Approximation Algorithms for Cycle Packing Problems

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Abstract

The *cycle packing number* $\nu_c(G)$ of a graph G is the maximum number of pairwise edge-disjoint cycles in G. Computing $\nu_c(G)$ is an NP-hard problem. We present approximation algorithms for computing $\nu_c(G)$ in both the undirected and directed cases. In the undirected case we analyze the modified greedy algorithm suggested in [4] and show that it has approximation ratio $O(\sqrt{\log n})$ where $n = |V(G)|$, and this is tight. This improves upon the previous $O(\log n)$ upper bound for the approximation ratio of this algorithm. In the directed case we present a $\sqrt{n}$-approximation algorithm. Finally, we give an $O(n^{2/3})$-approximation algorithm for the problem of finding a maximum number of edge-disjoint cycles that intersect a specified subset S of vertices. Our approximation ratios are the currently best known ones and, in addition, provide bounds on the *integrality gap* of standard LP-relaxations to these problems.

1 Introduction

We consider the following fundamental problem in Algorithmic Graph Theory. Given a graph (digraph) G, how many edge-disjoint (directed) cycles can be packed into G? Define the *cycle packing number* $\nu_c(G)$ of G to be the maximum size of a set of edge-disjoint cycles in G. The *maximum cycle packing problem* is to find a set of $\nu_c(G)$ edge-disjoint cycles in G. Problems concerning packing edge-disjoint or vertex-disjoint cycles in graphs and digraphs have been studied extensively (see, e.g., [1, 4, 9]).

It is well known that computing $\nu_c(G)$ (and hence finding a maximum cycle packing) is an NP-hard problem in both the directed and undirected cases. Even the very special case of deciding whether a graph (digraph) has a triangle decomposition is known to be NP-Complete (see, e.g. [6] for a more general theorem on the NP-Completeness of such decomposition problems). Thus, approximation algorithms are of interest. A ρ-*approximation algorithm* for a maximization problem is a polynomial time algorithm that produces a solution of value at least $1/\rho$ times the value of an optimal solution; ρ is called the *approximation ratio* of the algorithm.

A recent result of Caprara, Panconesi, and Rizzi [4] shows that by slightly modifying the greedy algorithm one obtains an $O(\log n)$-approximation algorithm for the undirected maximum cycle packing problem. Our first result is an improved analysis of this modified greedy algorithm showing that the approximation ratio is $O(\sqrt{\log n})$. In particular, we obtain the following result.

THEOREM 1.1. *There exists an $O(\sqrt{\log n})$ approximation algorithm for the undirected maximum cycle packing problem.*

We also prove that the approximation guarantee of the modified greedy algorithm is $\Omega(\sqrt{\log n})$. The approximation ratio in Theorem 1.1 is currently the best known one for the maximum undirected cycle packing problem.

Our next two results are for directed graphs.

THEOREM 1.2. *There exists a $\sqrt{n}$-approximation algorithm for the directed maximum cycle packing problem.*

The algorithms in Theorems 1.1 and 1.2 can be easily adjusted to the capacitated version of the problems as well, but, for simplicity of exposition, we state them for the uncapacitated case.

Finally, we consider the *maximum S-cycle packing* problem in directed graphs: given a directed graph G and a subset S of its vertices, find among the cycles that intersect S (henceforth, *S-cycles*) a maximum number $\nu_c(G, S)$ of edge-disjoint ones. We note that on directed simple graphs, the maximum S-cycle packing problem is a special case of the extensively studied *edge-disjoint paths* problem. See [5] for an $O(n^{4/5})$-approximation algorithm and [10] for an $O(n^{2/3}\log^{2/3} n)$-approximation algorithm for the edge-disjoint paths problem in directed graphs.

THEOREM 1.3. *There exists an $O(n^{2/3})$-approximation algorithm for the directed maximum S-cycle packing problem on simple digraphs.*

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Given a graph $G = (V, E)$, the *fractional cycle packing* in G is a function ψ from the subset $\mathcal{C}$ of cycles in G to $[0, 1]$ satisfying $\sum_{e \in C \in \mathcal{C}} \psi(C) \leq 1$ for each $e \in E$. Letting $|\psi| = \sum_{C \in \mathcal{C}} \psi(C)$, the *fractional cycle packing number* $\nu_c^*(G)$ of G is defined to be the maximum of $|\psi|$ taken over all fractional cycle packings ψ in G. The *cycle cover number* $\tau_c(G)$ of G is the minimum number of edges whose deletion makes G acyclic. Clearly, $\nu_c(G) \leq \nu_c^*(G) \leq \tau_c(G)$ for any graph/digraph G.

The approximation ratios in Theorems 1.1, 1.2, and 1.3 provide bounds on the integrality gap of the standard LP-relaxations to the problems. Specifically, each of the algorithms computes a packing $\mathcal{C}$ so that: $|\mathcal{C}|/\nu_c^*(G) = \Omega(1/\sqrt{\log n})$ in Theorem 1.1, $|\mathcal{C}|/\nu_c^*(G) \geq 1/\sqrt{n}$ in Theorem 1.2, and $|\mathcal{C}|/\tau_c(G, S) = \Omega(n^{-2/3})$ in Theorem 1.3, where $\tau_c(G, S)$ is the minimum number of edges needed to cover all S-cycles in G.

In the following three sections we prove Theorems 1.1, 1.2, and 1.3, respectively.

2 Proof of Theorem 1.1

As was mentioned in the introduction, we analyze the modified greedy algorithm suggested by Caprara et al. in [4]. Specifically, this algorithm performs iteratively the following steps:

1. While G contains a vertex v of degree ≤ 1, delete v (and the edge incident to v, if exists);

2. while G contains a vertex v of degree 2 with neighbors u and w, delete v and edges (v, u), (v, w) and replace them by a new edge (u, w);

3. find a shortest cycle C in G, add C to the constructed solution and remove its edges from G.

Steps 1,2 and 3 are repeated until there are no edges left in G.

2.1 Upper bound

THEOREM 2.1. *"Modified greedy" computes a cycle packing of size $\Omega(\nu_c^*(G)/\sqrt{\log n})$. Thus it is an $O(\sqrt{\log n})$-approximation algorithm.*

Proof. Note that Steps 1 and 2 of the algorithm do not change the value of an optimal solution. We split the execution of the algorithm into two phases. In Phase 1 the length of every added cycle does not exceed $\sqrt{\log |G|}$ (where here $|G|$ is the number of vertices in the *current* graph). Phase 2 starts when a cycle added to an approximate packing has length more than $\sqrt{\log |G|}$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the set of cycles added to the packing during Phase 1 and Phase 2, respectively. Fix an optimal fractional packing ψ^*, so $|\psi^*| = \nu_c^*(G)$. Let ψ_1^* be the restriction of ψ^* to cycles that intersect some cycle from $\mathcal{C}_1$, $\psi_2^* = \psi^* - \psi_1^*$. Since every cycle from $\mathcal{C}_1$ has length $\leq \sqrt{\log n}$

$$|\psi_1^*| \leq |\mathcal{C}_1| \sqrt{\log n}\ .$$

Let now G_2 be the graph in the beginning of the second phase, i.e. right before taking a cycle of length more than $\sqrt{\log |G_2|}$. At this stage G_2 has girth more than $\sqrt{\log |G_2|}$ and minimum degree at least 3. As ψ_2^* is a fractional packing in G_2, it follows that

$$|\psi_2^*| \leq |E(G_2)|/girth(G_2) < |E(G_2)|/\sqrt{\log |G_2|}\ .$$

Now, Bollobás and Thomason [3] proved that if a graph G satisfies $|E(G)| \geq (1+\epsilon)|V(G)|$ for an $\epsilon > 0$, then G contains a cycle of length at most $\frac{1}{\epsilon}\log(2\epsilon n)$. Recall that the minimal degree in G_2 is at least 3, implying $|E(G_2)| \geq \frac{3}{2}|G_2|$. Thus before getting a graph with less than $\frac{5}{4}|E(G_2)|$ edges, the algorithm will pick in G_2 cycles of length at most $4\log(8|G_2|)$, and therefore the number of such cycles will be at least

$$\frac{|E(G_2)| - \frac{5}{4}|G_2|}{4\log(8|G_2|)} \geq \frac{|E(G_2)|}{24\log(8|G_2|)}\ ,$$

resulting in:

$$|\mathcal{C}_2| \geq \frac{|E(G_2)|}{24\log(8|G_2|)} > \frac{|E(G_2)|}{25\log|G_2|}\ .$$

Comparing the obtained bounds for $|\psi_2^*|$ and $|\mathcal{C}_2|$ we conclude that

$$|\psi_2^*| \leq 25|\mathcal{C}_2|\sqrt{\log|G_2|} \leq 25|\mathcal{C}_2|\sqrt{\log n}\ .$$

Altogether,

$$\begin{aligned} |\psi^*| = |\psi_1^*| + |\psi_2^*| &\leq |\mathcal{C}_1|\sqrt{\log n} + 25|\mathcal{C}_2|\sqrt{\log n} \\ &\leq 25\sqrt{\log n} \cdot |\mathcal{C}|\ . \end{aligned}$$

□

2.2 A tight example

THEOREM 2.2. *The approximation ratio of "Modified greedy" is $\Omega(\sqrt{\log n})$.*

For the proof we will need the following technical lemma.

LEMMA 2.1. *Let G be a graph on n vertices of maximal degree at most 7. Let $V_0 \subseteq V(G)$. If $|V_0| \geq n/2$ then there exists a subset $U \subset V_0$ of size $|U| = \lceil \log n \rceil$, such that all vertices of U are at distance more than $\frac{1}{3} \log n$ from each other.*

Proof. Note that every vertex $v \in G$ is at distance at most k from at most $7 \cdot 6^{k-1} < 7^k$ vertices from G. Define an auxiliary edge set E_0 on V_0 so that $(u,v) \in E_0$ if $dist_G(u,v) \leq \frac{1}{3} \log n$. Let $H = (V_0, E_0)$. Then H is a graph on at least $n/2$ vertices of maximal degree $\Delta(H) < 7^{\frac{1}{3}\log n} < n^{0.95}$, and has therefore an independent set U of size at least $|V(H)|/(1+\Delta(H)) > \log n$. Each such independent set gives a required set of vertices in G. $\square$

The *k-sunflower* S^k is a cycle of length k (the *core cycle*) to each edge of which we attach a cycle of length $k+1$ (a *petal*), so that the petals are vertex-disjoint outside the core cycle. The number of vertices of S^k is k^2. Observe that the core is the shortest cycle in a k-sunflower, and removing its edges results in a cycle on k^2 vertices. We choose $k = \sqrt{\log n/3}$ and denote $t = k^2$ (we ignore floors and ceilings as they do not affect the asymptotic nature of our result).

Let now G_0 be a 3-regular graph on n vertices of girth more than $t = \frac{1}{3} \log n$. Such graphs exist for infinitely many values of n as proved by Erdős and Sachs [7]. We start with $G = G_0$, set $W = \emptyset$, $i = 1$, and repeat $n/(2t)$ times the following procedure:

1. Find a subset $U_i \subset V \setminus W$ such that $|U_i| = t$ and all vertices of U_i are at distance more than $\frac{1}{3} \log n$ from each other in G;

2. Insert a copy S_i of the k-sunflower in U_i, placing it arbitrarily within U_i;
update $W \leftarrow W \cup U_i$; $i \leftarrow i+1$.

Since the sets U_i are disjoint and the maximum degree of S^k is 4, the graph G has maximum degree at most 7 during the execution of the above procedure. Also, $|W| \leq \frac{n}{2t} \cdot t = \frac{n}{2}$, and therefore finding a required U_i at each step is possible due to Lemma 2.1. Let us denote by G^* the final graph of the above procedure.

CLAIM 2.1. *Let C be a cycle of length at most $\frac{1}{3} \log n$ in G^*. Then C is a cycle in one of the inserted k-sunflowers S_i.*

Proof: Since $girth(G_0) > \frac{1}{3} \log n$, C contains an edge $e \in E(G^*) - E(G_0)$. Let $i^* = \max\{i : E(C) \cap E(S_i) \neq \emptyset\}$. We claim that C is a cycle in S_{i^*}. Let G_{i^*} be the graph created during the above described procedure after having inserted the sunflower S_{i^*}. Obviously, $C \subset G_{i^*}$. If $E(C) \subset E(S_{i^*})$ we are done. Assume otherwise. Since U_{i^*} spans only the edges of S_{i^*} in G_{i^*}, at some point C leaves U_{i^*} and then returns back. Let $u_1, u_2 \in U_{i^*}$ be the vertices of U_{i^*} where C leaves and reenters U_{i^*}. By our choice of U_{i^*}, $dist_{G_{i^*}}(u_1,u_2) > \frac{1}{3} \log n$, implying $|C| > \frac{1}{3} \log n$, a contradiction. $\square$

Completing the proof of Theorem 2.2: We analyze the performance of the modified greedy algorithm on G^*. By Claim 2.1, the shortest cycles in G^* are the $n/(2t) = O(n/\log n)$ core cycles of the inserted sunflowers, which are vertex-disjoint. Hence the algorithm starts by picking all of them. After all core cycles have been removed, none of the sunflowers contains a cycle of length at most $\frac{1}{3} \log n$, and applying Claim 2.1 again we infer that the modified greedy algorithm will be able to add at most $|E(G^*)|/(\log n/3) = O(n/\log n)$ cycles, altogether ending up with $O(n/\log n)$ cycles. On the other hand, a feasible solution can be obtained by taking all petals of all inserted sunflowers, whose total number is $(n/(2t)) \cdot k = \Theta(n/\sqrt{\log n})$. It follows that the approximation ratio of the modified greedy on G^* is

$$\Omega\left(\frac{\frac{n}{\sqrt{\log n}}}{\frac{n}{\log n}}\right) = \Omega(\sqrt{\log n}) \ .$$

$\square$

3 Proof of Theorem 1.2

It will be convenient to describe the algorithm with a certain parameter ℓ, which will be eventually set to $\ell = \sqrt{n}$. The algorithm starts with $\mathcal{C}_1, \mathcal{C}_2 = \emptyset$ and in the end outputs $\mathcal{C}_1 \cup \mathcal{C}_2$.

Phase 1:
As long as there is a directed cycle of length $\leq \ell$, find such a cycle, add it to $\mathcal{C}_1$, and delete its edges from the graph.

Phase 2:
For each $v \in V$, compute a maximum size set $\mathcal{C}_2(v)$ of edge-disjoint directed cycles that contain v. Among the packings computed, let $\mathcal{C}_2$ be one of maximal size.

THEOREM 3.1. *For $\ell = \sqrt{n}$ the algorithm computes a packing $\mathcal{C}_1 \cup \mathcal{C}_2$ of size at least $\nu_c^*(G)/\sqrt{n}$.*

Proof. As in the proof of Theorem 2.1, let us fix an optimal fractional packing ψ^*, let ψ_1^* be the restriction of ψ^* to cycles that intersect some cycle from $\mathcal{C}_1$, $\psi_2^* = \psi^* - \psi_1^*$. Since every cycle from $\mathcal{C}_1$ has length $\leq \ell$ we have $|\psi_1^*| \leq \ell|\mathcal{C}_1|$. We claim that $|\mathcal{C}_2| \geq \ell|\psi_2^*|/n$. Thus by combining the bounds for $|\mathcal{C}_1|, |\mathcal{C}_2|$ and substituting $\ell = \sqrt{n}$ we get:

$$|\mathcal{C}_1| + |\mathcal{C}_2| \geq |\psi_1^*|/\ell + \ell|\psi_2^*|/n$$

$$= (|\psi_1^*| + |\psi_2^*|)/\sqrt{n} = |\psi^*|/\sqrt{n} \ .$$

To see that $|\mathcal{C}_2| \geq \ell|\psi_2^*|/n$, let G_2 be the graph at the beginning of Phase 2. For each $v \in V$ let $\psi_2^*(v)$ be the restriction of ψ_2^* to the cycles in G_2 containing v. Note that for every $v \in V$ we can compute $\mathcal{C}_2(v)$ using any max-flow algorithm and flow decomposition. By the integrality of an optimal flow from the Max-Flow Min-Cut Theorem, $|\mathcal{C}_2| \geq |\psi_2^*(v)|$ for every vertex v. Thus, since every cycle in G_2 has length $> \ell$, we have:

$$n|\mathcal{C}_2| \geq \sum_{v \in V} |\psi_2^*(v)| \geq \ell|\psi_2^*| \ .$$

□

Although we are unable to prove that $\Theta(\sqrt{n})$ is also a lower bound for the integrality gap of directed cycle packing, we conjecture this is the case. This conjecture is supported by the following construction showing that $\Theta(\sqrt{n})$ is a lower bound for the odd directed cycle packing problem (namely, the maximum number of edge-disjoint directed cycles of odd length).

PROPOSITION 3.1. *For infinitely many n, there exists a digraph G on n vertices, in which every pair of odd cycles has a common edge, and yet $\nu^*_{oddc}(G) = \Omega(\sqrt{n})$, where $\nu^*_{oddc}(G)$ is the fractional odd cycle packing number of G.*

Proof (sketch). Let N be an odd positive integer, and consider the digraph D_N whose vertices are (i,j) for $i,j = 1,\ldots,N$. The edges of D_N emanate from (i,j) to $(i+1,j)$ for $i = 1,\ldots,N-1$ and $j = 1,\ldots,N$ and from (i,j) to $(i,j+1)$ for $i = 1,\ldots,N$ and $j = 1,\ldots,N-1$. There are also edges from (i,N) to $(N+1-i,1)$. One can check that D_N does not have two *vertex-disjoint* odd directed cycles (we omit the details). To estimate from below the fractional odd cycle packing number of G, for each $1 \leq i \leq (N+1)/2$, define the cycle C_i as follows:

$$\begin{aligned} C_i \quad = \quad & ((i,1),(i,2),\ldots,(i,i),(i+1,i),(i+2,i),\ldots, \\ & (N+1-i,i),(N+1-i,i+1), \\ & (N+1-i,i+2),\ldots,(N+1-i,N),(i,1)) \end{aligned}$$

(i.e. C_i starts at $(i,1)$, goes horizontally till (i,i), then drops vertically to $(N+1-i,i)$ and then again goes horizontally till $(N+1-i,N)$ and finally returns to $(i,1)$). It is easy to see that each vertex of D_N belongs to at most two cycles C_i, and therefore, giving value $\psi(C_i) = 0.5$ to each cycle C_i, we obtain a fractional odd cycle packing of value $(N+1)/4$. Now, by replacing each vertex v of D_n with the path v_{in}, v_{mid}, v_{out} and replacing each edge (u,v) with the edge (u_{out}, v_{in}) we obtain a new graph D'_N with $3N^2$ vertices. Any set of edge-disjoint directed cycles in D'_N is also vertex-disjoint, and corresponds to a set of vertex-disjoint directed cycles in D_N. Furthermore, any odd (even) cycle in D_N corresponds to an odd (even) cycle in D'_N. Thus, by letting $n = 3N^2$ the desired construction follows. □

4 Proof of Theorem 1.3

In this section we consider simple digraphs only. The greedy algorithm for the maximum S-cycle packing problem repeatedly chooses a shortest S-cycle and removes its edges from the graph.

THEOREM 4.1. *Given a subset S of vertices of a simple digraph G, the greedy algorithm finds a set of at least $\tau_c(G,S)/(5n^{2/3})$ edge-disjoint directed S-cycles in G.*

Let $f(n,\ell)$ be the maximum of $\tau_c(G)$ taken over all simple digraphs G on n vertices with $girth(G) > \ell$. It is easy to see that if $\mathcal{C}$ is a cycle packing computed by the greedy algorithm on G, then $\tau_c(G) \leq \ell|\mathcal{C}| + f(n,\ell)$ for any positive integer ℓ. A similar statement holds for the analogous definition of $f(n,l)$ in the undirected case. In fact, a similar statement holds for the analogous vertex-disjoint (directed or undirected) cycle packing and cycle cover problems. In the undirected vertex-disjoint case Komlós [8] showed that $f(n,\ell) = \Theta(\frac{n}{\ell}\ln(n/\ell))$. In the directed vertex-disjoint case, Seymour [9] showed that $f(n,\ell) \leq 4\frac{n}{\ell}\ln(4n/\ell)\ln\log(4n/\ell)$. He also gave an example showing that $f(n,\ell) = \Omega(\frac{n}{\ell}\ln(n/\ell))$. In the edge-disjoint case, answering an earlier conjecture of Bollobás, Erdös, Simonovits, and Szemerédi [2], Komlós [8] established the asymptotically tight bound $f(n,\ell) = \Theta(\frac{n^2}{\ell^2})$ in undirected graphs.

We generalize this by defining $h(n,\ell)$ to be the maximum of $\tau_c(G,S)$ taken over all simple digraphs G on n vertices and $S \subseteq V(G)$ so that every S-cycle in G has length $> \ell$. Let $\tilde{\nu}(G,S)$ denote the size of an S-cycle packing computed by some run of the greedy algorithm.

LEMMA 4.1. *For any positive integer ℓ,*

$$\tau_c(G,S) \leq \ell\tilde{\nu}(G,S) + h(n,\ell) \leq (\ell + h(n,\ell))\tilde{\nu}(G,S).$$

Proof. Fix an optimal cover F with $|F| = \tau_c(G,S)$, and partition it into two sets F_1 and F_2, where F_1 are the edges contained in S-cycles of length $\leq \ell$ of the S-packing computed. Then $|F_1| \leq \ell\tilde{\nu}(G,S)$, since every S-cycle of length $\leq \ell$ in the packing computed contains at least one edge from F_1. On the other hand $|F_2| \leq h(n,\ell)$, by the optimality of $|F|$ and by the definition of $h(n,\ell)$. The result follows. □

For digraphs, the bound $h(n, \ell) = O((n^2/\ell^2)\log^2(n/\ell))$ can be deduced from [10, Theorem 1.1] where a more general problem was considered. We will show that $h(n, \ell) = \Theta(n^2/\ell^2)$ using the following lemma of Komlós [8].

Lemma 4.2. ([8], Lemma 3) *Let $a_0, a_1, \ldots, a_t$ be a sequence of non-negative real numbers, and denote $s_k = \sum_{i=0}^{k} a_i$. Then there exist $k \in \{0, \ldots, t-1\}$ such that $a_k a_{k+1} < \frac{2e}{t^2} s_k s_t$.* □

Corollary 4.1. *Let $a_0, a_1, \ldots, a_t$ be a sequence of integers, and denote $s_k = \sum_{i=1}^{k} a_i$ and $p = \lceil t/2 \rceil$. Suppose that $s_p \leq s_t/2$. Then there exists $k \in \{0, \ldots, p-1\}$ such that:*

$$a_k a_{k+1} < \frac{2e}{p^2} s_k s_p \leq \frac{4e}{t^2} s_k s_t.$$

□

Lemma 4.3. *Let S be a subset of vertices of a simple digraph G on n vertices so that every S-cycle in G has length $> \ell$. Then there exists an S-cycle edge-cover F with $|F| \leq 4e(n/\ell)^2$. Moreover, such F can be found in polynomial time.*

Proof. The proof is by induction on n. If G has no S-cycles, in particular if it has ℓ vertices or less, the statement is obvious. We can also assume that G is strongly connected; otherwise, validity of the result for every strongly connected component of G implies the result for G.

Since every S-cycle in G has length $> \ell$, there are vertices u, v with $u \in S$ and $v \in V(G)$ such that every (u, v)-dipath has length $\geq \ell$, and hence there is a partition of $V(G)$ into nonempty sets $X_0, \ldots, X_t$, where $t \geq \ell$, such that no edge of G has tail in X_i and head in X_j, for $j \geq i+2$. Let $a_i = |X_i|$ for $i = 0, \ldots, t$, and let s_k and p be as in Corollary 4.1. Notice that $s_t = n$. We may assume that $s_p \leq n - s_p$, since otherwise we may consider the reversed sequence of $a_0, \ldots, a_t$. By Corollary 4.1, there exists $k \in \{0, \ldots, p-1\}$ such that:

$$a_k a_{k+1} < \frac{4e}{t^2} s_k n.$$

Let F' be the edge cut consisting of the set of edges going from X_k to X_{k+1} (if we consider the reversed sequence, then we take also the "reversed" cut). Then, since G is simple

$$|F'| \leq a_k a_{k+1} < \frac{4e}{t^2} s_k n.$$

We delete F' and apply the inductive hypothesis to the subgraphs G_1 and G_2 of G induced by the corresponding parts $V_1 = X_1 \cup \cdots \cup X_k$ and $V_2 = X_{k+1} \cup \cdots \cup X_t$. Clearly, any S-cycle in $G - F'$ is entirely contained either in G_1 or in G_2.

To summarize, we can find a cut F' that divides G into two subgraphs G_1 and G_2, where G_i has n_i vertices, such that $n_1 + n_2 = n$ and $n_1 \leq n/2 \leq n_2$, and such that $|F'| \leq \frac{4e}{\ell^2} n_1 n$. We need to prove that:

$$|F'| + 4e\left(\frac{n_1^2}{\ell^2} + \frac{n_2^2}{\ell^2}\right) \leq 4e\frac{n^2}{\ell^2}.$$

Indeed,

$$|F'| + 4e\left(\frac{n_1^2}{\ell^2} + \frac{n_2^2}{\ell^2}\right) \leq \frac{4e}{\ell^2}(n_1 n + n_1^2 + n_2^2)$$

$$< \frac{4e}{\ell^2}(2n_1 n_2 + n_1^2 + n_2^2) = 4e\frac{n^2}{\ell^2}.$$

□

The bound in Lemma 4.3 is tight up to a constant factor even for $S = V$, as can be seen by taking the blowup of a directed ℓ-cycle.

By Lemmas 4.1 and 4.3 we deduce:

Corollary 4.2. *Let S be a subset of vertices of a simple digraph G on n vertices. Then for any integer ℓ,*

$$\tau_c(G, S) \leq (\ell + 4e(n/\ell)^2)\tilde{\nu}(G, S).$$

In particular for $\ell = 2e^{1/3}n^{2/3}$ we have $\tau_c(G, S) \leq 3e^{1/3}n^{2/3}\tilde{\nu}(G, S) < 5n^{2/3}\tilde{\nu}(G, S)$ and this also completes the proofs of Theorems 4.1 and 1.3. □

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