importance.

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