Planarity, colorability and minor games

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Abstract

Let m and b be positive integers and let $\mathcal{F}$ be a hypergraph. In an (m, b) Maker-Breaker game $\mathcal{F}$ two players, called Maker and Breaker, take turns selecting previously unclaimed vertices of $\mathcal{F}$. Maker selects m vertices per move and Breaker selects b vertices per move. The game ends when every vertex has been claimed by one of the players. Maker wins if he claims all the vertices of some hyperedge of $\mathcal{F}$; otherwise Breaker wins. An (m, b) Avoider-Enforcer game $\mathcal{F}$ is played in a similar way. The only difference is in the determination of the winner: Avoider loses if he claims all the vertices of some hyperedge of $\mathcal{F}$; otherwise Enforcer loses.

In this paper we consider the Maker-Breaker and Avoider-Enforcer versions of the planarity game, the k-colorability game and the K_t-minor game.

1 Introduction

Let m and b be two positive integers. We are given a set X and a hypergraph $\mathcal{F} \subseteq 2^X$. During the (m, b) positional game $\mathcal{F}$, two players take turns claiming previously unclaimed elements of X. In every round, the first player claims m elements, and then the second player claims b elements. The set X is called the "board"; m and b are the biases of the first and second players respectively. For the purposes of this paper $\mathcal{F}$ is assumed to be monotone increasing. We investigate positional games with two different kinds of rules for determining the winner.

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In a *Maker/Breaker-type* positional game, the two players are called Maker and Breaker and $\mathcal{F}$ is referred to as the family of "winning sets". Maker wins the game if the set M he has claimed by the end of the game (i.e., when every element of the board has been claimed by one of the players) is a winning set, that is $M \in \mathcal{F}$. Otherwise Breaker wins. Observe that, since $\mathcal{F}$ is assumed to be monotone increasing, the game could essentially be stopped as soon as Maker occupied a *minimal* (with respect to inclusion) winning set $F \in \mathcal{F}$; the position on $X \setminus F$ can no longer influence the outcome of the game. Hence, Breaker wins if and only if he claims at least one element in every minimal winning set. Since a monotone increasing family and the family of its minimal elements uniquely determine each other, often, when there is no risk of confusion, we use $\mathcal{F}$ for the family of minimal winning sets as well. A classical example of the Maker-Breaker setting is the popular boardgame HEX.

In an *Avoider/Enforcer-type* positional game, the players are called Avoider and Enforcer and $\mathcal{F}$ is called the family of "losing sets". Avoider wins the game if the set A he has claimed by the end of the game (i.e., when every element of the board has been claimed by one of the players) is not a losing set, that is, $A \notin \mathcal{F}$. Otherwise Enforcer wins. Again, observe that the game could be stopped as soon as Avoider fully occupies a minimal losing set, since then Enforcer surely wins no matter how the game proceeds. Having this in mind, we will sometimes use $\mathcal{F}$ to denote the family of minimal losing sets.

The identity of the player making the first move does not affect the asymptotics of our results. However, for convenience, we will assume that Maker (Avoider) is the first player in all Maker/Breaker (Avoider/Enforcer) games we discuss.

In this paper, we investigate Maker/Breaker and Avoider/Enforcer positional games played on the board $E(K_n)$ – the edges of the complete graph on n vertices.

Maker-Breaker games The study of positional games on the edges of a (complete) graph was initiated by Lehman [20] who, in particular, proved that in the $(1,1)$ game Maker can easily build a spanning tree (by "easily" we mean that he can do so within $n-1$ moves). Chvátal and Erdős [10] suggested to "even out the odds" by giving Breaker more power, that is, by increasing his bias. They determined that the $(1,b)$ game $\mathcal{T}_n$ of "Connectivity", where the family $\mathcal{T}_n$ of minimal winning sets consists of the edge-sets of all spanning trees of K_n, is won by Maker even when the bias b of Breaker is as large as $cn/\log n$ for some small constant $c > 0$, while for some large enough constant $C > 0$, Breaker wins if his bias is at least $Cn/\log n$. They also showed that the $(1,1)$ game $\mathcal{H}_n$ of "Hamiltonicity", where Maker's goal is to build a Hamiltonian cycle (the family $\mathcal{H}_n$ of minimal winning sets consists of the edge-sets of all Hamiltonian cycles of K_n), is won by Maker for sufficiently large n. They conjectured that in fact Maker can win the $(1,b)$ Hamiltonicity game for some b that tends to infinity with n. This was proved by Bollobás and Papaioannou [9], who showed that Maker wins Hamiltonicity against a bias of $\Omega(\log n/\log\log n)$. Finally, Beck [1] gave a winning strategy for Maker against a bias of $\Omega(n/\log n)$.

Following [14] it will be convenient to introduce the following notation. For a family $\mathcal{F}$ of winning sets, let $b_{\mathcal{F}}$ be the non-negative integer for which Breaker has a winning strategy in the $(1, b)$ game $\mathcal{F}$ if and only if $b \geq b_{\mathcal{F}}$. Note that $b_{\mathcal{F}}$ is well-defined for any (monotone increasing) family $\mathcal{F}$ (unless $\mathcal{F}$ contains a hyperedge of size at most one). By the above, $b_{\mathcal{T}_n} = \Theta(n/\log n)$ and $b_{\mathcal{H}_n} = \Theta(n/\log n)$.

There is an intriguing relation between the threshold biases of the Connectivity and Hamiltonicity games and the threshold probability at which the random graph $G(n,p)$ first possesses these properties. The number of edges of the random graph $G(n,p)$ around this threshold $p = \log n/n$ has the same order of magnitude as the number of edges Maker has in his graph after playing against the threshold bias $b_{\mathcal{T}_n}$ or $b_{\mathcal{H}_n}$. This phenomenon (known as the Erdős paradigm) was pointed out by Beck [1], where this observation is attributed to Erdős (indeed the first such result appeared already in [10]). Since then, several other games have been found to have a threshold bias which is closely linked to a meaningful random graph threshold related to that particular game (see, e.g., [2, 4, 5, 6, 25]). Note, that the asymptotic values of the threshold biases $b_{\mathcal{T}_n}$ and $b_{\mathcal{H}_n}$ are not known in general. It can still turn out that these threshold biases are asymptotically equal to the inverse of the corresponding sharp thresholds for the appropriate properties of random graphs.

In this paper we investigate this connection further and find three more instances where such an intuition proves to be correct. Let $\mathcal{NP}_n$, $\mathcal{NC}_n^k$, $\mathcal{M}_n^t$ consist of the edge-sets of all non-planar graphs on n vertices, the edge-sets of all non-k-colorable graphs on n vertices, and the edge-sets of all graphs on n vertices containing a K_t-minor (a graph G contains an H-minor if H can be obtained from G by the deletion and contraction of some of its edges), respectively. Obviously, all three families are monotone increasing.

In the game $\mathcal{NP}_n$, which we call the "planarity game" (although perhaps "non-planarity" would have been a more appropriate name), Maker's goal is to claim all edges of a non-planar graph. In Theorem 2.1 we show that the corresponding threshold bias $b_{\mathcal{NP}_n}$ is asymptotically $n/2$. Note that this result is a consequence of the result for the minor game we obtain in Theorem 4.1, as the presence of a K_5-minor guarantees non-planarity. Nevertheless, we include an alternative, direct proof of Theorem 2.1 which approaches planarity from a different angle, and, in our opinion, is more instructive and serves as a good example of how Maker can win by adopting the role of Avoider in a different game.

Coming back to the relation between the probability thresholds of random graphs and the thresholds for the game bias, we note that the threshold probability $p_{\mathcal{NP}_n}$ of non-planarity in the random graph is $1/n$, that is, $p_{\mathcal{NP}_n} = \Theta(1/b_{\mathcal{NP}_n})$. However, the number of edges of $G(n, p_{\mathcal{NP}_n})$ is concentrated around *one half* of the number of edges in Maker's graph in the $(1, b_{\mathcal{NP}_n}, \mathcal{NP}_n)$ game (the non-planarity threshold is sharp, so such a comparison makes sense).

In the "k-colorability game", the family of winning sets is $\mathcal{NC}_n^k$, that is, Maker wins the game if he claims a non-k-colorable graph. In Theorem 3.1 we show that the threshold bias is linear for a fixed k by establishing that there are absolute constants c_1, c_2 such that $c_1 \frac{n}{k \log k} \leq b_{\mathcal{NC}_n^k} \leq c_2 \frac{n}{k \log k}$ for any k. For the special case $k = 2$, that is, the bipartite game, a more accurate result was proved by Bednarska and Pikhurko [8]. They showed that $(1-1/\sqrt{2}-o(1))n \leq b_{\mathcal{NC}_n^2} \leq \lceil n/2 \rceil - 1$. Again, one can compare these results with the corresponding random graph threshold $p_{\mathcal{NC}_n^k} = \frac{2k \log k}{n}$ for non-k-colorability and find the reciprocal relation.

Finally, we turn to a more general setting. In the K_t-minor game the family of winning sets is $\mathcal{M}_n^t$; hence, Maker wins the game if he is able to claim the edges of some K_t-minor of K_n. This game is in some sense a generalization of both the planarity and the colorability games we have discussed. Indeed, Wagner's Theorem gives a full description of planarity via the language of forbidden minors. Furthermore, if a graph is not r-colorable then it contains a K_s-minor for $s = r/(c\sqrt{\log r})$ where c is an absolute constant (see [17], [26]), and the famous and long standing Hadwiger conjecture asserts that in fact it contains a K_r-minor. The opposite implication is trivially false as for every positive integer n, the complete bipartite graph $K_{n,n}$ admits a K_n-minor.

In Theorem 4.1 we prove that the corresponding threshold bias $b_{\mathcal{M}_n^t}$ is asymptotically $n/2$ for every $3 \leq t \leq c\sqrt{n/\log n}$, for an appropriate constant c. Note that for the case $t = 3$, an accurate threshold bias follows from a result of Bednarska and Pikhurko [7], as a graph is K_3-minor free if and only if it is a forest (see the exact statement in Theorem 2.2 in Section 2).

Avoider-Enforcer games. At first sight, the definition of Avoider/Enforcer games seems less natural than that of Maker/Breaker games and accordingly the theory is much less developed. We, however, argue that besides being interesting in their own right (see [14] for a comprehensive discussion), they are essential in studying Maker/Breaker games. Avoider/Enforcer games arise naturally whenever one would like to play (what looks like) Maker/Breaker games on a *monotone decreasing family*. For example, the set $\mathcal{P}_n$ of planar graphs is a monotone decreasing family. If for example, "Maker's" goal is to keep his graph planar to the end of the game, then he can be thought of as Avoider, playing an Avoider/Enforcer game on the (monotone increasing) family of losing sets $2^{E(K_n)} \setminus \mathcal{P}_n = \mathcal{NP}_n$.

Moreover, for certain Maker/Breaker games the best known Maker strategies involve building a pseudo-random graph with certain parameters (see [13, 15]). It is proved that the particular pseudo-random properties of Maker's graph imply the graph-theoretic properties in question, entailing his win. Here, pseudo-random properties include bounds on the number of edges between pairs of disjoint vertex sets from *below* and from *above*. Hence, in such a game the family $\mathcal{F}$ is the intersection of a monotone increasing family and a monotone decreasing family.

Avoider/Enforcer games were studied in [3, 14, 21, 22, 23]. Similarly to Maker/Breaker games, one would like to define for each monotone increasing family $\mathcal{F}$ the Avoider/Enforcer threshold bias $f_{\mathcal{F}}$. A reasonable choice for $f_{\mathcal{F}}$ would be the non-negative integer for which Avoider wins the $(1, b)$ game $\mathcal{F}$ if and only if $b \geq f_{\mathcal{F}}$. While the similar threshold $b_{\mathcal{F}}$ does exist for Maker/Breaker games on (essentially) any hypergraph, for Avoider/Enforcer games it generally does not (see [14]).

Following [14], it will be convenient to introduce the following notation. For a hypergraph $\mathcal{F}$ we define the *lower threshold bias* $f^-_{\mathcal{F}}$ to be the largest integer such that Enforcer can win the $(1,b)$ game $\mathcal{F}$ for every $b \leq f^-_{\mathcal{F}}$, and the *upper threshold bias* $f^+_{\mathcal{F}}$ to be the smallest non-negative integer such that Avoider can win the $(1,b)$ game $\mathcal{F}$ for every $b > f^+_{\mathcal{F}}$. Except for certain degenerate cases, $f^-_{\mathcal{F}}$ and $f^+_{\mathcal{F}}$ always exist and satisfy $f^-_{\mathcal{F}} \leq f^+_{\mathcal{F}}$. Whenever $f^-_{\mathcal{F}} = f^+_{\mathcal{F}}$, the threshold bias $f_{\mathcal{F}}$ of the Avoider-Enforcer game $\mathcal{F}$ does exist and satisfies $f_{\mathcal{F}} = f^+_{\mathcal{F}}$.

In [14] the existence of the threshold $f_{\mathcal{T}_n}$ was established for the connectivity game and it was proved that $f_{\mathcal{T}_n}$ is roughly $n/2$. For the perfect matching game and Hamiltonicity game it was shown that $f^-_{\mathcal{M}_n} = \Omega(n/\log n)$ and $f^-_{\mathcal{H}_n} = \Omega(n \log\log\log\log n/\log n \log\log\log n)$, respectively. Note that it is not known if the threshold bias for any of these natural games exists.

In this paper, we give bounds on the lower and upper threshold biases for the planarity, k-colorability and K_t-minor games. Note that, unlike the games of Connectivity and Hamiltonicity, these Avoider-Enforcer games seem to be more natural than their Maker-Breaker analogs.

In Theorem 2.3 we prove that Avoider can keep his graph planar against any bias which is larger than $2n^{5/4}$, whereas Enforcer wins when playing with a bias no larger than $\frac{n}{2} - o(n)$. As in the case of the Maker-Breaker planarity game, the second part of this result is a direct consequence of our result for the minor game presented in Theorem 4.7. Nevertheless, we include an alternative, direct proof which relies on other properties of planar graphs. We believe it is more illustrative and gives more insight into the course of the game.

In Theorem 3.4 we show that playing against a bias of at least $2kn^{1+1/(2k-3)}$, Avoider can keep his graph k-colorable, whereas Enforcer wins if his bias is at most $\frac{n}{ck\log k}$.

In the Avoider-Enforcer version of the K_t-minor game, Avoider's task is to build a K_t-minor free graph. As in the planarity and colorability games, Avoider's goal in this game is very natural. Indeed, many graph-theoretic properties can be expressed in a "forbidden minor" fashion.

We show that, playing with a linear bias, Enforcer can make Avoider build a graph that admits a K_t minor for t which is as large as $c\sqrt{n/\log n}$. Moreover, in Theorem 4.7 we prove that even when playing with a bias which is arbitrarily close (from below) to $n/2$, Enforcer can make sure that by the end of the game Avoider will claim a K_t-minor for $t = n^{a_0}$, where $a_0 > 0$ is an appropriate constant.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. We also omit floor and ceiling signs whenever these are not crucial. All of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [11].

The rest of the paper is organized as follows: in Section 2 we discuss the Maker-

Breaker and Avoider-Enforcer planarity games, in Section 3 we discuss the colorability games and in Section 4 we discuss the minor games. Finally, in Section 5 we present some open problems.

2 Planarity games

2.1 The Maker-Breaker planarity game

The following theorem states that the threshold bias at which Maker's win turns into a Breaker's win in the planarity game is "around" $n/2$.

Theorem 2.1

$$b_{\mathcal{NP}_n} = \frac{n}{2} - o(n).$$

Proof Let $b \geq n/2$. The existence of a winning strategy for Breaker in the planarity game is an easy consequence of the following result of Bednarska and Pikhurko.

Theorem 2.2 *[7, Corollary 10] Suppose that CycleMaker and CycleBreaker select respectively* 1 *and* b *edges of* K_n *and CycleMaker wins if he builds a cycle. Then CycleMaker has a winning strategy (no matter who starts) if and only if* $b < \lceil n/2 \rceil$.

The assertion of Theorem 2.2 implies that with the bias $b \geq n/2$, Breaker can prevent Maker from building a cycle. It follows that at the end of the game Maker's graph will be a forest which is obviously planar.

Next, let $0 < \varepsilon < 1/3$ (the restriction $\varepsilon < 1/3$ is technical) and let $b \leq (1/2 - \varepsilon)\, n$, where $n = n(\varepsilon)$ is sufficiently large. We will provide Maker with a strategy for building a non-planar graph. Let $\alpha = \frac{2\varepsilon}{1-2\varepsilon}$ and let $\alpha_n = \alpha_n(\varepsilon)$ be the real number satisfying the equation

$$(1+\alpha_n)n = \frac{\binom{n}{2}}{(\frac{1}{2}-\varepsilon)n+1}.$$

Then $\lim_{n\to\infty} \alpha_n = \alpha$. Let m_n denote the number of edges that Maker will claim by the end of the game on K_n. We have $m_n - (1+\frac{\alpha}{2})n = \Omega(n)$.

Let $k = k(\varepsilon)$ be the smallest positive integer such that

$$\left(1+\frac{\alpha}{2}\right) > \frac{k}{k-2}.$$

Maker's goal is to avoid cycles of length smaller than k, which we will call "short cycles", during the first $\left(1+\frac{\alpha}{2}\right)n$ moves. If he succeeds, Maker's graph will at that point of the game have

$$\left(1+\frac{\alpha}{2}\right)n > \frac{k}{k-2}n$$

edges and girth at least k. But, it is well-known that a planar graph with girth at least k cannot have more than $\frac{k}{k-2}(n-2)$ edges. Hence, Maker's graph will already be non-planar, and he will win no matter how the game continues.

It remains to show that Maker can indeed avoid claiming a short cycle during the first $(1+\frac{\alpha}{2})n$ moves. His strategy is the following. For as long as possible he claims edges (u,v) that satisfy the following two properties:

(a) (u,v) does not close a short cycle;

(b) the degrees of both u and v in Maker's graph are less than $n^{1/(k+1)}$.

It suffices to prove that when this is no longer possible, that is, every remaining unclaimed edge violates either (a) or (b), Maker has already claimed at least $(1+\frac{\alpha}{2})n$ edges.

Every edge that violates property (b) must have at least one endpoint of degree $n^{1/(k+1)}$ in Maker's graph. Since Maker's graph at any point of the game contains at most $(1+\alpha)n$ edges, there are at most $2(1+\alpha)n^{1-1/(k+1)}$ vertices of degree at least $n^{1/(k+1)}$. Therefore, the number of edges that violate property (b) is at most

$$n \cdot 2(1+\alpha)n^{1-1/(k+1)} = o(n^2).$$

For any fixed $s < k$ and every vertex v, the number of paths of length s that have v as one endpoint is at most Δ^s, where Δ is the maximum degree in Maker's graph. If we assume that property (b) has not been violated, then $\Delta \leq n^{1/(k+1)}$. Therefore, there are at most

$$n \cdot \sum_{s=3}^{k-1} n^{s/(k+1)} = o(n^2)$$

edges that close a short cycle.

Thus, the total number of edges that violate (a) or (b) if claimed by Maker, is $o(n^2)$. On the other hand, after $(1+\frac{\alpha}{2})n$ moves have been played, the number of unclaimed edges is $\Theta(n^2)$. Hence, in the first $(1+\frac{\alpha}{2})n$ moves Maker can claim edges that satisfy properties (a) and (b), which means that he does not claim a short cycle. This completes the proof of the theorem. □

2.2 The Avoider-Enforcer planarity game

In the following theorem we give an upper bound and a lower bound for the threshold bias at which Enforcer's win turns into an Avoider's win in the planarity game.

Theorem 2.3

$$\frac{n}{2} - o(n) \leq f^{-}_{\mathcal{NP}_n} \leq f^{+}_{\mathcal{NP}_n} \leq 2n^{5/4}.$$

Proof Assume first that $b \geq 2n^{5/4}$. We will provide Avoider with a strategy for building a planar graph. The game is divided into four stages. Avoider's strategy is the following.

In the first stage, Avoider builds a matching by repeatedly claiming an edge that connects two vertices, neither of which is incident with any other edge previously claimed by him. The first stage ends when no such unclaimed edge remains and so Avoider cannot further extend his matching. We denote the set of vertices that are covered by Avoider's matching by M.

Next, in the second stage Avoider claims edges with one endpoint in M and the other in $V \setminus M$ such that throughout the second stage every vertex of $V \setminus M$ has degree at most one in Avoider's graph. The second stage ends when no such unclaimed edge remains.

In the third stage Avoider builds another matching on M. More precisely, he repeatedly claims edges that connect two vertices of M, neither of which is incident with any other edge previously claimed by him in the **third** stage. The third stage ends when no such unclaimed edge remains and Avoider cannot further extend this second matching.

In the fourth and final stage, Avoider claims edges arbitrarily to the end of the game. If we prove that in this stage Avoider will claim at most one edge, then the upper bound of the theorem will follow. Indeed, the graph that is spanned by Avoider's edges from the first and third stages is a union of two matchings, that is, a union of disjoint paths and cycles. Furthermore, if we add Avoider's edges from the second stage to this graph, then we simply add "hanging" edges (edges with one endpoint having degree one). Clearly, if we now add any single edge that may have been claimed by Avoider in the fourth stage to that graph, it remains planar.

Let e be the number of edges that Avoider claims in the entire game. By the end of the first stage, Enforcer must have claimed all the edges with both endpoints in $V \setminus M$. Since Avoider's matching on M consists of at most e edges, we have $|V \setminus M| \geq n - 2e$ and therefore Enforcer has already claimed at least $\binom{n-2e}{2}$ edges. It follows that there are at most

$$\binom{n}{2} - \binom{n-2e}{2} \leq 2en$$

unclaimed edges left in the graph and Avoider will claim at most $\frac{2en}{b}$ of them. In the second stage, Avoider claims edges between M and $V \setminus M$. When this is no longer possible, every unclaimed edge between M and $V \setminus M$ is incident with a vertex of $V \setminus M$ which has degree one in Avoider's graph. It follows that, at this point, the number of unclaimed edges between M and $V \setminus M$ is at most

$$2e \cdot \frac{2en}{b} = \frac{4e^2 n}{b}.$$

In the third stage, Avoider builds his second matching on M. When this is no longer possible, the number of unclaimed edges with both endpoints in M is at most

$$\binom{2e}{2} - \binom{2e - 4en/b}{2} \leq \frac{8e^2 n}{b}.$$

To see this, it is enough to observe that the number of vertices that are incident with the second matching is at most $4en/b$, and that all edges with endpoints in

M that are not adjacent to the second matching must be claimed by Enforcer after the third stage.

Putting everything together, the total number of unclaimed edges after the third stage is at most

$$\frac{4e^2n}{b} + \frac{8e^2n}{b} = \frac{12e^2n}{b}.$$

Since $e < n^2/2b$, we have that the number of edges to be played in the fourth stage is at most $\frac{12e^2n}{b} \leq \frac{3n^5}{b^3} \leq b$, which means that in the fourth stage Avoider will claim at most one edge.

Next, fix an $\varepsilon > 0$ and assume that $b \leq \frac{n}{2}(1-\varepsilon)$. We will provide Enforcer with a strategy, which guarantees that Avoider will occupy the edges of a non-planar graph. If $b \leq n/7$, then the number of edges Avoider claims in the entire game is at least $\lfloor \binom{n}{2} \cdot (b+1)^{-1} \rfloor > 3n$, and thus Avoider surely loses regardless of Enforcer's strategy. Hence, from now on we can assume that $b > n/7$. Let $k = k(\varepsilon)$ be the smallest positive integer such that $\frac{1}{1-\varepsilon/2} > \frac{k}{k-2}$. Enforcer's strategy will be to prevent Avoider from claiming a cycle of length smaller than k, which we will call a "short cycle". If he succeeds, then at the end of the game Avoider's graph will have at least

$$\left\lfloor \frac{\binom{n}{2}}{b+1} \right\rfloor \geq \frac{n}{1-\varepsilon/2} > \frac{k}{k-2}n$$

edges for sufficiently large n, and girth at least k. As we have mentioned before, a graph with such properties cannot be planar, thus Enforcer wins.

It remains to show that Enforcer can indeed prevent Avoider from claiming a short cycle. In order to do that we will use the following theorem of Bednarska and Łuczak.

Theorem 2.4 *[5, Theorem 1] For every graph G which contains at least three non-isolated vertices there exist positive constants c and n_0 such that, playing the $(1,q)$ game on K_n, G-Breaker can prevent G-Maker from building a copy of G provided that $n > n_0$ and $q > cn^{1/m_2(G)}$, where*

$$m_2(G) = \max_{\substack{H \subseteq G \\ v(H) \geq 3}} \frac{e(H)-1}{v(H)-2}.$$

For a cycle C_i of length i, we have $m_2(C_i) = \frac{i-1}{i-2}$. Therefore, there exist constants c_i, $i = 3, \ldots, k-1$ such that for sufficiently large n, Enforcer can prevent Avoider from claiming a copy of C_i, if the number of edges he is allowed to claim per move is at least $c_i n^{\frac{i-2}{i-1}}$. Since for sufficiently large n

$$\sum_{i=3}^{k-1} c_i n^{\frac{i-2}{i-1}} \leq \frac{n}{7} \leq b,$$

Enforcer can simultaneously prevent Avoider from claiming any short cycle C_i, $3 \leq i < k$, by simply playing all $k-3$ games in parallel. That is, after

Avoider claims an edge, Enforcer responds by claiming $c_3 n^{\frac{1}{2}}$ edges according to the strategy in the "triangle avoidance game", then he claims $c_4 n^{\frac{2}{3}}$ edges according to the strategy in the "4-cycle avoidance game", and so on. His different strategies, for the different cycle-games, might call for claiming the same edge more than once, in which case he just claims an arbitrary unclaimed edge instead. It is easy to see that this cannot harm him. This concludes the proof of the theorem. $\square$

3 k-colorability games

3.1 The Maker-Breaker k-colorability game

The following theorem states that the threshold bias at which Maker's win turns into a Breaker's win in the k-colorability game, where k is fixed and n is sufficiently large, is of order n. This is true for every $k \geq 2$. However, for convenience, and since the case $k = 2$ was treated in [8], we will assume that $k \geq 3$.

Theorem 3.1 *For every $k \geq 3$ and sufficiently large n, there exist constants s_k and s'_k such that*

$$s'_k n \leq b_{\mathcal{NC}_n^k} \leq s_k n,$$

where $s_k \sim \frac{2}{k \log k}$ and $s'_k \sim \frac{\log 2}{2k \log k}$ as $k \to \infty$.

Proof Assume first that $b \leq \frac{n}{ck \log k}$, where $c > 2/\log 2$. We will provide Maker with a strategy for building a non-k-colorable graph. Maker's goal will be to prevent Breaker from building a clique of size $\lceil n/k \rceil$, and this is enough to ensure his win. Indeed, Maker's graph is surely not k-colorable if it does not admit an independent set of size $\lceil n/k \rceil$.

Let $\mathcal{F}$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are the $\lceil n/k \rceil$-cliques of K_n. We name the players of the $(b, 1, \mathcal{F})$ game, CliqueMaker and CliqueBreaker. As was mentioned before, Maker wins the k-colorability game if he claims a vertex in every hyperedge of $\mathcal{F}$, that is, if he is able to win the $\mathcal{F}$ game as CliqueBreaker. We will use Beck's criterion, which is applicable to any Maker/Breaker-type game.

Theorem 3.2 *[1, Theorem 1] If*

$$\sum_{D \in \mathcal{H}} (1+q)^{-|D|/p} < \frac{1}{q+1},$$

then Breaker wins the (p, q) game $\mathcal{H}$.

We have

$$\begin{aligned}\sum_{D \in \mathcal{F}} 2^{-|D|/b} &\leq \binom{n}{\lceil n/k \rceil} 2^{-\binom{\lceil n/k \rceil}{2}/b} \leq (ek)^{\lceil n/k \rceil} 2^{-\binom{\lceil n/k \rceil}{2}/b} \\ &\leq 2^{\frac{n \log_2 e}{k} + \frac{n \log_2 k}{k} + \log_2(ek) - \frac{cn^2 k \log k}{2k^2 n} + \frac{nck \log k}{2kn}} = o(1).\end{aligned}$$

Hence, Maker can win the k-colorability game.

Assume now that $b \geq s_k n$, where s_k is a constant depending on k that will be determined later. We will provide Breaker with a strategy, to force Maker into building a k-colorable graph. We will make use of the following theorem of Kim.

Theorem 3.3 *[16, Corollary 1.2] If G is a graph with maximum degree Δ and girth at least 5, then*

$$\chi(G) \leq (1+\nu(\Delta))\frac{\Delta}{\log \Delta},$$

where $\nu(\Delta) \to 0$ as $\Delta \to \infty$.

Let Δ_0 be the maximal value of Δ for which

$$(1+\nu(\Delta))\frac{\Delta}{\log \Delta} \leq k$$

(if no such Δ_0 exists or if $\Delta_0 < 2$, then we take $s_k = 1/2$ and so Breaker wins by Theorem 2.2).

Since $\nu(\Delta) \to 0$ as $\Delta \to \infty$, we have that $\Delta_0 \sim k \log k$ as $k \to \infty$. Breaker's goal will be to force Maker to build a graph with maximum degree at most Δ_0 and girth at least 5. By Theorem 3.3 Maker's graph will then be k-colorable. In each move, Breaker will use $c_3 n^{1/2}$ of his edges to prevent Maker from building a triangle (recall that $m_2(C_3) = 2$), and $c_4 n^{2/3}$ of his edges to prevent Maker from building a cycle of length 4 ($m_2(C_4) = 3/2$), where c_3 and c_4 are the constants whose existence is guaranteed by Theorem 2.4. Breaker will use all of his remaining $b' := b - c_3 n^{1/2} - c_4 n^{2/3} = (1-o(1))b$ edges to make sure that the maximum degree in Maker's graph does not surpass Δ_0. Hence, if Maker claims the edge (u,v), then Breaker will claim $\frac{1}{2}b'$ edges incident with u and $\frac{1}{2}b'$ edges incident with v (if there are only $r < \frac{1}{2}b'$ unclaimed edges incident with u or v, then Breaker will claim all of them and additional $\frac{1}{2}b' - r$ arbitrary unclaimed edges). It follows that the maximum degree in Maker's graph will be at most

$$1 + \frac{n-1}{b'/2} \leq 1 + \frac{2n}{(1-o(1))b} \leq 1 + \frac{2}{s_k} + o(1),$$

where the $o(1)$ term tends to zero as n tends to infinity. Therefore, if $s_k = \lceil \frac{2}{\Delta_0 - 1.5} \rceil$, then Maker's graph will have maximum degree at most Δ_0. Hence, Breaker can force Maker to build a graph with maximum degree at most Δ_0 and girth at least 5, and thus he can win. Note that s_k, defined this way, satisfies $s_k \sim \frac{2}{k \log k}$ as $k \to \infty$. This concludes the proof of the theorem. □

3.2 The Avoider-Enforcer k-colorability game

In the following theorem we give an upper bound and a lower bound for the threshold bias at which Enforcer's win turns into an Avoider's win in the k-colorability game.

Theorem 3.4 *For every $k \geq 3$ there exists a constant s'_k, such that*

$$s'_k n \leq f^-_{\mathcal{NC}^k_n} \leq f^+_{\mathcal{NC}^k_n} \leq 2kn^{1+\frac{1}{2k-3}}.$$

Moreover, $s'_k \sim \frac{\log 2}{2k \log k}$ as $k \to \infty$.

Proof Assume first that $b \leq \frac{n}{ck \log k}$. We will provide Enforcer with a strategy which ensures that by the end of the game, Avoider's graph will not be k-colorable. Enforcer's goal will be to avoid building a clique of size $\lceil n/k \rceil$. If he achieves this goal, Avoider's graph will not contain an independent set of size $\lceil n/k \rceil$ and so will not be k-colorable; thus Enforcer will win. Let $\mathcal{F}$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are the $\lceil n/k \rceil$-cliques of K_n. We name the players of the $(b, 1, \mathcal{F})$ game CliqueAvoider and CliqueEnforcer. As mentioned above, Enforcer will win the k-colorability game if he will not claim all vertices in any hyperedge of $\mathcal{F}$, that is, if he is able to win as CliqueAvoider.

We will use the following criterion, applicable to any Avoider/Enforcer-type game.

Theorem 3.5 *[14, Theorem 1.1] If*

$$\sum_{D \in \mathcal{H}} \left(1 + \frac{1}{b}\right)^{-|D|} < \left(1 + \frac{1}{b}\right)^{-b},$$

then Avoider wins the $(b, 1)$ game $\mathcal{H}$.

We have

$$\begin{aligned} \sum_{D \in \mathcal{F}} \left(1 + \frac{1}{b}\right)^{-|D|} &\leq \binom{n}{\lceil n/k \rceil} \left(1 + \frac{1}{b}\right)^{-\binom{\lceil n/k \rceil}{2}} \leq (ek)^{\lceil n/k \rceil} 2^{-\binom{\lceil n/k \rceil}{2}/b} \\ &\leq 2^{\frac{n \log_2 e}{k} + \frac{n \log_2 k}{k} + \log_2(ek) - \frac{cn^2 k \log k}{2k^2 n} + \frac{nck \log k}{2kn}} = o(1). \end{aligned}$$

Applying Theorem 3.5 we conclude that there exists a winning strategy for CliqueAvoider, and thus Enforcer wins the k-colorability game.

Next, let $b > 2kn^{1+\frac{1}{2k-3}}$. We will provide Avoider with a strategy for building a $(k-1)$-degenerate graph (a graph G is called r-degenerate if there is an ordering of the vertices, $v_1, \ldots, v_n$, such that every vertex has at most r neighbors with a higher index). Clearly, that would entail Avoider's win in the k-colorability game as every $(k-1)$-degenerate graph is k-colorable.

Avoider will play several auxiliary minigames one after the other, never starting a new minigame before finishing the previous one, until all edges are claimed and the k-colorability game is over. Before we describe his strategy in detail, let us define two basic types of *minigames*.

Minigame Type I. For a set of vertices A, the (A)-*minigame* is played on those edges with both endpoints in A which are still unclaimed at the beginning of this minigame. Note that some edges within A may have already been claimed during previous minigames. We say that the vertices of A are *designated* to the (A)-minigame. When we say that *Avoider is playing the* (A)*-minigame*, we mean that Avoider is repeatedly claiming independent edges with both endpoints in A for as long as possible, that is, he extends a matching on A until it is no longer possible. When Avoider cannot further extend his matching, the (A)-minigame is over. At this point we denote the set of vertices of A incident with an edge, claimed by Avoider in **this** minigame by A_1, and let $A_2 = A \setminus A_1$. Note that by the end of the (A)-minigame, all edges with both endpoints in A_2 have already been claimed by one of the players.

Minigame Type II. Let A and B be two disjoint sets of vertices. The $(A : B)$-*minigame* is played on those edges with one endpoint in A and the other in B which are still unclaimed at the beginning of this minigame. Again, we assume that the (big) game is in progress, meaning that some of the edges between A and B may have already been claimed in previous minigames. We say that the vertices of B are *designated* to the $(A : B)$-minigame. When we say that *Avoider is playing the* $(A : B)$*-minigame*, we mean that Avoider is repeatedly claiming edges between A and B such that no vertex in B is incident with more than one of Avoider's edges claimed in **this** minigame. When this is no longer possible, the $(A : B)$-minigame is over. At this point, let B_1 denote the set of vertices of B that are incident with an edge claimed by Avoider in **this** minigame, and let $B_2 = B \setminus B_1$. Note that all edges with one endpoint in A and the other in B_2 have already been claimed by one of the players. The vertices in B_2 are called *finished*.

Now we can describe the way Avoider plays the game. We introduce *a minigame pool* $\mathcal{P}$, which is a dynamic collection of minigames that will be updated during the game – it will contain minigames waiting to be played by Avoider. At each moment $\mathcal{P}$ will contain at most one minigame of Type I and at most $k - 1$ minigames of Type II.

Avoider will maintain a partial ordering of the vertices, which he will refine whenever a minigame is over. In this partial ordering, the vertices designated to the same minigame will be incomparable to each other, the vertices designated to the lone minigame of Type I in the pool will be above all the other vertices and for any minigame $(A : B)$ of Type II, every vertex of A will be above every vertex of B.

Given a partial ordering, let the *up-degree* of v be the number of Avoider's edges (v, u) where either u is above v or they are incomparable.

To each minigame in the pool, we assign an integer parameter, that will help us keep track of the degeneracy of Avoider's graph throughout the game. Thus, instead of the (A)-minigame (or the $(A : B)$-minigame), we will consider the $(A)_l$-minigame (or the $(A : B)_l$-minigame) for an appropriate integer l. During play, Avoider maintains the following property: if a vertex is designated to a minigame with parameter l, then its up-degree is at most l.

In the beginning of the game $\mathcal{P}$ contains only one minigame – the $(V(K_n))_0$-minigame. The partially ordered set on $V(K_n)$ contains no relations.

We say that the size of an (A)-minigame is $\frac{1}{2}|A|^2$, and the size of an $(A : B)$-minigame is $|A| \cdot |B|$. Note that the size of a minigame is an upper bound on the number of edges it contains.

When the game is played, Avoider repeatedly chooses a minigame of the largest size in the pool $\mathcal{P}$, removes it from the pool, plays it to its end, and then updates $\mathcal{P}$ and the partial ordering as follows. If the minigame played was an $(A)_l$-minigame, then he places two new minigames into $\mathcal{P}$, the $(A_1)_{l+1}$-minigame and the $(A_1 : A_2)_l$-minigame. Furthermore, the designation of the vertices of A is lifted and replaced by that of the vertices of A_1 to the $(A_1)_{l+1}$-minigame and that of the vertices of A_2 to the $(A_1 : A_2)_l$-minigame. The partial order is refined by placing every vertex of A_1 above every vertex of A_2.

On the other hand, if the minigame played was an $(A : B)_l$-minigame, then Avoider places only the $(A : B_1)_{l+1}$-minigame back into $\mathcal{P}$. Furthermore, the designation of the vertices of B is lifted – the vertices of B_2 are finished and the vertices of B_1 are designated to the $(A : B_1)_{l+1}$-minigame. The partial order is not affected.

This shows that indeed in every stage of the game $\mathcal{P}$ will contain at most (in fact, exactly) one minigame of Type I.

Note that at any point of the game, every unclaimed edge is in exactly one of the minigames in $\mathcal{P}$. Moreover, every vertex of K_n will be either finished or designated to exactly one minigame in the pool.

After having played an $(A : B)$-minigame of Type II, the up-degree of the finished vertices, that is, the vertices of B_2, is fixed and in particular will not be increased in later stages of the game. This is because there are simply no more unclaimed edges which go to higher or incomparable vertices left. Indeed, the edges with both endpoints in B were all claimed during that minigame of Type I after which the vertices of B were designated to a Type II minigame. The edges (u, v), where $u \in B_2$ and v is above u were all claimed during the $(A : B)$-minigame. Furthermore, the up-degree of every vertex of B_2 was not changed during the $(A : B)$-minigame, so if the parameter of the $(A : B)$-minigame was l, then the up-degree of the vertices in B_2 is at most l at the end of the game.

It is clear that as long as Avoider follows this strategy and the parameter of every minigame in $\mathcal{P}$ is at most l, Avoider's graph is l-degenerate. Therefore, it suffices to prove that after the first minigame with parameter $k - 2$ is taken out of the pool to be played, Avoider plays at most one more move in the whole game. Note that whenever a minigame of Type II is played the size of the pool $\mathcal{P}$ is not changed, and whenever a minigame of Type I is played both the size of the pool $\mathcal{P}$ and the parameter of the new minigame of Type I are increased by one. It follows that proving the above will show that indeed, throughout the game, there will be at most $k - 1$ minigames of Type II in $\mathcal{P}$.

We will prove by induction on l that any minigame in the pool which has parameter $0 \leq l \leq k - 2$ is of size at most $n^2 \left(\frac{2k^2n^2}{b^2}\right)^l$. First, for the base step, note that the size of any minigame with parameter $l = 0$ is less than n^2. Now let us assume that l is an integer with $0 < l \leq k - 2$ and the induction hypotheses holds for all games with parameter less than l. For a minigame M in the pool with parameter l we consider three cases.

Case 1. M is an $(A_1)_l$-minigame that was inserted into the pool after the $(A)_{l-1}$-minigame has ended. Just before Avoider started playing the $(A)_{l-1}$-minigame there was no minigame in the pool of larger size. Since the total number of games in the pool was at most k, the total number of unplayed edges at that point was at most k times the size of the $(A)_{l-1}$-minigame. By the induction hypotheses, this is at most $kn^2\left(\frac{2k^2n^2}{b^2}\right)^{l-1}$. The number of edges Avoider will play during the $(A)_{l-1}$-minigame is certainly bounded from above by the total number of edges that Avoider will claim until the end of the whole k-colorability game, which is at most $\frac{kn^2}{b}\left(\frac{2k^2n^2}{b^2}\right)^{l-1}$. Avoider's strategy for the $(A)_{l-1}$-minigame guarantees that the set A_1 will be of size at most twice this much, and hence the $(A_1)_l$-minigame will be of size at most

$$\frac{1}{2}|A_1|^2 \leq \frac{1}{2}\left(\frac{2kn^2}{b}\left(\frac{2k^2n^2}{b^2}\right)^{l-1}\right)^2 \leq n^2\cdot\left(\frac{2k^2\cdot n^2}{b^2}\right)^l.$$

Case 2. M is an $(A_1 : A_2)_l$-minigame that was inserted into the pool after the $(A_1 \cup A_2)_l$-minigame has ended. The size of the $(A_1 : A_2)_l$-minigame is obviously bounded from above by the size of the $(A_1 \cup A_2)_l$-minigame, which we already upper-bounded in Case 1.

Case 3. M is an $(A : B_1)_l$-minigame that was inserted into the pool after the $(A : B)_{l-1}$-minigame has ended. As in Case 1, we can bound the number of edges Avoider will play during the $(A : B)_{l-1}$-minigame from above, by the total number of edges that Avoider will claim until the end of the whole k-colorability game. Thus, knowing that the $(A : B)_{l-1}$-minigame was of maximal size in $\mathcal{P}$ before it was played, we get that Avoider will make at most

$$\frac{k|A||B|}{b} \leq \frac{kn^2}{b}\left(\frac{2k^2n^2}{b^2}\right)^{l-1}$$

moves until the end of the game. Therefore, the size of B_1 is also at most that much. Since the size of A is at most n^2/b (the total number of vertices that can have positive degree in Avoider's graph), the total size of the $(A : B_1)_l$-minigame is at most $n^2\left(\frac{2k^2n^2}{b^2}\right)^l$. This concludes the induction step.

At the point when a minigame with parameter $k-2$ becomes the largest size in the pool, then the total number of edges to be played in the remainder of the game is at most $kn^2\left(\frac{2k^2n^2}{b^2}\right)^{k-2}$ which is less than b, meaning that Avoider will play at most one move before the game ends. However, at this point Avoider's graph is $(k-2)$-degenerate, so we are done. $\square$

Remark: The graph built by Avoider in the proof of Theorem 2.3 is clearly 3-colorable. It follows that if $k \geq 3$ and $b \geq 2n^{5/4}$, then Avoider can win the $(1, b)$ k-colorability game. For $k = 3$ this yields a better result than the one given in Theorem 3.4. Moreover, it is easy to see that if $b > n^{3/2}$, then Avoider can build a graph which consists of a matching and one additional edge; clearly such a graph is 2-colorable. Hence, using Enforcer's strategy from Theorem 3.4 we get

$$cn \leq f^{-}_{\mathcal{NC}^2_n} \leq f^{+}_{\mathcal{NC}^2_n} \leq n^{3/2},$$

for an appropriate constant $c > 0$.

4 Minor games

4.1 The Maker-Breaker minor game

In the Maker-Breaker version of the game, Maker's goal is to build a graph that contains a K_t-minor. The following theorem states that the threshold bias at which Maker's win turns into a Breaker's win in the K_t-minor game for every $3 \leq t \leq c\sqrt{n/\log n}$, for an appropriate constant c, is asymptotically $n/2$.

Theorem 4.1 *For every fixed $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$, such that if n is sufficiently large and $b \leq (1-\varepsilon)n/2$, then Maker has a winning strategy for the $(1,b)$ game $\mathcal{M}^t_n$ for every $t < C\sqrt{n/\log n}$.*

As a corollary we have the asymptotics of $b_{\mathcal{M}^t_n}$ for arbitrary fixed t.

Corollary 4.2 *Let $t \geq 3$ be a positive integer. Then*

$$b_{\mathcal{M}^t_n} = \frac{n}{2} - o_t(n).$$

The lower bound in the corollary follows from Theorem 4.1, while if $b \geq n/2$, then Breaker, as in the proof of Theorem 2.1, can force Maker to build a forest, which does not contain a K_t-minor for any $t \geq 3$.
In the proof of Theorem 4.1 we will use the following result of Kostochka and of Thomason.

Theorem 4.3 *([17], [26]) There exists a constant c' such that every graph of average degree at least $c'r\sqrt{\log r}$ admits a K_r-minor.*

Proof of Theorem 4.1 Assume that $b \leq (1-\varepsilon)n/2$. We will provide Maker with a strategy for building a graph that admits a large minor. Maker's strategy is divided into two stages. In the first stage that lasts exactly $m-1$ (where m is to be determined later) rounds, Maker builds a tree $T = (V, E)$ that satisfies the following properties:

1. $|V| = m \geq \varepsilon n$,
2. the degree of every $u \in V$ is at most 3,
3. there remain at least $\varepsilon^2 n^2/3$ unclaimed edges with both endpoints in V.

Maker's strategy for building such a tree is very simple: he starts by claiming an arbitrary edge and then, for as long as possible he claims a previously unclaimed edge (u, v) such that, in his current graph, u has degree 1 or 2 and v is isolated; such an edge, when exists, is chosen arbitrarily. Clearly this results in a tree with maximum degree at most 3. Now, assume that using this strategy Maker was able to build a tree on m vertices but could not extend it to a tree on $m + 1$ vertices (while maintaining the maximum degree criterion). This means that every edge (u, v) such that, in Maker's graph, u has degree 1 or 2 and v is isolated, must have been claimed by Breaker. Since Maker's graph is a tree, at least half its vertices have degree at most 2 and so Breaker must have claimed at least $\frac{m}{2}(n-m)$ edges. But at this point, Breaker has at most $m(1-\varepsilon)n/2$ edges entailing $m \geq \varepsilon n$. Furthermore, the number of edges with both endpoints in V that Breaker could have claimed is at most $m(1-\varepsilon)n/2-\frac{m}{2}(n-m) = \frac{m^2}{2} - \frac{m\varepsilon n}{2}$. It follows that there must be at least $\frac{m\varepsilon n}{2} - \frac{3m}{2} \geq \varepsilon^2 n^2/3$ unclaimed edges with both endpoints in V. This ends the first stage.

Before claiming edges in the second stage, Maker would like to partition T into roughly $\sqrt{n}$ connected components of roughly the same size. The following result of Krivelevich and Nachmias asserts that he can.

Lemma 4.4 *[18, Proposition 4.5] Let $G = (V, E)$ be a connected graph on r vertices with maximum degree at most k. Then for every positive integer l, there exist pairwise disjoint sets $V_1, \ldots, V_s \subseteq V$, with the following properties:*

1. *$lk \leq |V_i| \leq lk^2$ for every $1 \leq i \leq s$.*
2. *$\sum_{i=1}^{s} |V_i| \geq r - lk$.*
3. *$G[V_i]$ is connected for every $1 \leq i \leq s$.*

Using Lemma 4.4 with $r = m$, $l = \varepsilon^{-1}\sqrt{m}$ and $k = 3$ we conclude that at least $m - 3\varepsilon^{-1}\sqrt{m}$ of the vertices of T can be partitioned into parts $V_1, \ldots, V_s$ such that $3\varepsilon^{-1}\sqrt{m} \leq |V_i| \leq 9\varepsilon^{-1}\sqrt{m}$ and $T[V_i]$ is connected for every $1 \leq i \leq s$. Note that $\varepsilon\sqrt{m}/10 \leq s \leq \varepsilon\sqrt{m}/3$.

A pair (V_i, V_j) will be called *good* if there are at least $b + 1$ unclaimed edges (u, v) where $u \in V_i$ and $v \in V_j$. We claim that at least an $\varepsilon^2/20$ fraction of the total number of pairs is good. Indeed, assume for the sake of contradiction that there are less than $\varepsilon^2\binom{s}{2}/20$ good pairs, then there are at most

$$3m\varepsilon^{-1}\sqrt{m} + \sum_{i=1}^{s}\binom{|V_i|}{2} + b\binom{s}{2} + \varepsilon^2\binom{s}{2}(9\varepsilon^{-1}\sqrt{m})^2/20$$

unclaimed edges in V (the first term stands for the edges incident with vertices outside $\bigcup_{i=1}^{s} V_i$, the second term stands for edges inside the V_i's, the third term stands for unclaimed edges that might remain between any pair, even if it is not good, and the fourth term stands for edges between good pairs).For sufficiently large n this is strictly less than $\varepsilon^2 n^2/3$; this contradicts Maker's strategy for the first stage.

For every good pair (V_i, V_j), let $A_{i,j}$ be any set of $b+1$ unclaimed edges with one endpoint in V_i and the other in V_j. Trivially, by simply not claiming more than one edge from any $A_{i,j}$ (and not claiming edges outside the $A_{i,j}$'s for as long as possible), Maker can claim an edge of $A_{i,j}$ for at least half of the good pairs (V_i, V_j). In the second stage, Maker will use this strategy; denote Maker's graph at the end of the second stage by H. Consider the graph $\tilde{H}$ on the vertex set $V(\tilde{H}) = \{V_1, \ldots, V_s\}$, where (V_i, V_j) is an edge iff Maker has claimed an edge (x, y) such that $x \in V_i$ and $y \in V_j$. The average degree in $\tilde{H}$ is at least

$$\frac{\varepsilon^2 \binom{s}{2}/20}{s} \geq \varepsilon^3 \sqrt{m}/400 \geq \varepsilon^4 \sqrt{n}/400$$

and so by Theorem 4.3 it admits a K_t-minor for $t = C\sqrt{n/\log n}$, for an appropriate constant C. Since $T[V_i]$ is connected for every $1 \leq i \leq s$, H admits the same minor and the proof of Theorem 4.1 is complete. □

4.2 The Avoider-Enforcer minor game

In this game, Enforcer would like to make Avoider build a graph that admits a K_t-minor. As in the Maker-Breaker case, we allow t to be a function of n.

In the first theorem, we show that with a linear bias Enforcer can win the game for t which is as large as $c\sqrt{n/\log n}$.

Theorem 4.5 *If $b \leq n/19$, then Enforcer has a winning strategy for the $(1, b)$ game $\mathcal{M}_n^t$, for every $t < c\sqrt{n/\log n}$, where c is some absolute constant.*

In the proof of Theorem 4.5, we will use a result of Plotkin, Rao and Smith. Before we can state their result, we need the following definition.

Definition. Let $G = (V, E)$ be a graph on n vertices. A set $S \subset V$ is called a *separator* if every connected component of $G[V \setminus S]$ contains at most $2n/3$ vertices.

Theorem 4.6 *[24, Corollary 2.4] Let G be a graph on n vertices and let h be a function of n. If G does not have a separator of size at most $O(h\sqrt{n \log n})$, then G admits a K_h minor.*

Proof of Theorem 4.5 Assume that $b \leq n/19$; we will present a winning strategy for Enforcer. We will prove that Enforcer can make Avoider build a graph in which there is an edge between any two disjoint vertex sets of size at least $s = (1/3 - \varepsilon)n$ each, where $\varepsilon > 0$ is some small constant. Let $\mathcal{H}_n$ denote the hypergraph whose vertices are the edges of K_n and whose hyperedges are all the subgraphs of K_n, isomorphic to $K_{s,s}$. Enforcer's goal is to avoid claiming any hyperedge of $\mathcal{H}_n$. Using the criterion given by Theorem 3.5, we obtain

$$\sum_{D \in \mathcal{H}_n} \left(1 + \frac{1}{b}\right)^{-|D|} \leq \binom{n}{n/3}\binom{n}{n/3} 2^{-n^2/(9+\varepsilon')b} = o(1).$$

Thus, Enforcer can make sure that Avoider's graph will not contain a separator of size at most εn and so, by Theorem 4.6, Avoider's graph will admit a K_h minor for $h = \Theta(\sqrt{n/\log n})$. This concludes the proof of the Theorem. □

In the following theorem we show that even when the bias b is almost as large as $n/2$, Enforcer can still make Avoider claim all edges of a K_t-minor, where t is some constant power of n. On the other hand, we also prove an upper bound for the upper threshold bias of this game.

Theorem 4.7 *For every $\varepsilon > 0$ there exists a constant $a_0 = a_0(\varepsilon) > 0$, such that for every $4 \leq t \leq n^{a_0}$ we have*

$$\left(\frac{1}{2} - \varepsilon\right) n \leq f^-_{\mathcal{M}^t_n} \leq f^+_{\mathcal{M}^t_n} \leq 2n^{5/4}.$$

Before proving this theorem, we will state and prove a graph-theoretic lemma, which may also be of independent interest.

Lemma 4.8 *Let $G = (V, E)$ be a graph with average degree $2+\alpha$, for some $\alpha > 0$, and girth $g^* \geq \left(1 + \frac{2}{\alpha}\right)(4\log_2 t + 2\log_2 \log_2 t + c)$, where c is an appropriate constant. Then G admits a K_t-minor.*

Proof In the proof of the lemma we will use the following result of Kühn and Osthus (a similar result was also obtained by Diestel and Rempel [12]).

Theorem 4.9 *[19, Corollary 5] Let $t \geq 3$ be an integer. There exists a constant c such that every graph of minimum degree at least 3 and girth at least $4\log_2 t + 2\log_2 \log_2 t + c$ contains a K_t-minor.*

We repeatedly apply two deletion operations on G, which do not decrease the average degree and (trivially) do not decrease the girth. The first operation is the deletion of a vertex of degree at most one. Such an operation obviously does not decrease the average degree. In the second type of operation, given a path $u_1, u_2, \ldots, u_k$, with $k \geq 2 + 2/\alpha$, such that each of the internal vertices $u_2, u_3, \ldots, u_{k-1}$ has degree two in G, we remove $u_2, \ldots, u_{k-1}$. Again the average degree of the new graph is at least $2 + \alpha$. To verify this, let us assume that from a graph with e edges and v vertices satisfying $\frac{2e}{v} \geq 2 + \alpha$ we remove the internal vertices of a path with $k \geq 2 + 2/\alpha$ vertices. Then we obtain a graph with average degree at least

$$\frac{2(e - k + 1)}{v - (k - 2)} \geq 2 + \alpha,$$

as claimed. Let G_2 be the graph we obtain from G by repeated applications of these two operations. Since the average degree was not decreased in any step of the process, G_2 is not empty. It also follows that $\delta(G_2) \geq 2$, the girth of G_2 is at least g^* and every path of G_2, with internal vertices of degree two, is of length at most $1+2/\alpha$. Let G_3 denote the graph obtained from G_2 by contracting every path $u_1, u_2, \ldots, u_k$, such that u_i has degree two in G_2 for every $1 < i < k$, into a single edge. Again, this operation does not decrease the average degree and therefore G_3 is not empty. Clearly $\delta(G_3) \geq 3$. Moreover, since every such path in G_2 is of length at most $1+2/\alpha$ it follows that the girth of G_3 is at least $g = \frac{g^*}{1+2/\alpha} \geq 4\log_2 t + 2\log_2\log_2 t + c$. Applying Theorem 4.9 we conclude that G_3 admits a K_t-minor. Since G_3 was obtained from G by the deletion and contraction of edges (and the removal of isolated vertices), G admits the same minor and the proof is complete. $\square$

Proof of Theorem 4.7 Let $\varepsilon > 0$ and let $n = n(\varepsilon)$ be sufficiently large. Assume that $b \leq (1/2-\varepsilon)n$, and $4 \leq t \leq n^{a_0}$, where $a_0 = a_0(\varepsilon)$ is a "small" positive constant whose value will be determined later. If $b \leq n/19$, then by Theorem 4.5 Enforcer wins. Hence, from now on we can assume that $n/19 < b \leq (1/2-\varepsilon)n$.

Let $\alpha = \alpha(n,\varepsilon) > 0$ be the real number that satisfies

$$(1+\alpha)n = \frac{\binom{n}{2}}{(1/2-\varepsilon)n+1},$$

and let

$$\ell = 5a_0\left(1+\frac{2}{\alpha}\right)\log_2 n.$$

Enforcer's goal is to make sure that Avoider claims the edges of at most $o(n)$ cycles of length at most ℓ, throughout the game. In order to prove that Enforcer is able to do so, we will use a generalization of the Erdős-Selfridge Theorem, that applies to Maker-Breaker games in which the goal of Maker is to claim d (instead of just one) winning sets.

Theorem 4.10 *([1, 5]) If for positive integers d and b we have*

$$\sum_{A\in\mathcal{F}}(1+b)^{-|A|} < d,$$

then Breaker (as the first player) has a winning strategy in the $(1,b)$ Maker-Breaker game $\left\{\cup_{B\in F}B : F \in \binom{\mathcal{F}}{d}\right\}$ (a minimal hyperedge of $\left\{\cup_{B\in F}B : F \in \binom{\mathcal{F}}{d}\right\}$ is the union of d, not necessarily disjoint, hyperedges of $\mathcal{F}$).

Assume that on his first move, Avoider claims some edge e. Enforcer will claim the role of Breaker, as the first player, in the game on $E(K_n)\setminus\{e\}$, in which the winning sets are the cycle of length at most ℓ. We have

$$\begin{aligned}
\sum_{A\in\mathcal{F}}(1+b)^{-|A|} &\leq \sum_{A\in\mathcal{F}}\left(\frac{n}{19}\right)^{-|A|}\\
&\leq \sum_{3\leq i\leq \ell}\frac{(n)_i}{2i}\left(\frac{n}{19}\right)^{-i}\\
&\leq \sum_{3\leq i\leq \ell} 19^i\\
&\leq \ell\cdot n^{5a_0\left(1+\frac{2}{\alpha}\right)\log_2 19} = o(n),
\end{aligned}$$

where the last equality follows from an appropriate choice of a_0.

Applying Theorem 4.10, we conclude that in the entire game, Avoider will claim the edges of at most $o(n)$ cycles of length at most ℓ in $E(K_n) \setminus \{e\}$.

Let G_A denote Avoider's graph after he has played exactly $(1+\alpha)n$ moves. The average degree in G_A is $2+2\alpha$. We will prove that Enforcer has won the game already at this point. Remove the edge e, as well as one edge from every cycle of length at most ℓ in G_A; denote the resulting graph by G'_A. Clearly the girth of G'_A is at least ℓ. Moreover, the average degree in G'_A is at least $2+\alpha$ since we removed just $o(n)$ edges from G_A.

Since $\ell \geq \left(1+\frac{2}{\alpha}\right)(4\log_2 t + 2\log_2\log_2 t + c)$, where c is the constant given by Theorem 4.9, we conclude that G'_A admits a K_t-minor by Lemma 4.8. Clearly G_A admits the same minor.

If $b \geq 2n^{5/4}$, then playing as in the proof of Theorem 2.3 Avoider builds a graph that does not admit a K_4-minor. □

Remark: As in the remark following the proof of Theorem 3.4, if $b > n^{3/2}$, then Avoider can create a K_3-minor-free graph (i.e., a forest). Moreover, our strategy for Enforcer is valid also for $t = 3$.

5 Concluding remarks and open problems

Maker/Breaker thresholds. In Section 3 it was proved that $b_{\mathcal{NC}_n^k} = \Theta(n)$ for every fixed k. We believe that in fact the following stronger statement holds.

Conjecture 5.1 *There is a constant c, such that for every $k \geq 3$ and sufficiently large n we have*

$$b_{\mathcal{NC}_n^k} = \frac{(c+o(1))}{k\log k} n.$$

For both the planarity and the K_t-minor Maker/Breaker games, the second order terms are unknown; this is worth studying, in particular the dependence of the threshold $b_{\mathcal{M}_n^t}$ on t.

Asymptotic monotonicity of Avoider/Enforcer games. We say that an Avoider-Enforcer game $\mathcal{F}$ is *monotone*, if the existence of an Avoider's winning strategy for the $(1, q, \mathcal{F})$ game implies his win in the $(1, q+1, \mathcal{F})$ game, or equivalently, if $f^+_{\mathcal{F}} = f^-_{\mathcal{F}}$. Following [14], the function $f(n)$ is called an *asymptotic threshold bias* of the game $\mathcal{B}_n$ if both $f^-_{\mathcal{B}_n} = \Theta(f(n))$ and $f^+_{\mathcal{B}_n} = \Theta(f(n))$. If an asymptotic threshold bias exists, that is, if $f^-_{\mathcal{B}_n} = \Theta\left(f^+_{\mathcal{B}_n}\right)$, then the game $\mathcal{B}_n$ is called *asymptotically monotone*. In [14] it was conjectured that the perfect matching, and Hamiltonicity games are monotone. Here we conjecture the following.

Conjecture 5.2 *The Avoider-Enforcer non-planarity, non-k-colorability and K_t-minor games are asymptotically monotone.*

Avoider's strategies. For all of the Avoider-Enforcer games studied in this paper, there is a significant gap between the upper and lower bounds on the threshold biases. It would be interesting to close, or at least to reduce, these gaps. We do not believe that any of the upper bounds for $f^{+}_{\mathcal{F}}$, proved in this paper are tight. As a first step, it would be desirable to obtain an upper bound for the upper threshold bias $f^{+}_{\mathcal{M}^t_n}$ of the K_t-minor game which does depend on t.

The reason for the difficulty of finding upper bounds is due partly to the lack of a usable criterion for Avoider's win, similar to Theorem 3.5, in a $(1, q)$ game where $q > 1$. This is reminiscent of the difficulties that arise in the Hamiltonicity game $\mathcal{H}_n$ and in the perfect matching game $\mathcal{M}_n$ (see [14]). Let us recall that it is not even known whether Avoider can win $(1, n/100, \mathcal{M}_n)$ or $(1, n/100, \mathcal{H}_n)$. It seems quite reasonable that he can, knowing that Avoider wins $(1, n, \mathcal{M}_n)$ and $(1, n/2, \mathcal{H}_n)$ irregardless of his strategy.

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