Hamilton cycles in random graphs with a fixed degree sequence

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Abstract

Let $\mathbf{d} = d_1 \leq d_2 \leq \cdots \leq d_n$ be a non-decreasing sequence of n positive integers, whose sum is even. Let $\mathcal{G}_{n,\mathbf{d}}$ denote the set of graphs with vertex set $[n] = \{1, 2, \ldots, n\}$ in which the degree of vertex i is d_i. Let $G_{n,\mathbf{d}}$ be chosen uniformly at random from $\mathcal{G}_{n,\mathbf{d}}$. It will be apparent from Section 4.3 that the sequences we are considering will all be graphic. We give a condition on $\mathbf{d}$ under which we can show that **whp** $\mathcal{G}_{n,\mathbf{d}}$ is Hamiltonian. This condition is satisfied by graphs with exponential tails as well those with power law tails.

1   Introduction

Let $\mathbf{d} = d_1 \leq d_2 \leq \cdots \leq d_n$ be a fixed non-decreasing sequence of n positive integers, whose sum is even. Let $\mathcal{G}_{n,\mathbf{d}}$ denote the set of graphs with vertex set $V = [n] = \{1, 2, \ldots, n\}$ in which the degree of vertex i is d_i. Let $G_{n,\mathbf{d}}$ be chosen uniformly at random from $\mathcal{G}_{n,\mathbf{d}}$. It will be apparant from Section 4.3 that the sequences we are considering will all be graphic. When $d_i = r$ for $i \in [n]$ then this models a random r-regular graph $G_{n,r}$ and there is a large literature on this subject. We refer the reader to the survey by Wormald [15] for an excellent summary. By now we know much about the structure of random regular graphs.

For general $\mathbf{d}$, less is known. In many, but not all, cases we can estimate $|\mathcal{G}_{n,\mathbf{d}}|$; see Bender and Canfield [2], McKay and Wormald [9, 10]. We have the *configuration model* to study

them, Bollobás [3]. We know something of their connectivity properties, Molloy and Reed [12, 13] and Cooper [5]. (See also Cooper and Frieze [6] for the connectivity properties of random *digraphs* with a fixed degree sequence). They have been used in the context of *massive graph models* of telephone networks and the WWW, Aiello, Chung and Lu [1].

In a previous paper [7] we studied the chromatic number of $G_{n,\mathbf{d}}$. Let

$$D_k = d_n + d_{n-1} + \cdots + d_{n-k+1}$$

be the sum of the k largest degrees.

Let d denote the average degree and let

$$M_1 = D_n = dn \text{ and } M_2 = \sum_{i=1}^{n} d_i(d_i - 1) \leq \Delta M_1 \text{ where } \Delta = d_n.$$

We proved the following:

Theorem 1 *[7]*

1. *Suppose that there exist constants $1/2 < \alpha < 1$, ϵ, $K > 0$ and $\omega = \omega(n) \to \infty$ such that*

    (a)
    $$D_t \leq Kdn(t/n)^\alpha \tag{1}$$

    for $t \leq \epsilon n$.

    (b)
    $$\Delta^5 \leq M_2/\omega. \tag{2}$$

    Then there exists b_1 dependent only on α, ϵ, K such that **whp**[1]

    $$\chi(G_{n,\mathbf{d}}) \leq b_1 \frac{d}{\ln d}.$$

2. *Suppose only that $\Delta^4 \leq M_1/\omega$ (a weaker condition than 1(b)), then there exists b_2 such that* **whp**

    $$\chi(G_{n,\mathbf{d}}) \geq b_2 \frac{d}{\ln d}.$$

Condition 1(b) is required for the use the results of [11]. We will make the same assumption when we deal with Hamiltonicity. It may be possible to prove our results under the less stringent conditions of [10], but there are difficulties, as will be pointed to later.

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[1]A sequence of events $\mathcal{E}_n, n \geq 0$ is said to occur *with high probability* (**whp**) if $\lim_{n \to \infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

It is natural to ask whether there many types of degree sequences that satisfy the conditions of the first part of the theorem. It is easy to see that regular graphs are included. In [7] we showed that degree sequences satisfying (1) are important. We considered those with power law and exponential tails and showed that they satisfied the conditions of Theorem 1:

Power Law Tails: For integer $\ell \geq 1$ we let ν_ℓ denote the number of vertices of degree ℓ. Our assumption is that there are some constants $A > 0$ and $\zeta > 3$ such that for $\ell \geq (A/\epsilon)^{1/(\zeta-1)}$

$$\nu_\ell \leq \begin{cases} 0 & \ell \leq 1 \\ \lfloor A d \ell^{-\zeta} n \rfloor & 2 \leq \ell \leq n^{1/5}/\ln n \\ 0 & \ell > n^{1/5}/\ln n \end{cases}.$$

Here we have $\alpha = \frac{\zeta-2}{\zeta-1} > 1/2$.

Exponential Tails: For some constants $A > 0$ and $0 < \epsilon \ll \zeta < 1$ we have for $\ell \geq \lfloor \log_{1/\zeta}(Ad/\epsilon) \rfloor$

$$\nu_\ell \leq Ad\zeta^\ell n.$$

Note that **whp** the degree sequence of $G_{n,p}$, $p = c/n$, c constant, satisfies this condition.

In this paper we study the Hamiltonicity of $G_{n,\mathbf{d}}$. We prove the following:

Theorem 2 *Suppose that there exist constants $1/2 < \alpha < 1$, ϵ, $K > 0$ and $\omega = \omega(n) \to \infty$ such that (1) and (2) hold. Suppose also that*

A1 $d_1 \geq K_1 d^{1-1/4\alpha}$ *for sufficiently large K_1.*

A2 $d \leq n^\gamma$, *where γ is constant and $\gamma < \frac{2\alpha-1}{2\alpha+1}$.*

A3 d *is sufficiently large.*

Then $G_{n,\mathbf{d}}$ is Hamiltonian **whp**.

2 Configurations

We will work initially in the configuration model and then show how our result can be justified in the uniform model, i.e. $G_{n,\mathbf{d}}$. Let $W = [nd]$ be our set of *points* and let $W_i = [d_1 + \cdots + d_{i-1} + 1, d_1 + \cdots + d_i]$, $i \in [n]$, partition W. The function $\phi : W \to [n]$ is defined by $w \in W_{\phi(w)}$. Given a pairing F (i.e. a partition of W into $m = dn/2$ pairs) we obtain a (multi-)graph G_F with vertex set $[n]$ and an edge $(\phi(u), \phi(v))$ for each $\{u, v\} \in F$. Choosing a pairing F uniformly at random from among all possible pairings of the points of W produces a random (multi-)graph G_F.

This model is valuable because of the following easily proven fact: Suppose $G \in \mathcal{G}_{n,\mathbf{d}}$. Then

$$\mathbf{Pr}(G_F = G \mid G_F \text{ is simple}) = \frac{1}{|\mathcal{G}_{n,\mathbf{d}}|}.$$

It follows that if G is chosen randomly from $\mathcal{G}_{n,\mathbf{d}}$, then for any graph property $\mathcal{P}$

$$\mathbf{Pr}(G \in \mathcal{P}) \leq \frac{\mathbf{Pr}(G_F \in \mathcal{P})}{\mathbf{Pr}(G_F \text{ is simple})}. \tag{3}$$

In the next section it will be useful to consider that F is constructed by taking a random permutation ψ of W and then taking $F = \{e_j = \{\psi(2j-1), \psi(2j)\} : j = 1, 2, \ldots, m\}$. We can then for example use the following version of the Azuma-Hoeffding concentration inequality. Let random variable $Z = Z(\pi)$ be such that if π' is obtained from π by a single inversion then $|Z(\pi) - Z(\pi')| \leq \rho$. Then

$$\mathbf{Pr}(|Z - \mathbf{E}(Z)| \geq t) \leq 2e^{-t^2/m\rho^2}. \tag{4}$$

For a proof, see for example Lemma 11 of [8].

3   Non-Hamiltonian Degree Sequences

We observe next that there are simple degree sequences for which $G_{n,\mathbf{d}}$ is non-Hamiltonian **whp**.

Let $\nu = \lceil 2n/3 \rceil$ and let $d_1 = d_2 = \cdots = d_\nu = h$ and $d_{\nu+1} = d_{\nu+2} = \cdots = d_n = D$ for some h, D independent of n. Let $d = (\nu h + (n - \nu)D)/n$ be the average degree. In the configuration model we have $\mathbf{Pr}(G_F \text{ is simple}) = \Omega(1)$.

Let F be chosen randomly. Suppose that $D \gg h$. Let $V_1 = [\nu]$ and $V_2 = [n] \setminus V_1$. Next let

$$Z_1 = |\{j \in V_1 : j \text{ only has neighbours in } V_2\}|.$$

Now

$$\mathbf{E}(Z_1) = \nu \frac{((n-\nu)D)((n-\nu)D - 1) \cdots ((n-\nu)D - h + 1)}{(dn - 1)(dn - 3) \cdots (dn - 2h + 1)} \sim \frac{2}{3} \left(\frac{D}{D + 2h} \right)^h n.$$

An application of (4) can be used to show that $Z_1 \sim \frac{2}{3} \left(\frac{D}{D+2h} \right)^h n$ **whp**. Now choose D, h such that $\left(\frac{D}{D+2h} \right)^h > \frac{1}{2}$. Note that $D = Lh^2$ for large L will suffice.

Then **whp** G_F contains an independent subset inside V_1 with more than $|V_2|$ vertices. Clearly G_F is non-Hamiltonian in this case.

It is important to see how the above degree sequence violates the conditions of Theorem 2. We can choose K, ϵ, α such that (1) and (2) hold, but we will find that $d_1 = h$ is too small to satisfy Condition A1 of the theorem when d is large.

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4   Proof of Theorem 2

We randomly and independently colour the elements of W Red, Blue and Yellow, each with probability $1/3$. We use this colouring of W to induce a colouring of the edges of G_F. The Red edges will form an expander. The Blue edges will be used to ensure connectivity and the Yellow edges will be used to complete a Hamilton cycle. Let W_R be the set of Red elements and let $\widehat{F}_R = \{e \in F : e \cap W_R \neq \emptyset\}$ be the set of edges that are immediately coloured Red i.e. the edges for which at least one of its configuration points is Red. Let $\widehat{G}_R = (V, \widehat{F}_R)$ be the subgraph of G_F induced by $\widehat{F}_R$. For $S \subseteq F$ and $i \in [n]$ let $d(i, S) = |\{e \in S : e \cap W_i \neq \emptyset\}|$. Then let $V_0 = \left\{ i : d(i, \widehat{F}_R) \leq d_1/2 \right\}$ and let $F_R = \widehat{F}_R \cup \{e \in F : e \cap V_0 \neq \emptyset\}$ be the final set of Red G_F edges. Let $G_R = (V, F_R)$ be the subgraph of G_F induced by F_R. Note that G_R has minimum degree at least $d_1/2$.

Fix a configuration F. We first observe that with $Bin(n, p)$ being the binomially distributed random variable with parameters n and p,

$$\mathbf{Pr}(i \in V_0 \mid F) \leq \mathbf{Pr}\left(Bin\left(d_i, \frac{4}{9} + \frac{d_i}{M_1 - d_i} \right) \geq \frac{d_i}{2} \right),$$

where the $\frac{d_i}{M_1 - d_i}$ term accounts for the degree loss due to loops. Thus by Chernoff bounds,

$$\mathbf{E}(|V_0| \mid F) \leq \sum_{i=1}^{n} e^{-d_i/200}.$$

Now changing the colour of an element of W changes $|V_0|$ by at most 2 and so it follows from Azuma's inequality that for any $t > 0$,

$$\mathbf{Pr}\left(|V_0| \geq \sum_{i=1}^{n} e^{-d_i/200} + t \,\middle|\, F \right) \leq e^{-t^2/2dn}.$$

So we see that **whp**,
$$|V_0| \leq n_0 = ne^{-d_1/200} + (dn \log n)^{1/2}. \tag{5}$$

4.1   Expansion of G_R

For $X \subseteq F$ and $S \subseteq [n]$ let $N_X(S)$ be the set of vertices which are not in S, but have a G_X neighbour in S where $G_X = ([n], X)$. We abbreviate $N_{F_R}(S)$ to $N_R(S)$. Our aim is to prove

Lemma 1
$$\mathbf{Whp}\ |N_R(S)| \geq 2|S|\ for\ all\ S \subseteq [n], |S| \leq s_1 \tag{6}$$

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where

$$\frac{s_1}{n} = \min\left\{\left(\frac{d_1}{32dK^2e^{1+20/d_1}}\right)^{2\alpha-1-20/d_1}, (2K)^{-1/\alpha}, \epsilon\right\}. \tag{7}$$

At this point we remark that if K_1 is sufficiently large then Condition A1 of Theorem 2 implies

$$s_1^2 d_1^2 \geq 2000dn^2. \tag{8}$$

This inequality is enough to verify (17) below.

Proof
Case 1: $|S| \leq s_0$ where

$$s_0 = (12dK^2e^{1+20/d_1})^{-1/(2\alpha-1-20/d_1)}n.$$

Let $\mathcal{E}_S$ be the event that S contains at least $d_1|S|/20$ edges in G_F. Let $d_S = \sum_{i\in S} d_i$. Then

$$
\begin{aligned}
\mathbf{Pr}\left(\bigcup_{|S|\leq s_0}\mathcal{E}_S\right) &\leq \sum_{s=3}^{s_0}\sum_{|S|=s}\binom{d_S}{d_1s/10}\frac{(d_1s/10)!}{(d_1s/20)!2^{d_1s/20}}\left(\frac{1}{dn-d_1s/10}\right)^{d_1s/20} \tag{9}\\
&\leq \sum_{s=3}^{s_0}\sum_{|S|=s}\frac{d_S^{d_1s/10}}{(d_1s/20)!2^{d_1s/20}}\left(\frac{1}{dn-d_1s/10}\right)^{d_1s/20}\\
&\leq \sum_{s=3}^{s_0}\sum_{|S|=s}\left(\frac{d_S^2e}{d_1s/10}\times\frac{1}{dn-d_1s/10}\right)^{d_1s/20}\\
&\leq \sum_{s=3}^{s_0}\binom{n}{s}\left(\frac{12K^2d^2n^2(s/n)^{2\alpha}e}{d_1sdn}\right)^{d_1s/20}\\
&\leq \sum_{s=3}^{s_0}\left(\left(\frac{s}{n}\right)^{2\alpha-1-20/d_1}\frac{12dK^2e^{1+20/d_1}}{d_1}\right)^{d_1s/20}\\
&= o(1). \tag{10}
\end{aligned}
$$

Assume that $\mathcal{E}_X$ does not occur for $|X| \leq s_0$. G_R is a subgraph of G_F and so we can assume the corresponding event does not happen in G_R. Now suppose that $|S| \leq s_0/3$ and $|T| < 2|S|$ where $T = N_R(S)$. Let p, q be the number of Red G_F edges contained in S and from S to T respectively. Then $2p + q$ is equal to the total Red degree of S. Thus,

$$2p + q \geq d_1|S|/2 \text{ and } p \leq d_1|S|/20. \tag{11}$$

But this implies that $p + q \geq 9d_1|S|/20 > d_1|S \cup T|/20$, a contradiction.

Remark 1 *Suppose that we delete a set of edges with no vertex being incident to more than 20 and we add a set of edges with no vertex being incident to more than 25 of these.*

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Then (11) *would become* $2p+q \geq (d_1 - 40)|S|/2$ *and* $p \leq (d_1 + 500)|S|/20$. *But this implies that* $p + q \geq (9d_1 - 900)|S|/20 > (d_1 + 500)|S \cup T|/20$, *and we get the same contradiction. This remark will be used in translating our result from configurations to* $G_{n,\mathbf{d}}$. *We will use a switching argument that* **whp** *involves the deletion (and addition) of* $o(s_0)$ *edges.*

Case 2: $s_0/3 \leq |S| \leq s_1$.
Note first that $d_1 \gg \log d$ by assumption and we can then see from Condition A2 of Theorem 2 that $|V_0| \leq n_0 = o(s_0)$. We will assume therefore from now on that $s_0/3 \geq 100|V_0|$.

Fix sets S, T with $|T| = 2|S| = 2s$ and suppose that $N_R(S) \subseteq T$. Now $|V_0| \leq |S|/100$ and so if $\widehat{d}_S$ denotes the total degree of S in $\widehat{G}_R$ then $\widehat{d}_S \geq 99d_1 s/200$.

Let p, q be as above, but defined with respect to $\widehat{G}_R$. At this point we remark that $d_S \leq dn/2$, which follows from (1) and the second term in the expression (7) for s_1.

The probability S contains p edges of $\widehat{G}_R$ and there are q edges of $\widehat{G}_R$ from S to T is at most

$$(12)$$

$$
\begin{aligned}
\binom{d_S}{2p} \frac{(2p)!}{p!2^p} \left(\frac{1}{dn - 2p + 1} \right)^p \left(\frac{Kdn(2s/n)^\alpha}{dn - 2p} \right)^q &\leq \frac{2(dn)^{2p+q}(K(s/n)^\alpha)^{2p}}{p!2^p(dn-2p)^{p+q}}(K(2s/n)^\alpha)^q \\
&\leq (K(s/n)^\alpha)^{\widehat{d}_S} 2^{(1+\alpha)q} \left(\frac{dne}{p} \right)^p \\
&\leq (K(s/n)^\alpha)^{\widehat{d}_S} 2^{(1+\alpha)q} \left(\frac{2dne}{\widehat{d}_S} \right)^{\widehat{d}_S/2} \\
&\leq \left(\left(\frac{s}{n} \right)^{2\alpha - 1} \frac{32K^2de}{d_1} \right)^{\widehat{d}_S/2}.
\end{aligned}
$$

Explanation for (12): We choose $2p$ members of W to make up the $\widehat{G}_R$ edges of S in at most $\binom{d_S}{2p}$ ways. We choose a partition of these points into p pairs in $\frac{(2p)!}{p!2^p}$ ways. The quantity $1/(dn - 2p + 1)^p$ bounds the probability that these pairs exist in $\widehat{G}_R$. Then $(d_T/(dn - 2p))^q$ bounds the probability that the remaining q points are paired in T. We use (1) to bound d_T. We use $2p + q \leq d_S \leq dn/2$ to simplify the calculations.

So the probability that there exists such a pair S, T is at most

$$
\sum_{p,q} \sum_{s=s_0/3}^{s_1} \binom{n}{s}\binom{n}{2s} \left(\left(\frac{s}{n} \right)^{2\alpha-1} \frac{32K^2de}{d_1} \right)^{d_1 s/5} \tag{13}
$$

$$
\leq d^2 n^2 \sum_{s=s_0/3}^{s_1} \left(\left(\frac{s}{n} \right)^{2\alpha-1-20/d_1} \frac{32K^2de^{1+20/d_1}}{d_1} \right)^{d_1 s/5}
$$

$$
= o(1).
$$

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□

Remark 2 *Replacing $\binom{n}{2s}$ by $\binom{n}{2.001s}$ in (13) allows us to claim that* **whp** $|N_R(S)| \geq 2|S|$ *in Case 2, even after deleting $o(s_0)$ edges. More precisely, the sum in (13) will evaluate to $o(1)$. Thus we see that* **whp** $|N_R(S)| \geq 2.001|S|$ *for all $s_0/3 \leq |S| \leq s_1$. Then after the deletion of $o(s_0)$ edges we will still have $|N_R(S)| \geq 2|S|$ for all $s_0/3 \leq |S| \leq s_1$. This remark will be used in translating our result from configurations to $G_{n,\mathbf{d}}$. We will use a switching argument that* **whp** *involves the deletion (and addition) of $o(s_0)$ edges.*

Going back to our construction of G_R, let $\widehat{F}_B$ denote the set of pairs $\{v, w\} \in F$ in which v and w are both coloured Blue and let $F_B = \left\{ e = \{v, w\} \in \widehat{F}_B : \phi(v), \phi(w) \notin V_0 \right\}$ where ϕ is defined in Section 2. Let $W_B = \bigcup_{e \in F_B} e$ and $V_B = \phi(W_B)$ and note that F_B is a random pairing of W_B. Now let

$$V_1 = \{i \in [n] : d(i, F_B) \leq d_1/10\} \subseteq V_0 \cup N_F(V_0) \cup \left\{i \in [n] : d(i, \widehat{F}_B) \leq d_1/10\right\}.$$

Arguing as for (5) we have that **whp**

$$|N_F(V_0)| \leq nd_1 e^{-d_1/200} + (dn \log n)^{1/2} \text{ and } \left| \left\{i \in [n] : d(i, \widehat{F}_B) \leq d_1/10\right\} \right| \leq ne^{-d_1/1800},$$

in which case we can assume that

$$|V_1| \leq 2nd_1 e^{-d_1/200} + 2(dn \log n)^{1/2} \leq s_1. \tag{14}$$

Definition 1 *A graph $G = (V, E)$ is called a (k, c)-expander if $|N(U)| \geq c|U|$ for every subset $U \subseteq V(G)$ of cardinality $|U| \leq k$. (Here $N(S)$ is the set of vertices which are not in S, but have a neighbour in S).*

We have shown that **whp** G_R is an $(s_1, 2)$-expander. Thus **whp** each component of G_R has size at least $3s_1$. We can now show that adding the extra edges F_B will **whp** connect these components and thus show that $G_{RB} = G_R + F_B$ is connected **whp**. We will need to prove just a little more. Indeed, let $C_1, C_2, \ldots, C_\rho$, $\rho \leq n/3s_1$ be the components of G_R. We see from (14) that if $C_i' = C_i \setminus V_1$ then $|C_i'| \geq 2s_1$ for $i = 1, 2, \ldots, \rho$. If $x \in W_B$ and $\phi(w) \in C_i'$ then the probability it is paired with $y \in W_B, \phi(y) \in C_j$ is at least $\frac{s_1 d_1 - 5t}{5dn}$ given that we have made t pairings of $x \in W_B$. Thus by considering the first $s_1 d_1/10$ such pairings, we see that $|F_B \cap (C_i \times C_j)|$ dominates $Bin(s_1 d_1/10, s_1 d_1/10dn)$. Thus by a Chernoff bound,

$$\mathbf{Pr}(G_{RB} \text{ is not connected}) \leq$$
$$\mathbf{Pr}\left(\exists 1 \leq i < j \leq \rho : |F_B \cap (C_i \times C_j)| \leq \frac{s_1^2 d_1^2}{200dn}\right) \leq \frac{n^2}{9s_1^2} \exp\left\{-\frac{s_1^2 d_1^2}{800dn}\right\} = o(1). \tag{15}$$

4.2 Pósa's Lemma and its consequences

Definition 2 *Let $G = (V, E)$ be a non-Hamiltonian graph with a longest path of length ℓ. A pair $(u, v) \notin E(G)$ is called a* hole *if adding (u, v) to G creates a graph G' which is Hamiltonian or contains a path longer than ℓ.*

Lemma 2 *Let G be a non-Hamiltonian connected $(k, 2)$-expander. For every longest path of G there is a set A of size k and sets $B_a, a \in A$, each of size k such that (a, b) is a hole for each $b \in B_a$.*

Proof   Let $P = (v_0, \ldots, v_k)$ be a longest path in graph G. A *Pósa rotation* of P [14] with v_0 fixed gives another longest path $P' = (v_0, \ldots v_i v_k \ldots v_{i+1})$ created by adding edge (v_k, v_i) and deleting edge (v_i, v_{i+1}). Let $A = END_G(v_0, P)$ be the set of endpoints obtained by a sequence of Pósa rotations starting with P, keeping v_0 fixed and using an edge (v_k, v_i) of G.

Each vertex $a \in A$ can then be used as the initial vertex of another set of longest paths whose endpoint set is $B_a = END_G(v_j, P)$, this time using a as the fixed vertex, but again only adding edges from G.

The Pósa condition (see, e.g., [4], Ch.8.2)

$$|N(END_G(v, P))| \leq 2|END_G(v, P)| - 1$$

for $v \in END_G(P)$ together with the fact that G is a $(k, 2)$-expander implies that $|END_G(v, P)| > k$. The connectivity of G implies that closing a longest path to a cycle either creates a Hamilton cycle or creates a longer path. For every $v \in END_G(P)$ and for every $u \in END_G(v, P)$, a pair (u, v) is a hole. □

Going back to our construction of $\widehat{F}_B$, let $\widehat{F}_Y$ denote the set of pairs $\{v, w\} \in F$ in which v and w are coloured Yellow and let $F_Y = \left\{ e = \{v, w\} \in \widehat{F}_Y : \phi(v), \phi(w) \notin V_0 \right\}$. Let $W_Y = \bigcup_{e \in F_Y} e$ and $V_Y = \phi(W_Y)$ and note that F_Y is a random pairing of W_Y. Now let

$$V_2 = \{i \in [n] : d(i, F_Y) \leq d_1/10\}.$$

Note that

$$V_2 \subseteq V_0 \cup N_F(V_0) \cup \left\{ i \in [n] : d(i, \widehat{F}_Y) \leq d_1/10 \right\}.$$

Arguing as for (5) we have that **whp**

$$|V_2| \leq 2nd_1 e^{-d_1/200} + 2n^{3/4} \leq s_1/2. \tag{16}$$

Suppose that G_{RB} is not Hamiltonian. We start with a longest path P of G_{RB} and construct $A, B_a, a \in A$ as in Lemma 2 using only the edges of G_{RB}. Now choose $a \in A \setminus V_2$ and

one by one expose the F_Y pairings involving W_a. There are at least $d_1/10$ (Yellow,Yellow) points to be paired and the probability of a pairing with a point in $W_b, b \in B_a \setminus V_2$ is at least $p_t = \frac{s_1 d_1/20 - 2t - 1}{dn - 2t - 1} = \frac{s_1 d_1 - 40t - 10}{20dn - 40t - 20}$, given t previous attempts at such a pairing. If there is a pairing then we add the corresponding edge e and either complete a Hamilton cycle or find a longer path P' in $G_{RB} + e$.

We can then repeat this process with P'. We claim that **whp** we can continue this process until we have added enough Yellow edges to create a Hamilton cycle. To see this we couple the process with a sequence of Bernoulli trials where the probability of success is p_t. It is sufficient to show that **whp** there will be at least n successes before we make $t_1 = s_1 d_1/80$ trials. But $t \le t_1$ implies $p_t \ge \frac{s_1 d_1 - 40}{40dn - s_1 d - 1 - 40}$ and so the expected number of successes in the first t_1 trials is at least

$$\frac{t_1(s_1 d_1 - 40)}{40dn - s_1 d_1 - 40} n \ge 2n, \qquad see\ (8). \tag{17}$$

We can therefore claim that **whp** there are at least $3n/2$ successes in the first t_1 trials and our claim follows.

This shows that G_F is Hamiltonian **whp**. We now translate the result to $G_{n,\mathbf{d}}$.

4.3   From configurations to graphs

It is at this point that we appeal to some results from McKay and Wormald [11]. Where possible, we will use the terminology and notation of that paper. A *loop* of a pairing F is a pair $\{u, v\}$ such that $\phi(u) = \phi(v)$. A *double pair* of F is a pair of pairs $\{u_1, v_1\}, \{u_2, v_2\} \in F$ such that $\phi(u_1) = \phi(u_2)$ and $\phi(v_1) = \phi(v_2)$. A *double loop* of F is a pair of pairs $\{u_1, v_1\}, \{u_2, v_2\}$ such that $\phi(u_1) = \phi(v_1) = \phi(u_2) = \phi(v_2)$. A *triple pair* is a triple of pairs $\{u_i, v_i\}, i = 1, 2, 3$ such that $\phi(u_1) = \phi(u_2) = \phi(u_3)$ and $\phi(v_1) = \phi(v_2) = \phi(v_3)$.

Condition (b) of Theorem 1 has played no part as yet. We do however need it to apply the results of [11]. So, in the lemmas that follow, we will assume that Condition (b) of Theorem 1 holds.

Lemma 3 *(Lemma 2 of [11])*
The probability that F contains at least one triple pair is $O(\Delta^2 M_2^2/M_1^3) = o(1)$ and the probability of at least one double loop is $O(\Delta^2 M_2/M_1^2) = o(1)$.

Let now l denote the number of loops and r denote the number of double pairs in F.

Lemma 4 *(Lemma 3 of [11])*
If $\lambda(n) \to \infty$ then **whp**
$$l \le 2\Delta + \lambda \ and \ r \le \Delta^2 + \lambda. \tag{18}$$
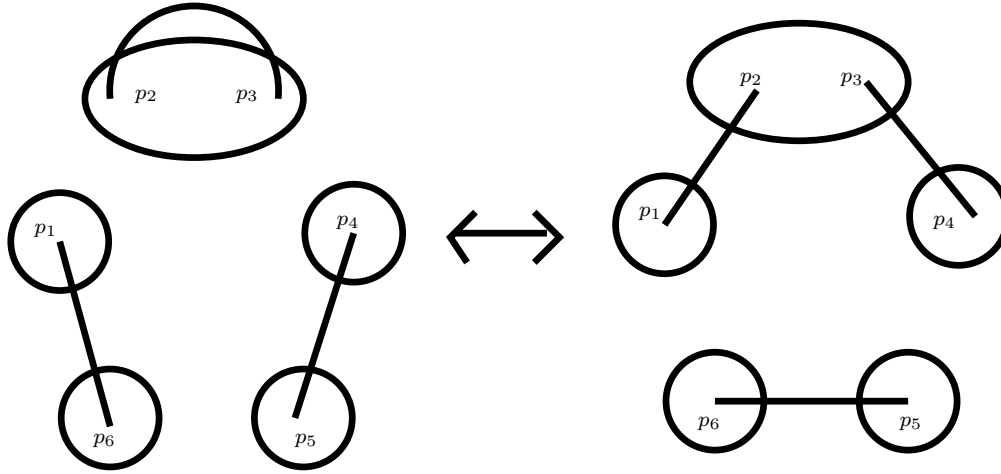
Figure 1:

We define the following two operations on a pairing: If $\phi(u) = i$ then we say that u is in *cell i*.

I *l*-switching.

Take pairs $\{p_1, p_6\}, \{p_2, p_3\}, \{p_4, p_5\}$ where $\{p_2, p_3\}$ is a loop, and $p_1, \ldots, p_6$ are in five different cells. Replace these pairs by $\{p_1, p_2\}, \{p_3, p_4\}, \{p_5, p_6\}$. In this operation, none of the pairs created or destroyed is permitted to be part of a double pair. (See Figure 1).

II *r*-switching.

Take pairs $\{p_1, p_5\}, \{p_2, p_6\}, \{p_3, p_7\}, \{p_4, p_8\}$ where $\phi(p_2) = \phi(p_3)$ and $\phi(p_6) = \phi(p_7)$, but the cells containing $p_1, p_2, p_4, p_5, p_6, p_8$ are all distinct. Replace these pairs by $\{p_1, p_2\}, \{p_3, p_4\}, \{p_5, p_6\}, \{p_7, p_8\}$. In this operation, none of the pairs created or destroyed (other than the pairs $\{p_2, p_6\}, \{p_3, p_7\}$) is permitted to be part of a multiple pair. (See Figure 2).

A *forward l*-switching is an *l*-switching as described, and a *backward l*-switching is the reverse operation. We use the same convention for *r*-switchings. Note that a forward *l*-switching always reduces the number of loops by one and does not create or destroy double pairs. Similarly, a forward *r*-switching reduces the number of double pairs by one and neither creates nor destroys loops.
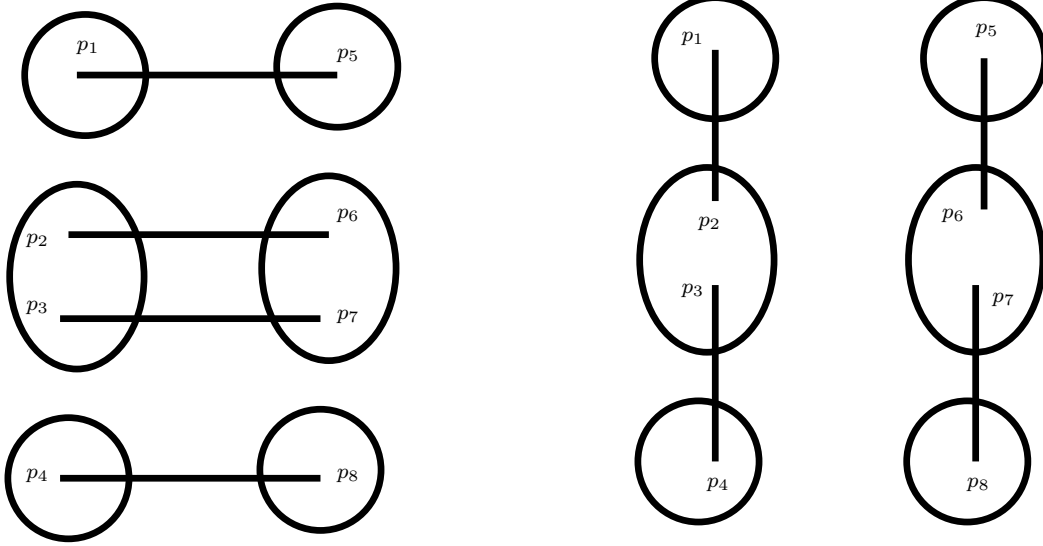
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Figure 2:

Now let $\mathcal{C}_{l,r}$ denote the set of pairings F with l loops, r double pairs and no triple pairs or double loops.

Lemma 5 *(Lemma 4 of [11])*
Denote an operation taking an element of $\mathcal{C}_{i,j}$ to an element $\mathcal{C}_{k,l}$ by $\mathcal{C}_{i,j} \to \mathcal{C}_{k,l}$. For each of the following operations, we bound the number, m, of ways of applying the operation to a fixed F.

(1) *Forward l-switching $\mathcal{C}_{l,r} \to \mathcal{C}_{l-1,r}$:*

$$2lM_1^2 \geq m \geq 2lM_1^2\left(1 - O\left(\frac{\Delta^2 + l + r}{M_1}\right)\right).$$

(2) *Backward l-switching $\mathcal{C}_{l-1,r} \to \mathcal{C}_{l,r}$:*

$$M_1 M_2 \geq m \geq M_1 M_2\left(1 - \frac{\Delta(6(l+2r) + \Delta l)}{M_2} - \frac{2\Delta(\Delta+2)}{M_1}\right).$$

(3) *Forward r-switching $\mathcal{C}_{0,r} \to \mathcal{C}_{0,r-1}$:*

$$4rM_1^2 \geq m \geq 4rM_1^2\left(1 - O\left(\frac{\Delta^2 + r}{M_1}\right)\right).$$

(4) *Backward r-switching $\mathcal{C}_{0,r-1} \to \mathcal{C}_{0,r}$:*

$$M_2^2 \geq m \geq M_2^2\left(1 - \frac{\Delta(16r + 9\Delta + 3 + \Delta^2)}{M_2}\right).$$

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Now consider the following algorithm for generating a member of $\mathcal{G}_{n,\mathbf{d}}$:

1. Generate a random pairing F.

2. If there is a double loop or a triple pair, output $\perp$ – construed as failure.

3. If the number of loops $l \geq 2\Delta + \log n$ or the number of double pairs $r \geq \Delta^2 + \log n$, output $\perp$ – construed as failure.

4. $F_0 \leftarrow F$.

5. For $i = 1$ to l choose a random forward l-switching on F_{i-1}, creating $F_i \in \mathcal{C}_{l-i,r}$.

6. For $i = l+1$ to $l+r$ choose a random forward r-switching on F_{i-1}, creating $F_i \in \mathcal{C}_{0,r-(i-l)}$.

7. Output $G^* = G_{F_{l+r}} \in \mathcal{G}_{n,\mathbf{d}}$.

For each l, r satisfying (18), with $\lambda = \log n$, and $G \in \mathcal{G}_{n,\mathbf{d}}$, there are by Lemma 5(2),(4)

$$(M_1 M_2)^l M_2^{2r} \left(1 + O\left(\frac{\Delta l(\Delta l + r)}{M_2} + \frac{\Delta^2 l}{M_1} + \frac{\Delta r(\Delta^2 + r)}{M_2} \right) \right)$$

sequences of switchings which yield G. Each of these has probability

$$((2M_1^2)^l l!(4M_1^2)^r r!)^{-1} \left(1 + O\left(\frac{l(\Delta^2 + l + r)}{M_1} + \frac{r(\Delta^2 + r)}{M_1} \right) \right)$$

of being followed by the algorithm, given l, r.

Thus if Condition 1(b) holds, then **whp** the algorithm outputs a graph in $\mathcal{G}_{n,\mathbf{d}}$ and

$$\mathbf{Pr}(G^* = G) = (1 + o(1)) \sum_{l=0}^{2\Delta+\log n} \sum_{r=0}^{\Delta^2+\log n} \frac{M_1^l M_2^{2r+l}}{2^{l+2r} M_1^{2(l+r)} l! r!} \mathbf{Pr}(l \text{ loops}, r \text{ double pairs})$$

and so for $G_1, G_2 \in \mathcal{G}_{n,\mathbf{d}}$

$$\mathbf{Pr}(G^* = G_1) = (1 + o(1)) \mathbf{Pr}(G^* = G_2).$$

Given this, we only have to show that **whp** G^* is Hamiltonian.

Let H_1 be the graph consisting of those edges of G_F that are *deleted* in going from G_F to G^*.

Lemma 6 Whp H_1 *has at most* $5\Delta^2 = o(n)$ *edges and has maximum degree at most 19.*

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Proof   The fact that H_1 has at most $5\Delta^2$ edges **whp** follows immediately from Lemma 4 and from the fact that each switching deletes at most 4 edges. Now every edge of F at distance ≥ 2 from the loop or double edge can be used as one of the two edges destroyed by the two types of switching. Thus vertex i has probability

$$O\left(\frac{d_i}{M_1 - \Delta^2}\right) = O\left(\frac{\Delta}{M_1}\right)$$

of being on an H_1-edge created by any switching, regardless of the history of the switchings to this point. So if H_1' is the subgraph of H_1 induced by these edges (i.e. the non-loops and multiple edges) then for some constant $c > 0$, assuming due to Lemma 4 that G^* satisfies (18):

$$\mathbf{Pr}(\Delta(H_1') \geq 10) \leq n\binom{\Delta^2 + 2\Delta + 2\log n}{10}\left(\frac{c\Delta}{M_1}\right)^{10} \leq \frac{c^{10}\Delta^{30}(\log n)^{10}n}{M_1^{10}}$$
$$\leq \frac{c^{10}(\log n)^{10}M_2^6 n}{\omega^6 M_1^{10}} \leq \frac{c^{10}(\log n)^{10}\Delta^6 n}{\omega^6 M_1^4} = o(1).$$

after using $\Delta = o(n^{1/3})$, from (2), and $M_1 \geq n$.

We can estimate the expected number of vertices incident with ≥ 10 multiple edges by

$$n\Delta^{20}\left(\frac{\Delta}{M_1}\right)^{10} \leq \frac{nM_2^6}{\omega^6 M_1^{10}} \leq \frac{n\Delta^4}{\omega^6 M_1^4} = o(1).$$

There are no double loops **whp** and so **whp** $\Delta(H_1) \leq 9 + 9 + 1 = 19$. $\square$

Remark 3 *It is probably a good time to remark that (1) with $k = 1$ implies that $\Delta \leq Kdn^{1-\alpha}$. And then we get from A2 of Theorem 2 that $\Delta^2 = o(s_0)$.*

Next let H_2 denote the graph induced by the set of edges added in going from G_F to G^*.

Lemma 7 Whp H_2 *has maximum degree at most 25.*

Proof   Except for loops and multiple edges, each edge of H_2 can be paired with an edge of H_1. Thus $\Delta(H_2) \leq \Delta(H_1) + \Delta_l + \Delta_m$ where **whp** $\Delta_l = 1$ is the maximum number of loops at a vertex and Δ_m is the maximum number of multiple edges at a vertex. Our result follows from $\Delta_m \leq 5$ **whp**. Indeed,

$$\mathbf{Pr}(\Delta_m \geq 6) \leq n\Delta^{12}\left(\frac{\Delta}{dn}\right)^6 = n\frac{\Delta^{18}}{(dn)^6} \leq n\frac{M_2^{18/5}}{(dn)^6} \leq n\frac{\Delta^{18/5}}{(dn)^{12/5}} \leq n^{-1/5}.$$

$\square$

Let us now see how this affects the argument used in Sections 4.1 and 4.2. Assume now that the edges of H_1 are deleted and the edges of H_2 are added.

Going back to Remarks 1 and 3 we see that the argument in Case 1 can handle the deletion of the edges of H_1 and the addition of the edges in H_2.

Going back to Remarks 2 and 3 we see that the argument in Case 2 can handle the deletion of the edges of H_1.

Going back to (17) we see that we can afford to give up $o(n)$ successes in the first t_1 trials due to deletion of the edges of H_1.

Going back to (15) we see that we can afford to delete $o(n)$ edges without disconnecting G_{RB}.

This completes the proof of Theorem 2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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